Probability of Derangements

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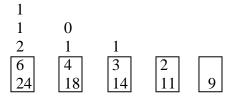
Table of Contents

List of Tables

List of Figures

Introduction

In "Secret Santa", people draw names from a hat to see who they will buy a present for. If a person picks his own name, he picks another name and throws his own name back in. If the last person draws his own name, there's a problem. What is that probability as a function of the number of people participating? Said another way, this series represents the numerators of a ratio, the denominator being $(n-1)!^2$ (A001044), in which the ratio gives the exact probability that the last person draws his own name from a set of names, given that each other person in the set has previously drawn randomly and in succession, but was not allowed to "keep" his own name.

The solution is most easily explained by way of example. Let's say there are 5 people, named 1-5. For the last person to choose "5," the first four people must draw 1-4 as a derangement, and there 9 ways of doing so: (sequence A047920)

```
1
1      0
2      1      1
6      4      3      2
24     18     14     11     9
```

Reading the bottom two rows, of the 24 possible arrangements, 6 of 24 are not possible because person 1 puts his own name back in. Similarly, 4 of the 18 remaining are eliminated by person 2, and 3 of 14 remaining are eliminated by person 3, 2 of 11 by person 4, and 9 "derangements" succeed getting through the first 4 names, leading to 9 ways for the 5th person to get his own name.

But what is the *probability* of each of these 9 derangements, or sequences? It boils down to understanding how many choices exist at each successive draw. The first person (i=1) can draw from 4 possibilities (N-i, namely, 2,3,4,5). The second person nominally has 3 to choose from (N-i, i=2), unless the first person drew number 2, in which case person 2 may draw 4 possibilities (1,3,4,5, or N-i+1). Similarly, the 3rd person has 2 or 3 to draw from, the 4th has 1 or 2, and the last person has 0 or 1. (0 choices means he's forced to choose his own name, which is "the problem.")

Continuing with this example, the probability of sequence 2,1,4,3,5 can be determined as follows:

       1st person: "2" is drawn from 4 possibilities (2,3,4,5 / **1 not yet taken**)
       2nd person: "1" is drawn from 4 possibilities (1,3,4,5 / **2 already taken**)
       3rd person: "4" is drawn from 2 possibilities (4,5 / 1-2 taken, **3 not yet taken**)
       4th person: "3" is drawn from 2 possibilities (3,5 / 1, 2, and **4 already taken**)
       5th person: "5" is forced (**5 not yet taken**)
…with probability $1/4 * 1/4 * 1/2 * 1/2 = 1/64$. This will be better represented as 9/576, because the LCM of all sequences is 576, or $(N-1)!^2$.

There is another sequence with the same probability, namely 2, 4, 1, 3, 5. It's the same because in both sequences, 2 is drawn before 2's turn, and 4 is drawn before 4's turn, but 1, 3 and 5's number is not drawn before their turns, respectively.

So, if there are N people, at the ith turn (i = 1..N), person i has either (N-i) or (N-i+1) choices, depending on whether his own name is chosen yet. A way to represent the underlying logic of the two cases above is "01010," where a 0 indicates that the person's number is not yet drawn, and a 1 indicates it is.

For the cases of interest, in which the Nth person is forced to choose his own name, the last digit of this pattern is therefore 0, by definition. Similarly, the 1st digit must be a 0, since *no* numbers are yet chosen. And also, the second to last digit must be a 1. So all the "problem" patterns start with 0 and end with 10.

Using the example of 5 people, the 2nd and 3rd digits can each be a 0 or 1, creating 4 target patterns:

    0-00-10
    0-01-10
    0-10-10
    0-11-10

A simple way to calculate the probability of any sequence with such a patterns is:

    01010 (pattern)
    <u>43210</u> (nominal number of choices per position)
    44220 (add the above two together for total choices per position)

Multiplying the non-zero digits together (4x4x2x2=64) gives 64, as above. Using 576 as the denominator, the numerators become {12, 8, 9, 6}. This is the sequence probability.

So, the probability of any sequence of a given pattern is:

    0-00-10   12 / 576
    0-01-10   8 / 576
    0-10-10   9 / 576
    0-11-10   6 / 576

So for N=5, the 9 derangements must be distributed over these 4 patterns. By enumeration, that distribution is shown to be:

    0-00-10   1 sequence
    0-01-10   5 sequences
    0-10-10   2 sequences
    0-11-11   <u>1 sequence</u>
                  9 total sequences.

In the list above, let's call {1, 5, 2, 1} the sequence frequency. The length of this vector is always 2^(N-3), as a direct consequence of the process being followed. A direct calculation of these frequency vectors for the first few N is shown in the following table:

People:	3	4	5	6	7
Frequencies:	1	1	1	1	1
		1	5	13	29
			2	6	14
			1	13	73
				2	6
				6	42
				2	18
				1	29
					2
					18
					8
					14
					2
					6
					2
					1

Table 1: Sequence Frequencies, A136301

Summarizing, for N=5, there is 1 case of 0-00-10 at probably 12/576, 5 of 0-01-10 at 8/576, 2 of 0-10-10 at 9/576, and 1 of 0-11-10 at 6/576. The total probability then that person 5 gets his own name is the sum of the frequencies of each pattern times the probability defined by each pattern, or **{1,5,2,1} * {12,8,9,6} / 576, or 76 / 576**.

Listing just the numerators of the probability ratios (sequence probability):

People:	3	4	5	6	7
Case 0	1	3	12	60	360
Case 1		2	8	40	240
Case 2			9	45	270
3			6	30	180
4				48	288
5				32	192
6				36	216
7				24	144
8					300
9					200
10					225
11					150
12					240
13					160
14					180
15					120

Table 2: Sequence Probability

Each value in Table 2: Sequence Probability can be determined by its associated pattern, as already shown. As a second example, for N=7, the pattern 0110010, or 0-1100-10, is represented above in column 7 row 12 (binary 1100), indicating that each case of that pattern has probability 240 / 6!^2, or 240 / 518400.

But the frequency table which indicates *how many* sequences have this pattern is more tricky, and fascinating. But first, back to answer the original question…

A136300
1, 0, 1, 5, 76, 1624, 52116, 2298708, 133929216, 9961180416, 921248743680, 103715841415680, 13967643016085760, 2217449301162071040, 409861043056032503040, 87262626872384643052800, 21202798521768886355558400, 5831660090586059239329792000, 1802587564536011525042697830400, 622185136016136818758343243366400, 238432510944919530702148531765248000, 100921590338431208107299482517995520000

Description: In "Secret Santa", people draw names from a hat to determine for whom they will buy a present. If a person picks his own name, he picks another name and throws his own name back into the hat. If the last person draws his own name, there's a problem. What is that probability as a function of the number of people participating? Said another way, this series represents the numerators of a ratio, the denominator being $(n-1)!^2$ (A001044), in which the ratio gives the exact probability that the last person draws his own name from a set of names, given that all other people had previously drawn randomly and in succession, but were not allowed to "keep" their own names.

Example: If there is one person, the chance of the last person getting his own name is 100%, or 1 over $0!^2$. For 2 people, it's $0 / 1!^2$. For 3 people, it's $1 / 2!^2$, creating a more interesting case. The possible drawings (sequences) are {2,1,3}, {2,3,1}, and {3,1,2}. All other sequences can't happen because a name is rejected and redrawn. But these 3 sequences don't have equal probability, rather, they are 25%, 25%, and 50% respectively. Of these, the first is the only one in which the last person draws his own name. In that case, the first person has a 50% chance of drawing a 2 or 3. If 2, the second person has a 50% chance of drawing 1 or 3, for a total outcome probability of 1/4. Similarly with 4 people, the chance is 5/36, followed by 76/576 for 5 people, etc. For the case of 5 people, the above equations boil down to this end calculation: {1,5,2,1} * {12,8,9,6} summed, or 12 + 40 + 18 + 6 = 76.

Formula

The integer sequence can be calculated in Mathematica, shown here to calculate the result for up to 22 people:

```
maxP = 22;

rows = Range[ 1, 2^(nP = maxP-3) ];

pasc = Table[ Binomial[ p+1, i] - If[ i >= p, 1,0], {p, nP}, {i, 0, p} ];

sFreq = Table[0,{maxP-1}, {2^nP} ]; sFreq[[2;;maxP-1, 1]] = 1;
For[ p=1, p<=nP, p++, For[ s=1, s<=p, s++, \
        rS=Range[2^(s-1)+1, 2^s]; \
        sFreq[[p+2,rS]]=pasc[[p+1-s,1;;p+2-s]].sFreq[[ s;;p+1,1;;2^(s-1)]] ] ];
(* TableForm[ Transpose[ sFreq ] ]*)

sProb = Table[  p+ 2-BitGet[ rows-1, p-1], {p, nP} ] ;
sProb = Table[ Product[ sProb[[i]], {i, p} ], {p, nP} ]  * Table[ If[ r <= 2^p, 1, 0], {p, nP}, {r,rows} ];

rslt = Flatten[ Prepend[ Table[ sProb[[p]] . sFreq[[p+2]], {p, nP} ], {1,0,1} ] ]
prob = N[ rslt / Array[ (#-1)!^2&, maxP ] ]
```

Notes:
1) The role of columns / rows is reversed from this paper, for faster calculation
2) pasc is the modified binomial distribution
3) sFreq is [A136301](#), Table 1: Sequence Frequencies
4) sProb is Table 2: Sequence Probability
5) rslt is the final numerators
6) prob is the final probabilities
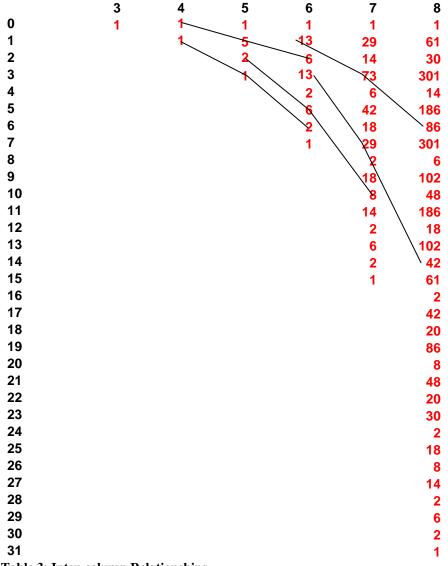
I also have this in excel, which has enough accuracy to calculate the result for up to 12 people. Email me at bparsonnet@comcast.net for a copy.

Historical note
The first time I wrote a program to calculate this sequence was in 1979, using an HP3000 mainframe computer. I calculated the result for up to 10 people, taking 7 full days of non-stop computing. I still have the print out. The Mathematica version above calculates to 22 people, and completes in about 30 seconds on my laptop. (That increase in performance correlates well with Moore's law, increasing speed by approximately 2^28 in about 30 years.)

Point 1: Inter-column relationships

Here are the frequency vectors for 3 – 8 people. The excel file has the table calculated through 13 people. In the interest of readability, I will show only sections of that table at times.

	3	4	5	6	7	8
0	1	1	1	1	1	1
1		1	5	13	29	61
2			2	6	14	30
3			1	13	73	301
4				2	6	14
5				6	42	186
6				2	18	86
7				1	29	301
8					2	6
9					18	102
10					8	48
11					14	186
12					2	18
13					6	102
14					2	42
15					1	61
16						2
17						42
18						20
19						86
20						8
21						48
22						20
23						30
24						2
25						18
26						8
27						14
28						2
29						6
30						2
31						1

Table 3: Inter-column Relationships

Referencing any value in the frequency table by [r,c] (row, column), then
$$[r,c] + [2r+1,c+1] = [4r+2, c+2]$$

Examples shown above are 1+2=2, 1+5=6, 2+6=8, 13+73=86, and 13+29=42.

Point 2: Intra-column relationships

	3	4	5	6	7	8
0	1	1	`1	1	1	1
1		1	5	13	29	61
2			2	6	14	30
3			1	13	73	301
4				2	6	14
5				6	42	186
6				2	18	86
7				1	29	301
8					2	6
9					18	102
10					8	48
11					14	186
12					2	18
13					6	102
14					2	42
15					1	61
16						2
17						42
18						20
19						86
20						8
21						48
22						20
23						30
24						2
25						18
26						8
27						14
28						2
29						6
30						2
31						1

Table 4: Intra-column Relationships

For all rows r **where mod(r,4)=1, [r,c] = 2[r+1,c] + [r-1,c]**. For examples, $13 = 2*6 + 1$, or $102=2*48+6$, or $102=2*42+18$, or $48 = 2*20+8$

There are countless more relationships of this type (linear combinations of numbers within one column yielding another, repeating at row increments of some power of 2), this being the simplest. A more complicated one is true for mod(r,8)=3, where the row's value = 3 * (r1 + r3) + r-3. ("r1" means the next row down, etc.) Eg, column 8 row 19, $86 = 3*(8+20)+2$.

Point 3: Value Pairing Pattern

	Row (binary)	1's compliment	Reversed bits	Row pairing	8 People
0	00000	11111	11111	31	1
1	00001	11110	01111	15	61
2	00010	11101	10111	23	30
3	00011	11100	00111	7	301
4	00100	11011	11011	27	14
5	00101	11010	01011	11	186
6	00110	11001	10011	19	86
7	00111	11000	00011	3	301
8	01000	10111	11101	29	6
9	01001	10110	01101	13	102
10	01010	10101	10101	21	48
11	01011	10100	00101	5	186
12	01100	10011	11001	25	18
13	01101	10010	01001	9	102
14	01110	10001	10001	17	42
15	01111	10000	00001	1	61
16	10000	01111	11110	30	2
17	10001	01110	01110	14	42
18	10010	01101	10110	22	20
19	10011	01100	00110	6	86
20	10100	01011	11010	26	8
21	10101	01010	01010	10	48
22	10110	01001	10010	18	20
23	10111	01000	00010	2	30
24	11000	00111	11100	28	2
25	11001	00110	01100	12	18
26	11010	00101	10100	20	8
27	11011	00100	00100	4	14
28	11100	00011	11000	24	2
29	11101	00010	01000	8	6
30	11110	00001	10000	16	2
31	11111	00000	00000	0	1

Table 5: Value Pairing Pattern

Each column shows numbers in pairs, where the row of the paired number can be found by starting with the row number, converting to binary representation, doing a one's compliment, and reversing the order of bits to determine the row number for the paired entry. The pairs for column 8 are shown above. The pairings are always symmetrical

In cases were the "reversed 1's complement" equals the original row number, the value for that row is "paired" with itself. For example, in column 7, row 0011 maps to itself, and thereby has no other paired column. See next point.
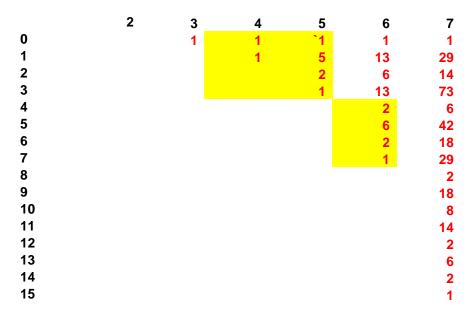
Point 4: Self-paired values

For odd numbered columns, the length of the row in binary representation is even. That means that some of the row numbers can "pair" to themselves. For example, for N=7, the row pairing vector is {15, 7, 4, 3, 2, 5, 9, 1, 14, 6, 10, 2, 12, 4, 8, 0}, where the rows indicated in red map to themselves and have no "pair."

The first unpaired value in the odd columns is always the highest value for that column. And as a sequence themselves are A048144 (re: Complete Bipartite Graphs), with the row number following an obvious 2^n pattern:

	3	4	5	6	7	8	9
0	1	1	`1	1	1	1	1
1		1	5	13	29	61	125
2			2	6	14	30	62
3			1	13	73	301	1081
4				2	6	14	30
5				6	42	186	690
6				2	18	86	330
7				1	29	301	2069
8					2	6	14
9					18	102	414
10					8	48	200
11					14	186	1394
12					2	18	86
13					6	102	834
14					2	42	374
15					1	61	1081
16						2	6
17						42	222
18						20	108
19						86	834
20						8	48
21						48	504
22						20	228
23						30	690
24						2	18
25						18	258
26						8	120
27						14	414
28						2	42
29						6	222
30						2	90
31						1	125

Table 6: Self-Paired Values
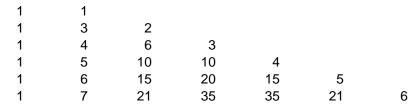
Now note that the bottom half of any column equals the horizontal sum of the two previous columns…

	2	3	4	5	6	7
0		1	1	`1	1	1
1			1	5	13	29
2				2	6	14
3				1	13	73
4					2	6
5					6	42
6					2	18
7					1	29
8						2
9						18
10						8
11						14
12						2
13						6
14						2
15						1

{1,1,0,0} + {1,5,2,1} = {2,6,2,1}. This is true for all columns. Plus, the sum of the bottom section is A00255 (related to permutation substrings).

More interesting, though, is that this is a simple case of a more general pattern based on the binomial distribution. This is the most important relationship, indicating how the table can be calculated in its entirety from previous columns, recursively.

First, a quick modification to the Pascal's Triangle, in which 1 is subtracted from the two rightmost values in each row:

1	1					
1	3	2				
1	4	6	3			
1	5	10	10	4		
1	6	15	20	15	5	
1	7	21	35	35	21	6

Table 7: Modified Pascal

The value for each entry in the frequency table is the dot product of one of the above rows and a section of the frequency table. The selection of the two vectors follows an elegant pattern, albeit a little tedious to explain.

	Row	High Bit	w/o high bit	2	3	4	5	6	7
0	0000				1	1	`1	1	1
1	0001	0	0		1		5	13	29
2	0010	1	0				2	6	14
3	0011	1	1				1	13	73
4	0100	2	0					2	6
5	0101	2	1					6	42
6	0110	2	2					2	18
7	0111	2	3					1	29
8	1000	3	0						2
9	1001	3	1						18
10	1010	3	2						8
11	1011	3	3						14
12	1100	3	4						2
13	1101	3	5						6
14	1110	3	6						2
15	1111	3	7						1
	A000523		A053645						

In any column, the top row = 1; the next row is based on the first; the next 2 rows on the first 2; the next 4 on the first 4, the next 8 on the first 8, and so on. (This view treats a block of values at once. If you were to look each row individually, the input row is shown by A053645.) The width of the input section is (N-2) – HighBit. The rectangles show the relationship between each section of rows in column vs. their respective input regions.

Each section is then multiplied by the modified Pascal vector (Table 7: Modified Pascal) of the corresponding length. For example, rows 4-7 of column 7 can be calculated as follows:

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 13 \\ 0 & 2 & 6 \\ 0 & 1 & 13 \end{bmatrix} * \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 42 \\ 18 \\ 19 \end{bmatrix}
$$

…and so on.

I find it remarkable that this set of relationships coexists with the very different types of relationships identified previously. Eg, look at the two 13's in column 6: they are paired by row number, but one is [0,1,1,1] * [1,4,6,3], and the other is [0,1,5] * [1,3,2]. Etc.

Point 6: Stirling Transforms

Left justify the frequency table (remove leading zeros from any row), a portion of which is shown here:

0	1	1	1	1	1	1
1	1	5	13	29	61	125
2	2	6	14	30	62	126
3	1	13	73	301	1081	3613
4	2	6	14	30	62	126
5	6	42	186	690	2346	7602
6	2	18	86	330	1142	3738
7	1	29	301	2069	11581	57749
8	2	6	14	30	62	126
9	18	102	414	1470	4878	15582
10	8	48	200	720	2408	7728
11	14	186	1394	8130	41474	195426
12	2	18	86	330	1142	3738
13	6	102	834	5070	26466	126462
14	2	42	374	2370	12662	61362
15	1	61	1081	11581	95401	673261
16	2	6	14	30	62	126
17	42	222	870	3030	9942	31542

Table 8: Left-Justified Sequence Frequency

The inverse stirling transform provides an interesting result:

0	1	0	0	0	0
1	1	4	0	0	0
2	2	4	0	0	0
3	1	12	36	0	0
4	2	4	0	0	0
5	6	36	72	0	0
6	2	16	36	0	0
7	1	28	216	576	0
8	2	4	0	0	0
9	18	84	144	0	0
10	8	40	72	0	0
11	14	172	864	1728	0
12	2	16	36	0	0
13	6	96	540	1152	0
14	2	40	252	576	0
15	1	60	900	5760	14400
16	2	4	0	0	0
17	42	180	288	0	0

Table 9: Inverse Stirling Transform of Left-Justified Frequency

Note the length of each row follows an obvious pattern, terminating with zeros. Furthermore, the GCD of each column C is C!^2 (particularly interesting, given the left-justification adjustment).

Dividing each column by the GCD:

1	0	0	0	0
1	1	0	0	0
2	1	0	0	0
1	3	1	0	0
2	1	0	0	0
6	9	2	0	0
2	4	1	0	0
1	7	6	1	0
2	1	0	0	0
18	21	4	0	0
8	10	2	0	0
14	43	24	3	0
2	4	1	0	0
6	24	15	2	0
2	10	7	1	0
1	15	25	10	1

Etc…

Table 10: Stirling Numbers of Second Kind, from Sequence Frequencies

Note that the row positions 1,2,4,8… generate the Stirling numbers of the second kind.

This table has a number of interesting properties, such as every 4th row is twice the next + previous. Again, there exist a great number of this relationships of this type, but I find it especially surprising since the table was generated from frequency table after being realigned via left-justification.

Point 7: "Pièce de résistance"

Every row in the above table can be represented by a linear combination of the stirling rows above it: (In the table below, Column 1 shows the multiplier for the first stirling row, column 2 for the second, etc.):

1	0	0	0	0
0	1	0	0	0
1	1	0	0	0
0	0	1	0	0
1	1	0	0	0
1	3	2	0	0
0	1	1	0	0
0	0	0	1	0
1	1	0	0	0
5	9	4	0	0
2	4	2	0	0
1	4	6	3	0
0	1	1	0	0
0	1	3	2	0
0	0	1	1	0
0	0	0	0	1
1	1	0	0	0
13	21	8	0	0
6	10	4	0	0
13	34	30	9	0
2	4	2	0	0
6	18	18	6	0
2	7	8	3	0
1	5	10	10	4
0	1	1	0	0
0	5	9	4	0
0	2	4	2	0
0	1	4	6	3
0	0	1	1	0
0	0	1	3	2
0	0	0	1	1

Table 11: Equations for each row based on Stirling

- The yellow sections follow an obvious – yet different – power of 2 pattern, and spell out the modified Pascal table. Note that below each yellow section are recurrences of the modified Pascal table. For example, while not detailed in this write-up, look at the right-most column.
- Embedded within each single column is the entire original sequence frequency table (Table 1: Sequence Frequencies) in vertical form. It is most obvious in the first column, where there is nothing but the original sequence frequencies separated by zeros.

So, Columnar Sequence Frequencies ->. Left justify -> Inverse Stirling Xform -> GCD -> Linear Combinations -> Single Column Sequence Frequencies AND modified Pascal.

Point 8: Row Equations of form $\Sigma k_i * i^n$

The above table can be generated a second way, with other surprising intermediate patterns. Instead of using an inverse stirling transform on the left-justified frequency table, determine the coefficients that put each row into the form $a*1^n + b*2^n + c*3^n \dots$

1	0	0	0	0	0
-3	2	0	0	0	0
-2	2	0	0	0	0
7	-12	6	0	0	0
-2	2	0	0	0	0
6	-18	12	0	0	0
4	-10	6	0	0	0
-15	50	-60	24	0	0
-2	2	0	0	0	0
6	-30	24	0	0	0
4	-16	12	0	0	0
-14	86	-144	72	0	0
4	-10	6	0	0	0
-12	66	-102	48	0	0
-8	38	-54	24	0	0
31	-180	390	-360	120	0
-2	2	0	0	0	0
6	-54	48	0	0	0
4	-28	24	0	0	0
-14	158	-360	216	0	0
4	-16	12	0	0	0
-12	120	-252	144	0	0
-8	68	-132	72	0	0
30	-330	1020	-1200	480	0
4	-10	6	0	0	0
-12	102	-186	96	0	0
-8	56	-96	48	0	0
28	-286	834	-936	360	0
-8	38	-54	24	0	0
24	-222	606	-648	240	0
16	-130	330	-336	120	0
-63	602	-2100	3360	-2520	720

1) Each column C (other than the first) now has a GCD = C!
2) Note the sequence of factorials along the upper right edge

Point 9: Stirling of Second Kind

Dividing each column in the above table by its GCD creates:

1	0	0	0	0	0
-3	1	0	0	0	0
-2	1	0	0	0	0
7	-6	1	0	0	0
-2	1	0	0	0	0
6	-9	2	0	0	0
4	-5	1	0	0	0
-15	25	-10	1	0	0
-2	1	0	0	0	0
6	-15	4	0	0	0
4	-8	2	0	0	0
-14	43	-24	3	0	0
4	-5	1	0	0	0
-12	33	-17	2	0	0
-8	19	-9	1	0	0
31	-90	65	-15	1	0
-2	1	0	0	0	0
6	-27	8	0	0	0
4	-14	4	0	0	0
-14	79	-60	9	0	0
4	-8	2	0	0	0
-12	60	-42	6	0	0
-8	34	-22	3	0	0
30	-165	170	-50	4	0
4	-5	1	0	0	0
-12	51	-31	4	0	0
-8	28	-16	2	0	0
28	-143	139	-39	3	0
-8	19	-9	1	0	0
24	-111	101	-27	2	0
16	-65	55	-14	1	0
-63	301	-350	140	-21	1

- Again the yellow sections show the stirling numbers of the second kind, but with the first column missing.
- The new first column as an interesting pattern. The stirling rows are obvious (1, -3, 7, -15, 31, -63, …). The rows in between stirling rows are -2 * the same length section before it. Eg: [-2, 6, 4, -14, 4, -12, -8] = -2 * [1, -3, -2, 7, -2, 6, 4].
- And again, if you represent each row as a linear combination of the yellow rows above it, you recreate (Table 11: Equations for each row based on Stirling) from Point 7 which regenerates the modified Pascal table, and the original sequence frequency table.

Results

| People	Probability
1	1.0000000000000000
2	0.0000000000000000
3	0.2500000000000000
4	0.1388888888888890
5	0.1319444444444440
6	0.1127777777777780
7	0.1005324074074070
8	0.0904946145124717
9	0.0823823696145125
10	0.0756457860922147
11	0.0699601678634417
12	0.0650929057547925
13	0.0608764331768653
14	0.0571864406383405
15	0.0539286981954046
16	0.0510304102987377
17	0.0484344083463231
18	0.0460951435768213
19	0.0439758588666862
20	0.0420465521576293
21	0.0402824835695692
22	0.0386630634659079

Table 12: Probability Table
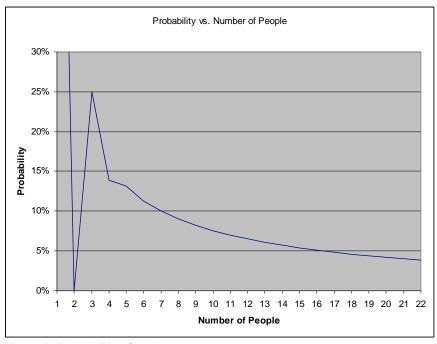
Figure 1: Probability Graph

A136301

1,1,1,1,5,2,1,1,13,6,13,2,6,2,1,1, 29, 14, 73, 6, 42, 18, 29, 2, 18, 8, 14, 2, 6, 2, 1, 1, 61, 30, 301, 14, 186, 86, 301, 6, 102, 48, 186, 18, 102, 42, 61, 2, 42, 20, 86, 8, 48, 20, 30, 2, 18, 8, 14, 2, 6, 2, 1, 1, 125, 62, 1081, 30, 690, 330, 2069, 14, 414, 200, 1394, 86, 834, 374, 1081, 6, 222, 108, 834, 48, 504, 228, 690, 18, 258, 120, 414, 42, 222, 90, 125, 2, 90, 44, 374, 20, 228, 104, 330, 8, 120, 56, 200, 20, 108, 44, 62, 2, 42, 20, 86, 8, 48, 20, 30, 2, 18, 8, 14, 2, 6, 2, 1, 1, 253, 126, 3613, 62, 2346, 1142, 11581, 30, 1470, 720, 8130, 330, 5070, 2370, 11581, 14, 870, 428, 5354, 200, 3360, 1580, 8130, 86, 1926, 920, 5354, 374, 3138, 1382, 3613, 6, 462, 228, 3138, 108, 1980, 936, 5070, 48, 1152, 552, 3360, 228, 1980, 876, 2346, 18, 570, 276, 1926, 120, 1152, 516, 1470, 42, 570, 264, 870, 90, 462, 186, 253, 2, 186, 92, 1382, 44, 876, 416, 2370, 20, 516, 248, 1580, 104, 936, 416, 1142, 8, 264, 128, 920, 56, 552, 248, 720, 20, 276, 128, 428, 44, 228, 92, 126, 2, 90, 44, 374, 20, 228, 104, 330, 8, 120, 56, 200, 20, 108, 44, 62, 2, 42, 20, 86, 8, 48, 20, 30, 2, 18, 8, 14, 2, 6, 2, 1

Description: The frequency of occurrence for each pattern of all outcomes for a Secret Santa drawing, in which each person draws a name in sequence but is not allowed to pick his own name, and for which the last person does draw his own name. For much more detail, see A136300. The sequence is best represented as a series of columns 1..n, where each column n has 2^(n-1) rows.

Formula: H(r,c) = sum of H(T(r),L(r)+j) * M(c-T(r)-1,j) for j = 0..c-L(r)-1, where M(y,z) = binomial distribution (y,z) when y - 1 > z, and (y,z)-1 when (y-1)<=z, and T(r) = A053645, and L(r) = A000523.

Formula in Mathematica:
```
maxP = number of rows to calculate
rows = Range[ 1, 2^(nP = maxP-3)  ];
pasc = Table[ Binomial[ p+1, i] - If[ i >= p, 1,0], {p, nP}, {i, 0, p} ];
sFreq = Table[0,{maxP-1}, {2^nP} ]; sFreq[[2;;maxP-1, 1]] = 1;
For[ p=1, p<=nP, p++, For[ s=1, s<=p, s++, \
       rS=Range[2^(s-1)+1, 2^s]; \
       sFreq[[p+2,rS]]=pasc[[p+1-s,1;;p+2-s]].sFreq[[ s;;p+1,1;;2^(s-1)]] ] ];
TableForm[Transpose[sFreq]]
```

Example: For 5 people, there are 9 target outcomes for which the last person gets his own name. But these 9 outcomes share only 4 possible probabilities, namely, 12/576, 8/576, 9/576, and 6/576. For the 9 outcomes, they distribute over these 4 probabilities with the distribution 1, 5, 2, 1 for a total likelihood of 76/576. As to the calculation of the table itself...look at H(3,5) which has the unique value of 73. The final step in the formula described above is the dot product of two vectors: {0, 1, 5, 13} * {1, 4, 6, 3} = 73. Vector M is {1,4,6,3} is similar to the binomial distribution for 4 items, but the case (4,3) is 3 instead of 4 (as described by the formula).