Toothpicks and Live Cells

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We understand a **toothpick** to be a compact unit subinterval of the real line. At time 1, place a toothpick in the xy-plane with endpoints at $(0, \pm 1/2)$. Both endpoints are *exposed* and must be *covered* at time 2. This is done by simultaneously placing a new toothpick with endpoints at $(\pm 1/2, 1/2)$ and a new toothpick with endpoints at $(\pm 1/2, -1/2)$. New toothpicks at odd times are always vertical; new toothpicks at even times are always horizontal. Any old endpoint is exposed if it is neither the endpoint nor the midpoint of any other existing toothpick. If exposed, it must be covered by the midpoint of a new toothpick without delay. At time 3, four new toothpicks are needed; likewise for times 4 and 5. At time 6, eight new toothpicks are required (see Figure 1); at time 7, twelve are required. No toothpicks are ever removed [1].

Let $T(n)$ denote the total number of toothpicks at time n. For $k \geq 0$, we have the following recursion:

$$T\left(2^k + j\right) = \begin{cases} \frac{1}{3}\left(2^{2k+1} + 1\right) & \text{if } j = 0, \\ T\left(2^k\right) + 2T\left(j\right) + T\left(j+1\right) - 1 & \text{if } 1 \leq j \leq 2^k - 1. \end{cases}$$

No simple formula for $T(n)$ is known; it is not well behaved asymptotically in the sense that [2]

$$0.4513058284... = c = \liminf_{n\to\infty} \frac{T(n)}{n^2} < \limsup_{n\to\infty} \frac{T(n)}{n^2} = \frac{2}{3}.$$

We understand **cells** to be the basis elements of an infinite planar square lattice. Neighbors of each cell are defined to be the four squares that share an edge with it (see Figure 2). At time 1, a single cell is alive. At time $n > 1$, a cell newly comes to life if and only if exactly one of its neighbors is alive and older (that is, alive at time $n - 1$). Once a cell is alive, it remains alive forever [3, 4, 5, 6, 7, 8, 9].

Let $U(n)$ denote the total number of live cells at time n. For $n \geq 1$, a simple formula applies:

$$U(n) = \frac{1}{3}\left(4\sum_{m=0}^{n-1} 3^{b(m)} - 1\right)$$

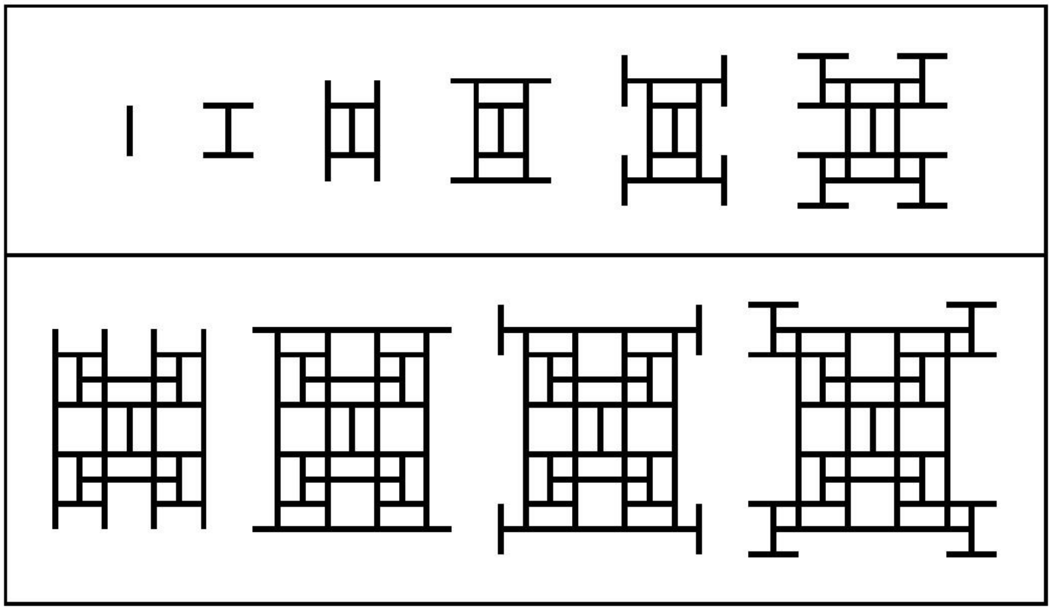

Figure 1: Toothpicks $\{T(n)\}_{n=1}^{10} = \{1, 3, 7, 11, 15, 23, 35, 43, 47, 55\}$, from [1].

where $b(m)$ is the number of ones in the binary expansion of m. We have seen such exponential sums of digital sums before [10] and find [11]

$$0.9026116569... = \liminf_{n\to\infty} \frac{U(n)}{n^2} < \limsup_{n\to\infty} \frac{U(n)}{n^2} = \frac{4}{3}.$$

The fact that 4/3 is the limit superior has been known for years [4]; by contrast, no one seems to have studied the limit inferior until now. Is this quantity equal to $2c$? Why should the toothpick and Ulam-Warburton automata be so closely related? Sloane [12] provided an overview of associated ideas.

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References

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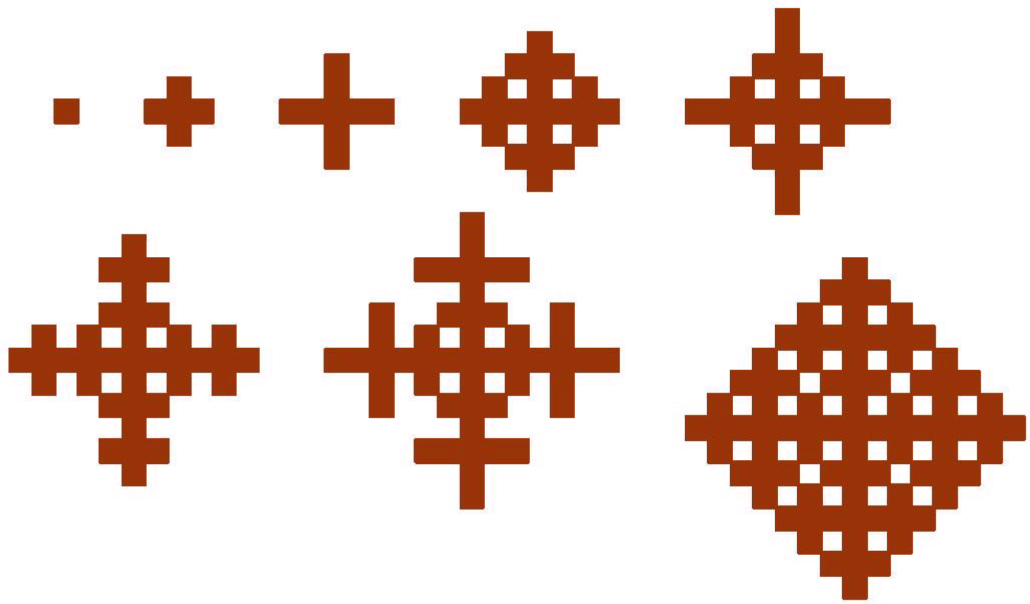

Figure 2: Live cells $\{U(n)\}_{n=1}^{8} = \{1, 5, 9, 21, 25, 37, 49, 85\}$, from [9].

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