Proof of an explicit formula for Bower's CycleBG transform

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In this note, we prove an explicit formula for the `CycleBG` transform introduced by Christian G. Bower in 2005 in the documentation of sequences A106362, A106364, A106365, A106366, A106367, A106368, and A106369 in the OEIS. Given an ordinary generating function

$A(x) = \sum_{k\geq 1} a_k x^k$ of a sequence of numbers $(a_n)_{n=1}^{\infty}$,

the `CycleBG` transform of A is defined by

$$T(A)(x) = A(x) + \text{invMOEBIUS}(A(x^2) - A(x) + \text{invEULER}(\text{Carlitz}(A)(x))),$$

where the `Carlitz` transform of A is defined by

$$\text{Carlitz}(A)(x) = \frac{1}{1 - \sum_{k=1}^{\infty}(-1)^{k+1}A(x^k)}.$$

We prove that

$$T(A)(x) = A(x) - \sum_{k=0}^{\infty} A(x^{2k+1}) + \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log(\text{Carlitz}(A)(x^k)). \tag{1}$$

Proof. If we let $\sum_{k=1}^{\infty} b_k x^k = \text{invEULER}(\text{Carlitz}(A)(x))$, then

$$\text{EULER}\left(\sum_{k=1}^{\infty} b_k x^k\right) = \text{Carlitz}(A)(x).$$

If we also let

$$\sum_{k=1}^{\infty} \frac{d_k}{k} x^k = \log(\text{Carlitz}(A)(x)), \tag{2}$$

then

$$b_n = \frac{1}{n} \sum_{s|n} \mu\left(\frac{n}{s}\right) d_s \quad \text{for } n \in \mathbb{Z}_{>0}, \tag{3}$$

where $\mu(\cdot)$ is the Möbius function. See Bernstein and Sloane [1, pp. 60-61]. In addition,

$$\text{MOEBIUS}((T(A)-A)(x)) = A(x^2) - A(x) + \sum_{k=1}^{\infty} b_k x^2.$$

If we let

$$A(x^2) - A(x) = \sum_{k=1}^{\infty} e_k x^k \quad \text{and} \quad (T(A)-A)(x) = \sum_{k=1}^{\infty} f_k x^k, \tag{4}$$

and use equation (3) and the material in Bernstein and Sloane [1, p. 60], we get

$$\begin{aligned} f_n = \sum_{s|n}(e_s + b_s) &= \sum_{s|n} e_s + \frac{1}{n}\sum_{s|n}\frac{n}{s}\sum_{t|s}\mu\left(\frac{s}{t}\right) d_t \\ &= \sum_{s|n} e_s + \frac{1}{n}\sum_{s|n}\left(\sum_{t|s} t\mu\left(\frac{s}{t}\right)\right) d_{n/s} \\ &= \sum_{s|n} e_s + \frac{1}{n}\sum_{s|n}\phi(s) d_{n/s}. \end{aligned} \tag{5}$$

It follows then from equations (2), (4), and (5) that

$$\begin{aligned} (T(A)-A)(x) &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{s|n}\phi(s) d_{n/s} x^n + \sum_{n=1}^{\infty}\sum_{s|n} e_s x^n \\ &= \sum_{s=1}^{\infty}\frac{\phi(s)}{s}\sum_{r=1}^{\infty}\frac{d_r}{r}(x^s)^r + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} e_s (x^r)^s \\ &= \sum_{s=1}^{\infty}\frac{\phi(s)}{s}\log(\text{Carlitz}(A)(x^s)) + \sum_{r=1}^{\infty}(A(x^{2r}) - A(x^r)), \end{aligned}$$

from which we can easily prove equation (1). □

References

[1] M. Bernstein and N. J. A. Sloane (1995), "Some canonical sequences of integers," *Linear Algebra and its Applications*, Vol. **226-228**, 57-72.