NUMERICAL CONSTRUCTION OF BHASKARA PAIRS

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ABSTRACT. We construct integer solutions (a, b) to the coupled system of diophantine equations $a^2 + b^2 = x^2$ and $a^3 + b^3 = y^2$ for fixed ratios b/a.

1. Pair of Coupled Nonlinear Diophantine Equations

1.1. Scope.
Following a nomenclature of Gupta we define:

Definition 1. *A Bhaskara pair is a pair* (a, b) *of integers that solve the system of Diophantine equations*

$$a^2 + b^2 = x^3 \wedge a^3 + b^3 = y^2 \tag{1}$$

for some pair (x, y)*.*

Remark 1. *Values of* a *and* b *are gathered in the OEIS* [7, A106319,A106320].

The symmetry of the equations indicates that without loss of information we can assume $0 \le a \le b$.

We will not look into solutions where a or b are rational integers (*fractional* Bhaskara pairs).

The two equations can be solved individually. The applicable literature on that subject however will be ignored in the subsequent quick analysis [1, 3, 2].

Given any solution (a, b), other solutions (as^6, bs^6) are derived by multiplying both a and b by a sixth power of an integer s, multiplying at the same time x by s^4 and y by s^9. One may call the solutions where a and b have no common 6-full divisor (i.e., have no divisor that is an integer multiple of a sixth power larger than 1) *fundamental* Bhaskara pairs.

1.2. Primitive Solutions.
A first family of solutions is found assuming $a = 0$. This reduces the equations to

$$b^2 = x^3 \wedge b^3 = y^2. \tag{2}$$

x^3 must be a perfect cube, so in the canonical prime power factorization of x^3 all exponents of the primes must be multiples of three. Also in the canonical prime power factorization of b^2 all exponents must be even. So the first equation demands that the exponents on both sides must be multiples of $[2, 3] = 6$, where square brackets $[., .]$ indicate the least common multiple. In consequence all b must be perfect cubes. Likewise the second equation demands that the exponents of b^3 and of y^2 are multiples of 6. In consequence all b must be perfect squares. Uniting both requirements, all b must be perfect 6th powers. And this requirement is obviously also sufficient: perfect 6th powers [7, A001014] generate Bhaskara pairs:

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k	$1+k^2$	$1+k^3$
1	2	2
2	5	3^2
3	2×5	$2^2\times 7$
4	17	5×13
5	2×13	$2\times 3^2\times 7$
6	37	7×31

TABLE 1. Prime factorizations of $1+k^2$ and $1+k^3$

Theorem 1. *All integer pairs* $(0, n^6)$, $n \in \mathbb{Z}_0$ *are Bhaskara pairs. The associated right hand sides are* $(x, y) = (n^4, n^9)$.

1.3. **Bhaskara Twins.** Another family of solution are the Bhaskara twins defined by solutions with $a = b$ [7, A106318]:

$$2a^2 = x^3 \wedge 2a^3 = y^2. \tag{3}$$

Working modulo 2 in the two equations requires that x^3 and y^2 are even, so x and y must be even, say $x = 2\alpha$, $y = 2\beta$. So

$$a^2 = 4\alpha^3 \wedge a^3 = 2\beta^2. \tag{4}$$

The first equation requires by the right hand side that in the canonical prime power factorization of both sides the exponents of the odd primes are multiples of 3 and the exponent of the prime 2 is $\equiv 2 \pmod 3$; it requires by the left hand side that all exponents are even. So the exponents of the odd primes are multiples of 6, and the exponent of 2 is $\equiv 2 \pmod 6$. So from the first equation $a = 2^{1+3\times}3^{3\times}5^{3\times}\cdots$, which means a is twice a third power. The notation $3\times$ in the exponents means "any multiple of 3."

The second equation in (4) demands by the right hand side that the exponents of the odd primes are even and the exponent of 2 is $\equiv 1 \pmod 2$. Furthermore by the left hand side all exponents are multiples of 3. This means all exponents of the odd primes are multiples of 6, and the exponent of the prime 2 is $\equiv 3 \pmod 6$ So from the 2nd equation $a = 2^{1+2\times}3^{2\times}5^{2\times}\cdots$, which means a must be twice a perfect square. Uniting both requirements, a must be twice a 6th power. It turns out that obviously that requirement is also sufficient to generate solutions:

Theorem 2. *All integer pairs* $(2n^6, 2n^6)$, $n \in \mathbb{Z}_0$ *are Bhaskara pairs. The associated right hand sides are created by* $(x, y) = (2n^4, 4n^6)$.

2. Integer Ratios of the two Members

2.1. **Prime Factorization.** There may be solutions where $a \mid b$, so $b = ka$ for some integer $k > 1$. The previous section, Equation (3), covers the solutions of the special case $k = 1$.

$$(1+k^2)a^2 = x^3 \wedge (1+k^3)a^3 = y^2 \tag{5}$$

The equations have prime factorizations that are only special with respect to the prime facto decomposition of $1+k^2$ and $1+k^3$:

Define prime power exponents c_i, d_i, a_i, x_i and y_i as follows, where p_i is the i-th prime:

$$1+k^2 = 2^{c_1}3^{c_2}5^{c_3}\cdots = \prod_i p_i^{c_i}, \tag{6}$$

$$1+k^3 = 2^{d_1}3^{d_2}5^{d_3}\cdots = \prod_i p_i^{d_i}, \tag{7}$$

$$a = 2^{a_1}3^{a_2}5^{a_3}\cdots = \prod_i p_i^{a_i}, \tag{8}$$

$$x = 2^{x_1}3^{x_2}5^{x_3}\cdots = \prod_i p_i^{x_i}, \tag{9}$$

$$y = 2^{y_1}3^{y_2}5^{y_3}\cdots = \prod_i p_i^{y_i} \tag{10}$$

Insertion of these prime factorizations into (5) requires for all $i \ge 1$

$$c_i + 2a_i = 3x_i \tag{11a}$$

$$d_i + 3a_i = 2y_i \tag{11b}$$

for unknown sets of a_i, x_i, y_i and known c_i, d_i (if k is fixed and known). For some sufficiently large i (i larger than the index of the largest prime factor of $[1+k^2, 1+k^3]$) we have $c_i = d_i = 0$ because the least common multiples are finite once k is fixed. For these

$$2a_i = 3x_i \tag{12a}$$

$$3a_i = 2y_i \tag{12b}$$

The first equation requires $2 \mid x_i$ and $3 \mid a_i$ for sufficiently large i. The second equation requires $3 \mid y_i$ and $2 \mid a_i$ for sufficiently large i. The combination requires also $6 \mid a_i$ for sufficiently large i, so essentially (apart from factors) a is a 6th power, x is a square and y is a cube.

In practice we use the Chinese Remainder Theorem (CRT) for all i, whether the c_i or d_i are zero or not [6, 4]. Multiply (11a) by 3 and (11b) by 2,

$$3c_i + 6a_i = 9x_i \wedge 2d_i + 6a_i = 4y_i \tag{13}$$

such that the two factors in front of the a_i are the same, and work modulo 9 in the first equation and modulo 4 in the second:

$$6a_i \equiv -3c_i \pmod 9; \tag{14a}$$

$$6a_i \equiv -2d_i \pmod 4. \tag{14b}$$

We compute $6a_i \pmod{9 \times 4}$ by any algorithm [5], so a_i is determined $\pmod 6$ and creates a fundamental solution.

The results will be illustrated for a set of small k in Tables 2–10. There are 4 columns, the prime index i, the exponents c_i and d_i defined by the prime factorization of $1+k^2$ and $1+k^3$, and the factor $p_i^{a_i}$ generated by the CRT, which occurs besides the $p_i^{6\times}$. The cases (rows) where $c_i = d_i = 0$ are not tabulated; they would be absorbed in the 6th powers of non-fundamental solutions.

i	c_i	d_i	$p_i^{a_i}$
2	0	2	3^0
3	1	0	5^4

TABLE 2. The Chinese remainder solutions for $k = 2$. Fundamental solution $a = 5^4$.

i	c_i	d_i	$p_i^{a_i}$
1	1	2	2^4
3	1	0	5^4
4	0	1	7^3

TABLE 3. The Chinese remainder solutions for $k = 3$. Fundamental solution $a = 2^4 \times 5^4 \times 7^3$.

i	c_i	d_i	$p_i^{a_i}$
3	0	1	5^3
6	0	1	13^3
7	1	0	17^4

TABLE 4. The Chinese remainder solutions for $k = 4$. Fundamental solution $a = 5^3 \times 13^3 \times 17^4$.

2.2. **k=2.** Looking at the second line of Table 1 we have only special contributions for primes $p_2 = 3$ and $p_3 = 5$: $c_1 = 0, d_1 = 2$, $c_2 = 1, d_2 = 0$; the $i \geq 4$ are free.

The Chinese remainder evaluation for $i = 1$ gives $6a_i \equiv 0 \pmod{4 \times 9}$, $a_i \equiv 0 \pmod 6$. So the exponent of the 3 is a multiple of 6, as the general case.

The Chinese remainder evaluation for $i = 2$ with $c_i = 1$, $d_i = 0$ gives $6a_i \equiv 24 \pmod{4 \times 9}$, i.e., $a_i \equiv 4 \pmod 6$. So the exponent of the $p_2 = 5$ is a multiple of 6 plus an extra 4.

In summary, the necessary condition from the prime power analysis is that a is of the form $a = 5^4 s^6$, $s \in \mathbb{Z}_0$ if $b = 2a$. Insertion of that form into the equations turns out that all of these are indeed solutions:

Theorem 3. *All solutions of the form $(a, b = 2a)$ are given by the set of $a = 5^4 s^6$ with non-negative integers s, where $(x, y) = (5^3 s^4, 3 \times 5^6 s^9)$.*

2.3. $k = 3$**.** From the third line of the table we have the extra cases of Table 3. So the solutions require $a = 2^4 \times 5^4 \times 7^3 s^6$ and all these turn also out to be sufficient:

Theorem 4. *All solutions of the form $(a, b = 3a)$ are given by the set of $a = 2^4 \times 5^4 \times 7^3 s^6$ with non-negative integers s, where $(x, y) = (2^3 \times 5^3 \times 7^2 s^4, 2^7 \times 5^6 \times 7^5 s^9)$.*

2.4. $k = 4$**.** The primes of the third line of table 1 generate the extra cases of Table 4. So the solutions require $a = 5^3 \times 13^3 \times 17^4 s^6$ and all these turn also out to be sufficient:

Theorem 5. *All solutions of the form $(a, b = 4a)$ are given by the set of $a = 5^3 \times 13^3 \times 17^4 s^6$ with non-negative integers s, where $(x, y) = (5^2 \times 13^2 \times 17^3 s^4, 5^5 \times 13^5 \times 17^6 s^9)$.*

i	c_i	d_i	$p_i^{a_i}$
1	1	1	2^1
2	0	2	3^0
4	0	1	7^3
6	1	0	13^4

TABLE 5. The Chinese remainder solutions for $k = 5$. Fundamental solution $a = 2 \times 7^3 \times 13^4$.

i	c_i	d_i	$p_i^{a_i}$
4	0	1	7^3
11	0	1	31^3
12	1	0	37^4

TABLE 6. The Chinese remainder solutions for $k = 6$. Fundamental solution $a = 7^3 \times 31^3 \times 37^4$.

i	c_i	d_i	$p_i^{a_i}$
1	1	3	2^1
3	2	0	5^2
14	0	1	43^3

TABLE 7. The Chinese remainder solutions for $k = 7$.

i	c_i	d_i	$p_i^{a_i}$
2	0	3	3^3
3	1	0	5^4
6	1	0	13^4
8	0	1	19^3

TABLE 8. The Chinese remainder solutions for $k = 8$.

2.5. $k \geq 5$**.** Further solutions $(a, b = ka)$ with $k = 5 \dots 10$ are gathered in Tables 5–10.

Theorem 6. *All solutions $(a, b = ka)$ for factors $k = 5 \dots 10$ are obtained from tables 5–10 in the form $a = \prod p_i^{a_i} s^6$, where the pre-factor (the fundamental solution) is the product of the prime powers in the last column of the associated table.*

As a sort of summary of Section 2, collecting results for $k \leq 30$ shows that the b-values are (a superset) of [7, A106320] { 2, 128, 1250, 1458, 8192, 31250, 80000, 93312, 235298, 524288, 911250, 1062882, 2000000, 3543122, 5120000, 5971968, 9653618, 10290000, 15059072, 19531250, 22781250, 27827450, 33554432, 48275138, 58320000, 68024448, 94091762, 97964230, 128000000, 147061250, 171532242, 226759808, 296071778, 327680000, 382205952, 488281250 }

3. Fractional Ratios of the Two Members

Besides the cases where b is an integer multiple of a there is also room for cases where the values in the pair (a, b) have some non-integer ratio $b/a = k/u > 1$ with

i	c_i	d_i	$p_i^{a_i}$
1	1	1	2^1
3	0	1	5^3
13	1	0	41^4
21	0	1	73^3

TABLE 9. The Chinese remainder solutions for $k = 9$. Fundamental solution $a = 2 \times 5^3 \times 41^4 \times 73^3$.

i	c_i	d_i	$p_i^{a_i}$
4	0	1	7^3
5	0	1	11^3
6	1	0	13^3
26	1	0	101^4

TABLE 10. The Chinese remainder solutions for $k = 10$. Fundamental solution $a = 7^3 \times 11^3 \times 13^3 \times 101^4$.

coprime $(k, u) = 1$. Eq. (5) turns into

$$(1 + k^2/u^2)a^2 = z^3 \wedge (1 + k^3/u^3)a^3 = y^2; \tag{15}$$

$$(u^2 + k^2)a^2 = u^2 z^3 \wedge (u^3 + k^3)a^3 = u^3 y^2. \tag{16}$$

We generalize (6) and (7) to $u \geq 1$ and define

$$u^2 + k^2 = 2^{c_1} 3^{c_2} 5^{c_3} \cdots = \prod_i p_i^{c_i}, \tag{17}$$

$$u^3 + k^3 = 2^{d_1} 3^{d_2} 5^{d_3} \cdots = \prod_i p_i^{d_i}, \tag{18}$$

$$u = 2^{u_1} 3^{u_2} 5^{u_3} \cdots = \prod_i p_i^{u_i}. \tag{19}$$

The uniqueness of the prime power expansions of (16) requires

$$c_i + 2a_i = 3x_i + 2u_i; \tag{20a}$$

$$d_i + 3a_i = 2y_i + 3u_i. \tag{20b}$$

Unimpressed we promote the analysis as at the end of Section 2.1, multiply the two equations by 3 and 2, and derive the Chinese remainder equations:

$$6a_i \equiv 6u_i - 3c_i \pmod 9; \tag{21a}$$

$$6a_i \equiv 6u_i - 2d_i \pmod 4. \tag{21b}$$

The CRT guarantees that an integer solution $6a_i$ exists, because 9 and 4 are relatively prime. Furthermore the result will always be a multiple of 6 (hence a_i an integer), because from the first of the equations read modulo 3 we deduce that $6a_i$ is a multiple of 3, and from the second read modulo 2 that $6a_i$ is a multiple of 2:

Lemma 1. *For each ansatz of the ratio $b/a = k/u$, the algorithm generates a conjectural, unique fundamental (i.e., 6-free, smallest) solution a.*

To show that these products of the CRT are also solving the coupled Diophantine equations, we need to show that the step from (20) to (21) is reversible, so that

i	c_i	d_i	u_i	$p_i^{a_i}$
1	0	0	1	2^1
3	0	1	0	5^3
4	0	1	0	7^3
6	1	0	0	13^4

TABLE 11. The Chinese remainder solutions for $k/u = 3/2$.

i	c_i	d_i	u_i	$p_i^{a_i}$
1	0	0	3	2^3
6	0	2	0	13^0
7	2	0	0	17^2
9	0	1	0	23^3

TABLE 12. The Chinese remainder solutions for $k/u = 15/8$.

all solutions of (21) also fulfill (20). Indeed we can find a multiple of 9 and add it to the right hand side of the equivalence (21a) such that it becomes an equality, and can find a multiple of 4 and add it to the right hand side of the equivalence (21b) such that it becomes an equality. Dividing the two equations by 3 and 2, respectively, turns out to be a constructive proof that the $3x_i$ and $2y_i$ exist, and that they are multiples of 3 and 2:

Theorem 7. *For each ansatz of the ratio $b/a = k/u$, the algorithm generates a unique fundamental (i.e., 6-free, smallest) solution a.*

The simplest application is the ansatz $k/u = 3/2$ with the solution displayed in Table 11. The fundamental solution is $(a, b = 3a/2) = (2 \times 5^3 \times 7^3 \times 13^4, 5^4 \times 7^3 \times 13^4) = (2449105750, 3673658625)$. A solution with smaller b is obtained by $k/u = 15/8$ as illustrated by Table 12.

Systematical exploration of ratios k/u sorted along increasing numerator k generates Table 13.

a	b	k/u
2	2	1
625	1250	2
3430000	10290000	3
2449105750	3673658625	3/2
22936954625	91747818500	4
56517825	75357100	4/3
19592846	97964230	5
3327950899994	8319877249985	5/2
3437223234	5728705390	5/3
104677490484	130846863105	5/4
19150763710393	114904582262358	6
2745064044632305	3294076853558766	6/5
3975350	27827450	7
936110884878	3276388097073	7/2
26869428369750	62695332862750	7/3
4813895358057500	8424316876600625	7/4
329402537360	461163552304	7/5
54709453541096250	63827695797945625	7/6
3305810795625	26446486365000	8
113394176313	302384470168	8/3
689223517385	1102757627816	8/5
978549117961625	1118341849099000	8/7
274817266734250	2473355400608250	9
41793444127641250	188070498574385625	9/2
176590156053048868	397327851119359953	9/4
6143093188763230	11057567739773814	9/5
601306443010000	773108283870000	9/7
6758920534667005000	7603785601500380625	9/8
104372894488263401	1043728944882634010	10
458710390065569889	1529034633551899630	10/3
8357399286061919849	11939141837231314070	10/7
49927726291701142521	55475251435223491690	10/9
11221334146768	123434675614448	11
4801442438	26407933409	11/2
33528490382546250	122937798069336250	11/3
5247317639775500	14430123509382625	11/4
1712007269488880	3766415992875536	11/5
13496488877215427538	24743562941561617153	11/6
587831133723750	923734638708750	11/7
58661465201996135000	80659514652744685625	11/8

Table 13. The fundamental solutions for b/a ratios up to numerator $k = 11$.

Multiplications with 6th powers and sorting along increasing b leads from Table 13 to Table 14. Primitive solutions with $a = b$ or $a = 0$ ($k/u = 1$ or $k/u = \infty$) are not listed. The fundamental solutions are flagged by $s = 1$. The list is not proven to be complete up to the maximum b, because only a limited number of ratios $b/a = k/u$ were computed.

Table 14: Non-primitive Bhaskara pairs with $b \le 2.5 \times 10^{11}$ after scanning the b/a ratios up to numerators $k \le 1500$. [7, A106322]

a	b	k/u	s
625	1250	2	1
40000	80000	2	2
455625	911250	2	3
2560000	5120000	2	4
3430000	10290000	3	1
9765625	19531250	2	5
3975350	27827450	7	1
28130104	52743945	15/8	1
29160000	58320000	2	6
56517825	75357100	4/3	1
19592846	97964230	5	1
73530625	147061250	2	7
163840000	327680000	2	8
219520000	658560000	3	2
332150625	664301250	2	9
625000000	1250000000	2	10
254422400	1780956800	7	2
1107225625	2214451250	2	11
920414222	2235291682	17/7	1
1800326656	3375612480	15/8	2
2449105750	3673658625	3/2	1
1866240000	3732480000	2	12
3617140800	4822854400	4/3	2
3437223234	5728705390	5/3	1
3016755625	6033511250	2	13
1253942144	6269710720	5	2
2500470000	7501410000	3	3
4705960000	9411920000	2	14
9725113750	11493316250	13/11	1
7119140625	14238281250	2	15
2898030150	20286211050	7	3
10485760000	20971520000	2	16
4801442438	26407933409	11/2	1
15085980625	30171961250	2	17
20506845816	38450335905	15/8	3
14049280000	42147840000	3	4
21257640000	42515280000	2	18
41201494425	54935325900	4/3	3
29403675625	58807351250	2	19

a	b	k/u	s
14283184734	71415923670	5	3
40000000000	80000000000	2	20
56364477246	81415356022	13/9	1
22936954625	91747818500	4	1
53603825625	107207651250	2	21
16283033600	113981235200	7	4
104677490484	130846863105	5/4	1
70862440000	141724880000	2	22
58906510208	143058667648	17/7	2
53593750000	160781250000	3	5
92522430625	185044861250	2	23
115220905984	216039198720	15/8	4
9863101250	226851328750	23	1
156742768000	235114152000	3/2	2
119439360000	238878720000	2	24

4. Summary

We have shown that for each ratio b/a a unique smallest (fundamental) solution of the non-linear coupled diophantine equations (1) exists, which can be constructed by modular analysis via the Chinese Remainder Theorem. We constructed these explicitly for a limited set of small ratios.

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