Using Powers of Phi to represent Integers (Base Phi)

If you have already looked at the page where we showed how to represent integers using the Fibonacci numbers, and you have also read about the numerical properties of powers of Phi then this page takes you a stage further - writing the integers in base Phi!

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Powers of Phi

Here is part of the table of numerical properties of powers of Phi:

Here Phi = 1·6180339... = phi^{-1}
and phi = 0·6180339... = Phi − 1 = 1/Phi = Phi^{-1}

Phi power	phi power	A + B Phi	C + D phi	real value
Phi^{5}	phi^{-5}	3 + 5 Phi	8 + 5 phi	11·0901699..
Phi^{4}	phi^{-4}	2 + 3 Phi	5 + 3 phi	6·8541019..
Phi^{3}	phi^{-3}	1 + 2 Phi	3 + 2 phi	4·2360679..
Phi^{2}	phi^{-2}	1 + 1 Phi	2 + 1 phi	2·6180339..
Phi^{1}	phi^{-1}	0 + 1 Phi	1 + 1 phi	1·6180339..
Phi^{0}	phi^{0}	1 + 0 Phi	1 + 0 phi	1·0000000..
Phi^{-1}	phi^{1}	-1 + 1 Phi	0 + 1 phi	0·6180339..
Phi^{-2}	phi^{2}	2 - 1 Phi	1 - 1 phi	0·3819660..
Phi^{-3}	phi^{3}	-3 + 2 Phi	-1 + 2 phi	0·2360679..

Phi^{-4}	phi^{4}	5 - 3 Phi	2 - 3 phi	0·1458980..
Phi^{-5}	phi^{5}	-8 + 5 Phi	-3 + 5 phi	0·0901699..

We can capture these relationships precisely in two formulae:

$$Phi^{n} = Fib(n-1) + Fib(n)\ Phi$$

$$Phi^{n} = Fib(n+1) + Fib(n)\ phi$$

It is not difficult to prove (by Induction) that these formulae are indeed correct. They both apply to negative n as well, if we extend the Fibonacci series backwards:

..., -8, 5, -3, 2, -1, 1, 0, 1, 1,2, 3, 5, 8, ...

where we still have the **Fibonacci property**: Fib(n) = Fib(n-1) + Fib(n-2) but it now holds ***for all values of n, positive, zero and negative***!

Another property of this extended Fibonacci series of numbers is that

Fib(−n) = **−**Fib(n) for **even** n
Fib(−n) = Fib(n) for **odd** n

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Integers as sums of powers of Phi

In the table of powers of phi above, you will have noticed that the same multiples of Phi occur, sometimes positive and sometimes negative. For example, **2 phi** occurs in both Phi^{3} = 3 + 2 phi and Phi^{-3} = -1 + 2 phi. If we *subtract* these two powers, the *multiples* of phi will disappear and leave us with an integer.
Similarly, **3 phi** occurs in both Phi^{4} = 5 + 3 phi and Phi^{-4} = 2 - 3 phi. If we *add* these two powers, again the *multiples* of phi will cancel out and leave an integer.

Here are some more examples:

$$Phi^{1} + Phi^{-2} = (1 + 1\ phi) + (1 - 1\ phi) = 2$$
$$Phi^{2} + Phi^{-2} = (2 + 1\ phi) + (1 - 1\ phi) = 3$$
$$Phi^{4} + Phi^{-4} = (5 + 3\ phi) + (2 - 3\ phi) = 7$$

So we have expressed the integers 2, 3 and 7 as **a sum of powers of Phi**.

Because Phi^{0} is just 1, we can ***add 1*** (=Phi^{0}) to those numbers above and so represent 3 (again), 4 and 8 as a sum of powers of Phi.

We can also add combinations of these numbers and get other ones too. In all of them, we are writing the integer as a sum of *different* powers of Phi.

$$4 = 3 + 1 = (Phi^{2} + Phi^{-2}) + Phi^{0}$$
$$8 = 7 + 1 = (Phi^{4} + Phi^{-4}) + Phi^{0}$$

$$9 = 2 + 7 = (Phi^1 + Phi^{-2}) + (Phi^4 + Phi^{-4})$$
$$10 = 3 + 7 = (Phi^2 + Phi^{-2}) + (Phi^4 + Phi^{-4})$$

This reminds us of expressing numbers as :

- sums of powers of 2 (binary), or
- sums of powers of 3 (ternary), or
- sums of powers of 8 (octal) and, of course, the usual way using
- sums of powers of 10 (decimal)!

All the above are powers of an integer (2, 3, 8 or 10) but the really unusual thing here is that we are taking powers of *Phi, an **irrational** number and adding them to get a pure integer*!

A natural question now is:

Are all integers representable as sums of powers of Phi?

The answer is **Yes!** The number n is just n copies of Phi^0 added together!!! ☺
So let's rephrase the question...
What we *really* meant to ask was how to do this **using *only* powers of Phi and *not repeating any power more than once* in the sum** (which is what we did in the examples above).

Things to do

1. $1 = Phi^0$ and
   $1 = Phi^{-1} + Phi^{-2}$ and
   By expanding Phi^{-n} (= phi^n) as $Phi^{-(n+1)}+Phi^{-(n+2)}$ how many more ways can you find to sum powers pf Phi to a total of 1 if no power of Phi can be used more than *once*? e.g.
   $Phi^{-2} = Phi^{-3}+Phi^{-4}$ so
   $1 = Phi^{-1} + Phi^{-2}$ expands to
   $1 = Phi^{-1} + Phi^{-3} + Phi^{-4}$
2. Try to express each of the following numbers as a sum of *different* powers of Phi each power occurring no more than once.
   You could check your answers in two ways:
   (a) on your calculator to see if you are approximately right but a more precise method is...
   (b) to use the exact values by translating all the powers of Phi into sums of integers and *multiples of Phi* using the formula
   $Phi^n = Fib(n+1) + Fib(n)\ phi$
   so that you can check that all the multiples cancel out:
   a. 5 as the sum of 2 and 3

   b. 5 as the sum of 4 and 1
      (use your answers to the first question using different representations of 1)
   c. 6
   d. 6 again, but find a different answer this time
   e. 9 Find THREE different answers!
   f. 10
   g. 11
   h. 12
   i. each of the numbers from 13 to 20
3. Of your representations of number 6 in the previous question, which answer has **the fewest powers of Phi**?
4. Find a table of answers for all the values from 1 to 20 but all your answers should have the fewest number of powers in them.

From your answers to the above questions, it may look like many numbers can be expressed in Base Phi. Do you think that ALL whole numbers can be?

> **If you do**, how would you try to *convince* someone of this?
> **If you do not**, which integer do you think does NOT have a Base Phi representation? (Are you sure?)

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Base Phi Representations

Let's use what we learned on the Fibonacci Bases Page to write down our sums-of-distinct-powers-of-Phi representations of a number. As in decimal notation, the columns represent the powers of the Base, but for us the base is Phi, not 10. We have negative powers of Phi as well as positive ones, so, just as in decimal fractions, we need a "point" to separate the positive powers of Phi from the negative ones.

So if 1·25 in decimal means

powers of 10:	...	3	2	1	0	.	-1	-2	...
					1	**.**	**2**	**5**	

$= 1 + 2\text{x}10^{-1} + 5\text{x}10^{-2}$

then
2 = $Phi^1 + Phi^{-2}$ so 2 in Base Phi is

powers of Phi:	...	3	2	1	0	.	-1	-2	...
				1	0	.	0	1	

= $Phi^1 + Phi^{-2}$

which we write as $2=10{\cdot}01_{Phi}$ to indicate that it is a Base Phi representation.

There are Base Phi Representations for all whole numbers

You might like to convince yourself that, by successively adding 1's, if necessary applying the Expanding 1's process, then we can always find a way of representing ANY integer as sum of distinct powers of Phi. By applying the Reducing 1's process as often as necessary, we can then always find a base Phi representation that has the minimum number of 1's and no two of them will be next to each other.

Using the digits 0 and 1 only, we can express every integer as a sum of some powers of Phi

Things to do

1. How unusual is this property? Could we express every integer as sum of powers of √2? (Hint: think about even powers of √2)
2. What about powers of **e**=2.71828182... or π=3.1415926535... or some other *irrational* value like Phi that has no integer power equal to an integer?

Reducing the number of 1's in a Base Phi Representation

We haven't used much of the theory about Fibonacci numbers yet (those formulae further up this page). There are some interesting and relevant facts in the Formula for powers of Phi that we saw on the Phi's Fascinating Figures page. One of these was

$$Phi^n = Phi^{n-1} + Phi^{n-2}$$

This tells us that, **if ever we find two consecutive 1's in a Base Phi representation, we can replace them by an additional one in the column to the left**

For instance,

$$3 = 2 + 1 = 10{\cdot}01_{Phi} + 1{\cdot}0_{Phi} = 11{\cdot}01_{Phi}$$

but we can replace the two consecutive 1's by a 1 in the phi^2 column:

$$3 = 100{\cdot}01_{Phi}$$

Let's call this the **Reducing 1's Process**.

*But what happens if we have **three or more 1s next to each other**?*
Find the leftmost 11 and start there as there will always be two consecutive ones that have a zero on their left.
This will replace the two ones by zeros and so any following 11 will not have a zero in front of them.
We can always start with the leftmost pair of ones and then repeat the Reducing 1's Process on the new form if necessary until we eliminate all "11" from our representation.

Repeatedly applying the Reducing 1's process means that we can reduce a Base Phi representation until eventually we have no pairs of consecutive 1's

The base Phi representation of N with **the least number of 1s** is called the **minimal** representation of N

It follows from this definition that

a. Every minimal base Phi representation has no consecutive 1s
b. The minimal representation has the least number of 1s
c. Every minimal representation has the greatest number of 0s ignoring any leading and trailing 0s
d. Every whole number has a minimal base Phi representation

Expanding the number of 1's in a Base Phi Representation

What if we get more than one of a certain power of Phi?
The solution here is to use the same formula but *backwards*, that is, **replacing a 1 by 1's in the two columns to the right**. So that, whenever we have

```
...100... we can replace it by ...011...
```

Let's call this **the Expanding 1's Process**.

EG 2 = 1+1	$= 1{\cdot}0_{Phi}+1{\cdot}0_{Phi}$ Expanding the second 1·0 into 0·11:
	$= 1{\cdot}0_{Phi}+0{\cdot}11_{Phi}$ Now we can add without getting more than 1 in any column:
	$= 1{\cdot}11_{Phi}$ and we are ready to apply the Reducing 1's process:
	$=10{\cdot}01_{Phi}$

But every representation **will end with a "1"**, which we can always expand into "011".
2 in base Phi is 10.01 and also 10.0011,
but we can expand the final 1 of 10.0011 to get the new form of 10.001011
and repeat on the final one again to give 10.00101011
and then 10.0010101011 and so on for ever!
All representations can be expanded to get an infinitely long tail of 010101...01011 !
To avoid this, we decide that

We will ignore base Phi representations that end ...011

But there is another case to consider too:
Any representation ending in ...101 is equivalent to ...10011 and now we can convert the initial 100 into 011 to get ...01111.
Note that the original and the final forms here have the same number of 0s.
For instance:

4 = 101.01 = 100.1111
10 = 1111.0101 = 1111.001111 = 1110.111111

This leads us to a decision to make about the base Phi representation for n which contains the most 1s: which we want to call *the maximal* base Phi representation.
For instance, 27 has the following representations with no consecutive 0s:-

27 = 111011.110101 is *the shortest base Phi representation of 27*.
Now we use the two expansions just explained to convert an ending of 101 into 01111:
27 = 111011.11**00**01111 but now another "100" appears which we can convert to reduce the number of 0s:
27 = 111011.1**011**1111 and this now has *the least number of 0s in any base Phi representation of 27*.

So here we define the **maximal** representation as follows

The base Phi representation of N that does not end in 011 and with **the greatest number of 1s** is called the **maximal** base Phi representation of N.

Several things follow from this definition:

a. Every maximal representation has no two consecutive 0s
b. Every maximal representation has the greatest number of 1s
c. Every maximal representation has the least number of 0s
d. Every whole number has a maximal base Phi representation

Note that if we had use "the least number of 0s" in the definition and not "the greatest number of 1s" then we would not have got a unique maximal representation:
for instance, 7 has representations 1010.1101 and 10101.01111 both of which contain the least number of 0s (three) but only the latter has the greatest number of 1s (seven) amongst all the base Phi reps of 7.

Comparing the Minimal amd Maximal Representations

Here is a table of the minimal and maximal base Phi representations of 1 up to 30:

N	Minimal rep no 11s fewest 1s	Maximal rep no 00s most 1s
1	1 .	1 .
2	10 . 01	1 . 11
3	100 . 01	10 . 1111
4	101 . 01	11 . 1111
5	1000 . 1001	101 . 1111
6	1010 . 0001	111 . 0111
7	10000 . 0001	1010 . 101111
8	10001 . 0001	1011 . 101111
9	10010 . 0101	1101 . 101111
10	10100 . 0101	1110 . 111111
11	10101 . 0101	1111 . 111111
12	100000 . 101001	10101 . 111111
13	100010 . 001001	10111 . 011111
14	100100 . 001001	11010 . 110111
15	100101 . 001001	11011 . 110111
16	101000 . 100001	11101 . 110111
17	101010 . 000001	11111 . 010111
18	1000000 . 000001	101010 . 10101111
19	1000001 . 000001	101011 . 10101111
20	1000010 . 010001	101101 . 10101111
21	1000100 . 010001	101110 . 11101111
22	1000101 . 010001	101111 . 11101111
23	1001000 . 100101	110101 . 11101111
24	1001010 . 000101	110111 . 01101111
25	1010000 . 000101	111010 . 10111111
26	1010001 . 000101	111011 . 10111111
27	1010010 . 010101	111101 . 10111111
28	1010100 . 010101	111110 . 11111111
29	1010101 . 010101	111111 . 11111111
30	10000000 . 10101001	1010101 . 11111111

1		1.
2	10.01	1.11
3	100.01	
4		11.1111
7	10000.0001	
11		1111.111111
18	1000000.000001	
29		111111.11111111

Some patterns are visible in the table above and shown on the left here:
2,1,3,4,7,11,18,29 are formed in the same way as the Fibonacci numbers, by adding the latest two to get the next, but instead of starting with 0 and 1 as we do for the Fibonacci Numbers, we start with 2 and 1.
They are called the Lucas numbers, denoted L(n), and almost always appear where the Fibonacci numbers do!
Sum the two Fibonacci numbers on either side of each Fibonacci number and you generate each Lucas number:

$L(n) = F(n-1) + F(n+1)$

The formula for the Lucas numbers involves Phi and phi too:
$L(n) = Phi^n + (-phi)^n = Phi^n + (-Phi)^{-n} = Phi^n + (-1)^n\ Phi^{-n}$ For instance

$L(2) = 3 = Phi^2 + Phi^{-2}$ and so is 100.01 in Base Phi.
$L(4) = 7 = Phi^4 + Phi^{-4}$ and so is 10000.0001 in Base Phi.

But for L(0) = 2, we get $Phi^0 + Phi^{-0}$ so the two powers of Phi are the same and 2 is 2 Phi^0.
The significance of the maximal representation is seen in the following table of sums all with a total of eleven:

```
1+     1.          2+   10.01        3+  100.01        4+  101.01
10 1110.111111     9  1101.101111    8  1011.101111    7  1010.101111
11 1111.111111     11 1111.111111    11 1111.111111    11 1111.111111
```

Since 11, the fifth Lucas number, L(5), has a maximal representation of 1111.111111 with *no zeros* then the 1s of the top number in the sum are the 0s in the bottom number.
Sometimes we need to change the final 1 (of either number) into 011 to make the number of phigits after the (phigital-)point the same as in 5+6=11 :

```
5+ 1000.1001  = 1000.100011+
6     111.0111      111.0111
11               1111.111111
```

The next number with a maximal representation with no zeros is L(7), 29, and you will notice the same patterns between 1 and 28, 2 and 27, 3 and 26, etc where the "h***0***les" in the larger numbers are "f***1***lled" in the smaller and vice-versa.

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All forms of Base Phi representation

By converting "100" to "011" (or vice-versa) in any Base Phi representation, we get another valid base Phi form.
As we saw above when converting to the maximal form (no consecutive 0s) we can always convert the final "1" to "011" and so continue the expansion of N in Base Phi for ever. So we will only count finite Base Phi forms which do not end in "011" to prevent this.
Here is a table of equivalent Base Phi forms for the values 1 to 11:

1	2	3	4	5	6	7	8	9	10	11
1	10.01 1.11	100.01 11.01 10.1111	101.01 100.1111 11.1111	1000.1001 1000.0111 110.1001 110.0111 101.1111	1010.0001 1001.0111 1001.1001 111.1001 111.0111	10000.0001 1100.0001 1011.0001 1010.1101 1010.101111	10001.0001 10000.1101 10000.101111 1101.0001 1100.1101 1100.101111 1011.1101 1011.101111	10010.0101 10010.001111 10001.1101 10001.101111 1110.0101 1110.001111 1101.1101 1101.101111	10100.0101 10100.001111 10011.0101 10011.001111 10010.111111 1111.0101 1111.001111 1110.111111	10101.0101 10101.001111 10100.111111 10011.111111 1111.111111

The counts here are 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5 all of which should look familiar to you by now! However the pattern does not continue and is not even solely Fibonacci numbers.

A Phigits Calculator

CALCULATOR

Clear Convert 1 up to 10 to Base Phi

minimal form only
maximal form only
show all forms
count all forms

Clear Convert 100000.011111 to Base 10

RESULTS CLEAR RESULTS

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Other names for Base Phi

Representations of an integer n as a sum of *different* powers of Phi are aclled the **Base Phi representation of n** on this page.
Other names that have been suggested are

- **Phigital**: compare with digital for Base Ten;
- **Phinary**: compare with Binary since we are also using just the digits 0 and 1 but to base Phi [with thanks to Marijke van Gans for this term];
- expressing a number in **Phigits** [With thanks to Prof Jose Glez-Regueral of Madrid for mentioning this one.]

Links and References

Some of the above originally appeared in an article by George Bergman, in the **Mathematics Magazine** 1957, Vol 31, pages 98-110, where he also gives pencil-and-paper methods of doing arithmetic in Base Phi.

C. Rousseau **The Phi Number System Revisited** in *Mathematics Magazine* 1995, Vol 68, pages 283-284.

Prof Alexey Stakhov investigates the applications of Fibonacci and Phi number systems for representing numbers in a computer rather than the familiar binary system. His web site has lots more information.

Fibonacci Home Page

Phi's Fascinating Figures

The Mathematical Magic of the Fibonacci Numbers

WHERE TO NOW???

The next topic is... The Golden String

This is the last page on this Topic.