Theory and further extension of sequence A110713

Preface

I use the pdf-format for better readability of the formulas.
Some months ago, I extended the sequence from n=16 to 25. Using the internal structure of the problem, based on combinatorics, I now can extend the sequence from n=26 to 33. But I am not going to publish it because I want to avoid discussions about Visual Basic as source code.
With a more or less plausible assumption, I extended the sequence much further (last page). I stopped the listing with n=54 because otherwise I would have had to insert a new column. The listing may just serve as a reference for future extensions.

My concern is developing a theory of the internal structure of the problem. Perhaps it helps extend the sequence by using means more powerful than Visual Basic.

Contents

I) Introduction

a(n) be the number of products built by n factors in A(n)={1,2,…,n}. A simple method to construct the sequence is to save all the products in memory and increase a(n) by 1 as soon as a new product occurs. As a(n), in the average, seems to grow exponentially only a few terms can be calculated that way.

First some definitions:
(I, D1) sequence of primes p(i) = 2,3,5,7,…
(I, D2) Number pi(n) of primes not greater than n: pi(1)=0, pi(2)=1, .., pi(6)=3,..
(I, D3a) B(n,h,1): set of factors not greater than n and, if being a prime, not greater than p(h), $h\le pi(n)$. Note: B(n,h,1) = A(n) for h=pi(n)
If n is prime and h<pi(n) then $n\notin B(n,h,1)$
(I, D3b) B(n,h,m): set of products built by m factors from B(n,h,1)
(I, D3c) F(n,h,m) = |B(n,h,m)|
Note: F(n,pi(n),n) = a(n)
If n is prime and h<pi(n) then F(n,h,m) = F(n-1,h,m), see (I, D3a)

The generalisation is useful because F(n,h,m) is an h^{th} degree polynomial with respect to m. This and the main formula will be proved later.

Main formula: $F(n,h,m) = \sum_{j=0}^{pi(n/2)} r(n,j)\binom{m+h-j}{h}$ for $pi(^n/_2) \le h \le pi(n)$

Special coefficients: r(n,0)=1, r(n,1)=n-pi(n)-1.
n<9: r(n,j)=0 for j>1

Listing of the coefficients and a(n) = F(n,pi(n),n) for n<9 according to the main formula:

n	r(n,0)	r(n,1)	a(n)
1	1	0	1
2	1	0	3
3	1	0	10
4	1	1	25
5	1	1	91
6	1	2	196
7	1	2	750
8	1	3	1485

The terms for n<8 will be verified explicitly in the following chapters.
Extended Listing: Last chapter

II) Constructing a formula for n=1,2,3 and first generalisation

n	B(n,h,m)	F(n,h,m)	a(n)
1	B(1,0,m) ={1}	F(1,0,m) =1	F(1,0,1)=1
2	B(2,1,m) ={1,2,4,…,2^m}	F(2,1,m) =m+1	F(2,1,2)= 3
3	B(3,1,m) = B(2,1,m) B(3,2,m) is the union of the Cartesian products {1,3,..,3^(m-k)}x{2^k} for k=0,1,2,..,m	F(3,1,m) = F(2,1,m) F(3,2,m) = m+1+m+ m-1 + ... + 1=½(m+1)(m+2)	F(3,2,3)=10

Summary for n=1,2,3:

$F(n,h,m) = \binom{m+h}{h}$, $pi(^n/_2) \le h \le pi(n)$, main formula verified for h=pi(n)!

The step from (3,1,m) to (3,2,m) can be generalised:
The set B(n,h,m) is the union of m subsets b(n,h,k) := B(n,h,k) \ B(n,h,k-1), 1≤k<=m, b(n,h,0)={1}. Each product in b(n,h,k) can be represented by k factors, but not by less. The number of such products is f(n,h,k) := F(n,h,k) - F(n,h,k-1), f(n,h,0)=1. For $pi(^n/_2) \le h \le pi(n)$ the transition from h to h+1 is easy: Any product in b(n,h+1,m) can, unambiguously, be represented as $x*p(h+1)^k$, x being an element of b(n,h,m-k) and p(i)=2,3,5,.. Let k run from 0 to m, so each element of b(n,h+1,m) will have a corresponding one in B(n,h,m) and vice versa.

⇨ f(n,h+1,m)= F(n,h,m) and $F(n,h+1,m) = \sum_{k=0}^{m} F(n,h,k)$, Result:

II, Lemma 1: F(n,h+1,m) is the antiderivative of F(n,h,m) for $pi(^n/_2) \le h < pi(n)$

using the definition:

(II, D4) Discrete antiderivative of a sequence g(m):

$$G(m) = \sum_{k=0}^{m} g(k) \quad \text{Compare: } G(x) = \int_0^x g(t)\,dt$$

As there will be no confusion with analysis the adjective “discrete” is dropped.

If n is prime, F(n,h,m)= F(n-1,h,m) for h= pi(n)-1, see (I, D3c). With lemma 1:
<u>II, Lemma 2</u>: If n is prime, F(n,pi(n),m) is the antiderivative of F(n-1,pi(n)-1,m).

This can be easily verified for n=3:

F(3,2,m) = $\binom{m+2}{2}$ is the antiderivative of F(3,1,m) = F(2,1,m) = $\binom{m+1}{1}$.

General rule concerning the antiderivative of a combination:

<u>(II, R1)</u> $\binom{m+k}{k} \to \binom{m+k+1}{k+1}$, corresponding summation: $\sum_{i=0}^{m}\binom{i+k}{k}=\binom{m+k+1}{k+1}$.

An equivalent summation will be needed later:

<u>(II, R2)</u> $\sum_{i=0}^{k}\binom{m+i-k}{i}=\binom{m+1}{k}$

III) Recursion

It ise useful to determine F(n,h,m), n>3, by recursion n→n-1. Definitions:

<u>(III, D5a)</u> D(n,h,m) = B(n,h,m) \ B(n-1,h,m)

Any product in D(n,h,m) can be represented by m factors in B(n,h,1). The maximum factor is necessarily n, otherwise the product would belong to B(n-1,h,m). Some of such products with m factors, however, do not belong to D(n,h,m) because they can be reduced to a representation with each factor being smaller than n. Other products are ambiguous because they have more than one representation.

Example n=6: B(6,2,1)= {1,2,3,4,6}, D(6,2,1)= {6}, D(6,2,2) = {3*6, 4*6, 6*6}. The factors 1 and 2 are redundant because the corresponding products are reduceable (1*6 and 2*6 can be reduced to 2*3 and 3*4 respectively) or ambiguous (for example: 2*6*6 =3*4*6). So the basic set of factors to build the products in D(6,2,m), m>1, is {3,4,6}.

<u>(III, D5b)</u> d(n,h,m) = |D(n,h,m)| = F(n,h,m) - F(n-1,h,m)

d(n,h,m) will be proved as polynomial with respect to m. So this property is passed on F(n,h,m).

Note: If n is prime and h<pi(n) then d(n,h,m) = 0, see (I, D3c)

IV) Constructing a formula for n=4,5,6 and further generalisations

n=4: First determine d(4,1,m). Any representation of a product in D(4,1,m) contains the factor 4. Otherwise, if all factors were smaller than 4, the product would be included in B(3,1,m).

m	1	2	3	…	m
D(4,1,m)	{4}	{2*4, 4*4}	{2*2*4, 2*4*4, 4*4*4}		
d(4,1,m)	1	2	3	…	m

The table can easily be continued: d(4,1,m) = m

F(4,1,m) = F(3,1,m) + d(4,1,m) = $\binom{m+1}{1}$ + $\binom{m}{1}$

Antiderivative: F(4,2,m) = $\binom{m+2}{2}+\binom{m+1}{2}$, <u>a(4) = F(4,2,4) = 25</u>

n=5: d(5,2,m) = 0 , Generally: d(n,h,m) = 0, if n is prime, see (III, D5b), annotation

$F(5,2,m) = F(4,2,m)$

Antiderivative: $F(5,3,m) = \binom{m+3}{3} + \binom{m+2}{3}$, <u>a(5) = F(5,3,5) = 91</u>

n=6: $d(6,2,m) = \binom{m+1}{2}$.

This is not quite as trivial as d(4,1,m), and a method should be used that can be generalised. The basic set of factors to build the products in D(6,2,m) is {3,4,6}. The factors 1 and 2 produce reduceable products, see (III, D5a), example n=6. At this point we need a rule of combinatorics:

<u>(IV, R3)</u> If m -not necessarily different- elements are chosen from N, the number of combinations is $\binom{m+N-1}{N-1} = \binom{m+N-1}{m}$ (with different elements: $\binom{N}{m}$)

There are 3 classes of products according to the smallest factor, and no such product is reduceable or ambiguous:

a) products of the form 3*..(m-2 factors chosen from 3,4,6)..*6

Number of combinations according to rule (IV, R3) with N=3, m→m-2: $a=\binom{m}{2}$

b) products of the form 4*..(m-2 factors chosen from 4,6)..*6 with N=2: $b=\binom{m-1}{1}$

c) product 6^m: c=1

$d(6,2,m) = \sum_{i=0}^{r} \binom{m+i-2}{i} = \binom{m+1}{2}$ according to (II, R2, k=2)

$d(6,3,m) = \binom{m+2}{3}$, antiderivative of d(6,2,m)

Result: $F(6,3,m) = F(5,3,m) + d(6,3,m) = \binom{m+3}{3} + 2\binom{m+2}{3}$, <u>a(6) = F(6,3,6) = 196</u>

n=7: d(7,3,m) = 0 , because n is prime (see n=5)

$F(7,3,m) = F(6,3,m)$

Antiderivative: $F(7,4,m) = \binom{m+4}{4} + 2\binom{m+3}{4}$, <u>a(7) = F(7,4,7) = 750</u>

<u>V) Polynomial aspects</u>

We deducted d(6,h,m), h=2 or h=3, as a sum of combinations which are polynomials with respect to m. The more elaborate example d(28,6,m) –see chapter VIII– shows the same. With the representative application of combinatorics in these examples it is evident that d(n,h,m)=F(n,h,m)–F(n-1,h,m), h=pi(n), generally is a polynomial. So, by recursion, it is obvious that F(n,h,m) is a polynomial, too. The degree is h, see chapter VII.

A h^{th} degree polynomial is fixed by h+1 basic values. Let us start with $h=h_1= pi(^n/_2)$ and try the Ansatz:

(V, A) $F(n,h_1,m) = \sum_{j=0}^{h_1} r(n,j)\binom{m+h_1-j}{h_1}$

with the given basic values $F(n, h_1,m)$, $0 \le m \le h_1$

The coefficients can be determined recursively as $\binom{m+h_1-j}{h_1} = \begin{cases} 1 \text{ for } j=m \\ 0 \text{ for } j>m \end{cases}$

The result is:

(V, 1) m=0: $F(n, h_1,0) = 1 \Rightarrow r(n,0) = 1$,

$0<m\le h_1$: $r(n,m) = F(n, h_1,m) - \sum_{j=0}^{m-1} r(n,j)\binom{m+h_1-j}{h_1}$

So the combinatorial ansatz (V, A) is correct.

Antideriving both sides of (V, A) leads to

(V, B) $F(n,h_1+1,m) = \sum_{j=0}^{h_1} r(n,j)\binom{m+h_1+1-j}{h_1+1}$ according to lemma 1 and rule (II, R1)

The result of continued antideriving is the main formula:

(V, C) $F(n,h,m) = \sum_{j=0}^{pi(n/2)} r(n,j)\binom{m+h-j}{h}$ for $pi(^n/_2)\le h\le pi(n)$

There is an analogous formula for $d(n,h,m)=F(n,h,m)-F(n-1,h,m)$:

(V, D) $d(n,h,m) = \sum_{j=1}^{pi(n/2)} \Delta r(n,j)\binom{m+h-j}{h}$, $\Delta r(n,j) = r(n,j)- r(n-1,j)$ or $r(n,j) = \sum_{k=1}^{n} c(k,j)$

The sum starts with j=1 because $d(n,h_1,0) = 0 \Rightarrow \Delta r(n,0) = 0$

If n is prime: $\Delta r(n,j) = 0$ because $d(n,h_1,m) = 0$, see (III, D5b), annotation.

Otherwise the coefficients can be determined recursively and analogous to (V, 1):

(V, 1’) m=1: $\Delta r(n,1) = d(n, h_1,1) = 1$

$1<m\le h_1$: $\Delta r(n,m) = d(n, h_1,m) - \sum_{j=1}^{m-1} \Delta r(n,j)\binom{m+h_1-j}{h_1}$

The main problem is to find $\Delta r(n,j)$, if n is not prime. The example d(28,6,m) in chapter VIII is good for explaining that it is polynomial, but not for numerical purpuses because too much individual work is to be done. A more effective algorithm will be described later.

VI) Constructing $F(n,h_1,m)$, $h_1= pi(^n/_2)$ for n>7

The underlined terms $d(n, h_1,m)$ are to be determined. “AD(..)” means “antiderivative of..”.

F(8,2,m)   = F(7,2,m) + d(8,2,m)
F(9,2,m)   = F(8,2,m) + d(9,2,m)
F(9,3,m)   = AD(F(9,2,m))
F(10,3,m) = F(9,3,m) + d(10,3,m)
F(11,3,m) = F(10,3,m) because 11 is prime
F(12,3,m) = F(11,3,m) + d(12,3,m)
F(13,3,m) = F(12,3,m) because 13 is prime
F(13,4,m) = AD(F(13,3,m))
F(14,4,m) = F(13,4,m) + d(14,4,m)
etc.

Antideriving (and thus increasing h_1) is applied if n = 2*prime–1

VII) An algorithm to determine d(n, h_1,m)

The following algorithm detects reduceable products, see (III, D5a), context. (VII, A), code: Visual Basic

```
Sub reduce_check(m, pr, p, red)
If m = 1 Then
  If pr < zahl Then red = True
Else
  x = pr ^ (1 / m)
  Do
    If pr Mod p = 0 Then
      Call reduce_check(m - 1, pr \ p, (p), red)
    End If
    p = p - 1
  Loop Until p < x
End If
End Sub
```

pr is a product built by m factors not greater than n and at least one of which equals n. Can pr be represented by m factors not greater than p? If so, the final value of red is "true" (initial value: "false").
p = n–1: if and only if red is true the product pr is reduceable.

The complete algorithm:
We construct the sets S_m of irreduceable products for m = 1,2,3,…,h_1= pi($^n/_2$).
S_1 = {n}; auxiliary set: T = {x / x∈B(n,h_1,1) ∧ x*n irreduceable}
S_2 = {x*n / x∈T}
construction of S_m, m>2, by iteration:

a) Build all products x*y / x∈T, y∈ S_{m-1}
b) Find ambiguous products x*y by sorting. Keep just one of such products respectively.
c) Find reduceable products by algorithm (VII, A) and erase them.
d) S_m is the set of remaining products. Their number is d(n,h_1,m) = |S_m|

Finally determine the coefficients Δr(n,j) with d(n,h_1,m), m = 1,2,3,…,h_1, see (V, 1')

Appendix

VIII) The polynomial degree of F(n,h,m)

Any product in B(n,h,m) represented by its prime factorisation can be written as a h-tuple (vector) of exponents. Example:
B(6,2,2) = {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36}, normal notation
= {00, 10, 01, 20, 11, 30, 02, 21, 40, 12, 31, 22}, vector notation
The cross sums (CS) of these products are 0,1,1,2,2,3,2,3,4,3,4,4. B(6,2,2) contains each number with CS ≤ 2, but not for example 3^4=(04) with CS=4. There are 6 numbers with CS ≤ 2 and 15 numbers with CS ≤ 4. This gives a rough estimate 6≤F(6,2,2)=12≤15.

Generally: Any h-tuple with CS ≤ m can be built by a sum of m h-tuples with CS=0 or CS=1 representing the factor 1 or one of the first h primes respectively. There are $\binom{m+h}{h}$ ways to choose m such h-tuples from N=h+1, see (III, R3).

The exponent k defined by $2^k \le n < 2^{k+1}$ is the greatest CS of numbers in B(n,h,1). So k*m is the greatest CS of numbers in B(n,h,m). This results in the estimate:

$$\binom{m+h}{h} \le F(n,h,m) \le \binom{k \cdot m+h}{h}$$

The lower limit is a h-th degree polynomial with respect to m, and so is the upper limit because k does not depend on m. As we know that F(n,h,m) is a polynomial, its degree is h.

IX) Deduction of d(28,6,m), polynomial aspect

Each product that belongs to B(28,6,m), but not to B(27,6,m), contains the factor 28. Some products with the factor 28, however, are not allowed because they can be reduced to a factor representation without 28. So all numbers smaller than 14 and the numbers 15 and 18 are redundant, see (III, D5a), example n=6, and will not be used as factors (above: R1, R2, R3). So the basic set of allowed factors is:
G = {14,16,20,21,22,24,25,26,27,28}.

ambiguous products :
A1) 14*24*28 = <u>16*21*28</u>
A2) 16*25*28 = <u>20*20*28</u>
A3) 20*20*27*28*28 = <u>21*24*24*25*28</u>

reduceable products :
R1) x*28 = 2x*14, x<14
R2) 15*28 = 20*21
R3) 18*28 = 21*24
R4) 14*27*28 = 21*21*24
R5) 16*27*28 = 21*24*24
R6) 20*20*27*28 = 21*24*24*25

If we neglect the other 3 reduceable products and choose just one factor representation of the ambiguous products (my choice is underlined), any product is represented by exactly one set of factors.
The complete list on the left was generated by an algorithm based on (VII, A). “Complete” means that all further ambiguous or reduceable products can be divided by a listed one.

N(k) be the number of products with k∈G as the smallest factor.

<u>k=14</u>
The set of restrictions is R(14) = {24,27,16*25}. These elements may not be combined with 14, see A1, R4, A2. First we drop the restriction 16*25. There are 8 allowed combination factors: 14,16,20,21,22,25,26,28.
In any product with m factors two of them are fixed (14 and 28) and m–2 may be arbitrarily chosen. The number of choices is $\binom{m+5}{7}$, see rule (IV, R3).

Considering the restriction 16*25, i.e. products with the factors 14,16,25,28, m–4 factors may be arbitrarily chosen. The number of choices is $\binom{m+3}{7}$ and has to be subtracted. So the number of products containing the smallest factor 14 is

$$N(14) = \binom{m+5}{7} - \binom{m+3}{7} = \binom{m+4}{6} + \binom{m+3}{6}$$

k= 16:
Set of restrictions: R(16) = {14,25,27}. 14 is a formal restriction taking into account that the product 14*16 is already included in N(14). So there are 10–3= 7 allowed combination factors. This leads to N(16) = $\binom{m+4}{6}$

k = 20:
Set of restrictions: R(20) = {14,16,20*27}, same structure as R(14)
N(20) = N(14)=$\binom{m+4}{6}+\binom{m+3}{6}$

k = 21,22,24,25,26,27,28
The numbers of allowed combination factors are 7,6,5,4,3,2,1 respectively because there are only formal restrictions.
Number of products: $\sum_{i=1}^{7}\binom{m+i-2}{i-1}=\binom{m+5}{6}$
Total number of products: d(28,6,m) =$\binom{m+5}{6}$+3$\binom{m+4}{6}$+2$\binom{m+3}{6}$
Remark: We had to analyse the factor 14 first because of the largest number of non-formal restrictions, then 16 with the second largest number, and so on. In other cases ($n\neq28$) the sortings by size and by the number of restrictions may be different .

IX Listing

Main formula: F(n,h,m) = $\sum_{j=0}^{pi(n/2)} r(n,j)\binom{m+h-j}{h}$ for $pi(^n/_2)\le h\le pi(n)$,
It is numerically easiest to use h= $pi(^n/_2)$ to determine the coefficients and h=pi(n) for the final solution a(n) = F(n,pi(n),n).
Special coefficients: r(n,0)=1, r(n,1)=n-pi(n)-1.
There is no simple formula for r(n,j), j>1.
Note that the upper limit of the sum above could, specially for larger values of n, be chosen smaller than $pi(^n/_2)$. Example n=33: $pi(^n/_2)$=6 could be replaced by 3.
For n=1 to 33 I determined r(n,j) completely (see listing below) so that the terms a(1)..a(33) are correct. There is a heuristic rule: r(n,k)=0 => r(n,j)=0 for n<34, j>k.
Extending this rule to n>33 I determined a(34)..a(54). These terms are not proved (see preface). The following list (arbitrarily) ends with n=54 because a new column r(n,4) would be needed for n>54.

List of the coefficients r(n,j) and the result a(n) = F(n,pi(n),n) for $n\le54$:

n	r(n,0)	r(n,1)	r(n,2)	r(n,3)	a(n)
1	1	0	0	0	1
2	1	0	0	0	3
3	1	0	0	0	10
4	1	1	0	0	25
5	1	1	0	0	91
6	1	2	0	0	196
7	1	2	0	0	750
8	1	3	0	0	1485
9	1	4	1	0	3025

10	1	5	2	0	5566
11	1	5	2	0	23387
12	1	6	2	0	38402
13	1	6	2	0	163268
14	1	7	3	0	284376
15	1	8	5	0	500004
16	1	9	6	0	795549
17	1	9	6	0	3575781
18	1	10	7	0	5657839
19	1	10	7	0	25413850
20	1	11	8	0	40027130
21	1	12	11	0	66010230
22	1	13	14	0	105164280
23	1	13	14	0	490429875
24	1	14	14	0	713491350
25	1	15	20	2	1232253906
26	1	16	24	3	1919584356
27	1	17	29	5	3008392024
28	1	18	32	7	4445580172
29	1	18	32	7	22070825348
30	1	19	33	6	30617017816
31	1	19	33	6	150656122552
32	1	20	36	6	218974671357
33	1	21	42	9	336680165965

34	1	22	48	12	506339002078	unproved
35	1	23	56	20	791327567498	n>33
36	1	24	57	20	1069566679831	
37	1	24	57	20	5520850763675	
38	1	25	64	25	8193064249227	
39	1	26	72	32	12187526329668	
40	1	27	75	32	16550134281422	
41	1	27	75	32	86692128597270	
42	1	28	78	32	118429199143850	
43	1	28	78	32	613655891426340	
44	1	29	83	35	867454603970850	
45	1	30	89	39	1228794668925180	
46	1	31	98	46	1777561938036780	
47	1	31	98	46	9328009191073440	
48	1	32	99	46	12530206630981935	
49	1	33	114	68	19594598894931495	
50	1	34	120	71	26893574051855736	
51	1	35	131	85	38804668147303731	
52	1	36	138	91	53140939763091120	
53	1	36	138	91	287743696002565704	
54	1	37	143	95	390177127526667712	