FOUNTAINS OF COINS AND SKEW FERRERS DIAGRAMS

Peter Bala, July 26 2019

Extending a result of Odlyzko and Wilf [5], we find the generating function for the number of fountains of coins according to the number of coins in the fountain, the number of coins in the bottom row of the fountain and by the numbers of coins in the even- and odd-numbered rows of the fountain. We discover a connection between fountains of coins enumerated by the numbers of coins in the even- and odd-numbered rows and skew Ferrers diagrams enumerated by area and width.

1. INTRODUCTION

An (n,k) fountain of coins is an arrangement of n coins in rows such that there are exactly k contingUous coins in the bottom row and such that each coin in a higher row touches exactly two coins in the next lower row. Let q mark the coins in a fountain and b mark the coins in the bottom row of a fountain. Odlyzko and Wilf [5] found an elegant continued fraction representation for the generating function of the number $f(n,k)$ of (n,k) fountains

$$\begin{aligned} F(q,b) &= \sum_{\substack{\text{all } (n,k) \\ \text{fountains}}} q^n b^k \\ &= \sum_{n,k\geq 0} f(n,k) q^n b^k \end{aligned}$$

in the form

$$F(q,b) = \frac{1}{1 -}\,\frac{qb}{1 -}\,\frac{q^2b}{1 -}\,\frac{q^3b}{1 -}\cdots .$$

See A047998. The number $f(n,k)$ of (n,k) fountains is also equal to the number of 231-avoiding permutations in the symmetric group S_k with exactly $n-k$ inversions [2, Proposition 4].

We number the rows of a fountain of coins, starting at 0 for the bottom row of the fountain. Our purpose in this note is to extend Odlyzko and Wilf's analysis of coin fountains to enumerate fountains according to the numbers of coins they contain in the even-numbered and the odd-numbered rows.

We define an (n,k,k_e,k_o) fountain of coins to be an (n,k) fountain such that k_e of the coins belong to the even-numbered rows of the fountain and $k_o = n - k_e$ coins belong to the odd-numbered rows of the fountain. Figure 1 gives an example of a $(24,10,15,9)$ fountain.

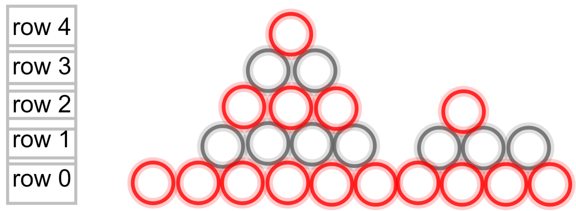

FIGURE 1. A $(24, 10, 15, 9)$ coin fountain

2. THE GENERATING FUNCTION FOR COIN FOUNTAINS

Let e mark the coins in the even-numbered rows of the fountain and b mark the coins in the odd-numbered rows of the fountain. We attach the weight $q^n b^k e^{k_e} o^{k_0}$ to an (n, k, k_e, k_o) coin fountain and define

$$F(q, b, e, o) = \sum_{\substack{\text{all } (n,k,k_e,k_o) \\ \text{fountains}}} q^n b^k e^{k_e} o^{k_0}$$

to be the generating function of the number of weighted coin fountains.

A coin fountain is defined to be a primitive fountain if its bottom row (row 0) contains $k \geq 1$ coins and its next-to-bottom row (row 1) contains $k-1$ coins (i.e., no empty positions). Let

$$G(q, b, e, o) = \sum q^n b^k e^{k_e} o^{k_o},$$

where the sum is taken over all primitive coin fountains, denote the generating function for the weighted primitive coin fountains. Clearly, every coin fountain factorises uniquely into an initial primitive fountain followed by a not-necessarily-primitive fountain (possibly empty). An arbitrary coin fountain can thus be viewed as a sequence of primitive coin fountains. This observation is equivalent to the relation between generating functions

$$F = 1 + GF. \tag{1}$$

Let $f(n, k, k_e, k_o)$ denote the number of (n, k, k_e, k_o) coin fountains and $g(n, k, k_e, k_o)$ denote the number of primitive (n, k, k_e, k_o) coin fountains. Removing the bottom row of k coins from a primitive (n, k, k_e, k_o) coin fountain produces a fountain of $n-k$ coins with (after renumbering the rows) $k-1$ coins in its bottom row (now row 0), k_o coins in the even-numbered rows and $k_e - k$ coins in the odd-numbered rows. This process is reversible. Consequently,

$$g(n, k, k_e, k_o) = f(n-k, k-1, k_o, k_e - k), \quad n \geq k \geq 1, k_e \geq k, k_o \geq 0.$$

which is equivalent to the following relation between generating functions:

$$G(q,b,e,o) \quad = \quad qbeF(q,qbe,o,e). \tag{2}$$

Note the switch of the e and o arguments. It follows from (1) and (2) that

$$\begin{aligned} F(q,b,e,o) &= \frac{1}{1-G(q,b,e,o)} \\ &= \frac{1}{1-qbeF(q,qbe,o,e)} \\ &= \cfrac{1}{1-\cfrac{qbe}{1-q^2beoF(q,q^2beo,e,o)}}. \end{aligned} \tag{3}$$

Succesive iterations of this identity lead to the formal Stieltjes continued fraction expansion

$$F = \frac{1}{1} - \frac{qbe}{1} - \frac{q^2beo}{1} - \frac{q^3be^2o}{1} - \frac{q^4be^2o^2}{1} - \frac{q^5be^3o^2}{1} - \frac{q^6be^3o^3}{1} - \cdots . \tag{4}$$

3. FOUNTAINS OF COINS AND SKEW FERRERS DIAGRAMS

Setting $q = 1$, $b = 1$ and $o = 1$ in (4) gives the generating function for the number of coin fountains having exactly n coins in the even-numbered rows as

$$\frac{1}{1} - \frac{e}{1} - \frac{e}{1} - \frac{e^2}{1} - \frac{e^2}{1} - \frac{e^3}{1} - \frac{e^3}{1} - \cdots \tag{5}$$

$$= 1 + e + 2e^2 + 4e^3 + 9e^4 + 20e^5 + 46e^6 + \cdots .$$

The sequence of coefficients $[1, 1, 2, 4, 9, 20, 46, ...]$ is A006958 (with an extra initial term equal to 1), which has the description 'Number of parallelogram polyominoes with n cells'. Another combinatorial interpretation for the sequence is stated there as the number of skew Ferrers diagrams of area n. The continued fraction in (5) is given by P. D. Hanna as the generating function for the sequence. Therefore we have the result:

Proposition 1 The number of skew Ferrers diagrams of area n is equal to the number of fountains of coins with n coins in the even-numbered rows of the fountain. □

We shall prove Hanna's observation shortly as a consequence of a more general result.

Setting $q = 1$ and $b = 1$ in (4) gives the bivariate generating function for the triangle $T(n,k)$ of the number of coin fountains with n coins in the even-numbered rows and k coins in the odd-numbered rows as the continued fraction

$$\frac{1}{1} - \frac{e}{1} - \frac{eo}{1} - \frac{e^2o}{1} - \frac{e^2o^2}{1} - \frac{e^3o^2}{1} - \frac{e^3o^3}{1} - \cdots. \tag{6}$$

Expanding this continued fraction into a power series in e with polynomial coefficients in the variable o produces the following table:

$T(n,k)$

$n\diagdown k$	1	o	o^2	o^3	o^4	o^5
1	1					
e	1					
e^2	1	1				
e^3	1	2	1			
e^4	1	3	4	1		
e^5	1	4	8	6	1	
e^6	1	5	13	17	9	1

This table, as it stands, doesn't appear in the OEIS but it does appear in row reversed form. We make the change of variables $e \to qw$ and $o \to 1/w$ to put the table into row reversed form, with bivariate generating function from (6) given by

$$\frac{1}{1} - \frac{qw}{1} - \frac{q}{1} - \frac{q^2w}{1} - \frac{q^2}{1} - \frac{q^3w}{1} - \frac{q^3}{1} - \cdots. \tag{7}$$

The expansion of (7) as a power series in q gives rise to the row reversal of the previous table:

$T(n,n-k)$

$n\diagdown k$	w^0	w^1	w^2	w^3	w^4	w^5	w^6
q^0	1						
q^1		1					
q^2		1	1				
q^3		1	2	1			
q^4		1	4	3	1		
q^5		1	6	8	4	1	
q^6		1	9	17	13	5	1

We willl show this table is A161492 (with an extra term 1 included in position (0,0)) - the triangular array showing the number of skew Ferrers diagrams with area n (marked by q) and width k columns (marked by w).

Delest and Fédou [4] give the generating function $G(q,w)$ of skew Ferrers diagrams by area and width as a ratio of q-series:

$$G(q,w) \quad = \quad \frac{\displaystyle\sum_{n\geq 0} \frac{(-1)^n q^{\binom{n+2}{2}} w^{n+1}}{(1-q^{n+1})\displaystyle\prod_{k=1}^{n}(1-q^k)^2}}{\displaystyle\sum_{n\geq 0} \frac{(-1)^n q^{\binom{n+1}{2}} w^n}{\displaystyle\prod_{k=1}^{n}(1-q^k)^2}} \tag{8}$$

$$= qw + q^2(w+w^2) + q^3(w+2w^2+w^3) + q^4(w+4w^2+3w^3+w^4) + \cdots .$$

Adding 1 to the generating function $G(q,w)$ (i.e., we include the empty skew Ferrers diagram) we find, after a short calculation,

$$1 + G(q,w) \quad = \quad \frac{\displaystyle\sum_{n\geq 0} \frac{(-1)^n q^{\frac{(n^2+3n)}{2}} w^n}{\displaystyle\prod_{k=1}^{n}(1-q^k)^2}}{\displaystyle\sum_{n\geq 0} \frac{(-1)^n q^{\frac{(n^2+n)}{2}} w^n}{\displaystyle\prod_{k=1}^{n}(1-q^k)^2}}. \tag{9}$$

The ratio of q-series in (9) may be expressed as a continued fraction by specialising a result in Ramanujan's lost notebook. Applying [1, Entry 6.2.3, p. 148, with $a=-w$, $b=-1$ and $\lambda=0$] gives the continued fraction representation

$$1 + G(q,w) \quad = \quad \frac{1}{1} - \frac{qw}{1} - \frac{q}{1} - \frac{q^2w}{1} - \frac{q^2}{1} - \frac{q^3w}{1} - \frac{q^3}{1} - \cdots . \tag{10}$$

We see that (7) and (10) agree. Thus we have established the following refinement of Proposition 1 connecting fountains of coins and skew Ferrers diagrams:

Proposition 2 The number of skew Ferrers diagrams of area n and width k is equal to the number of fountains of coins with n coins in the even-numbered rows and $n-k$ coins in the odd-numbered rows of the fountain. □

Question Is there a bijective proof of this result?

We conclude with two other continued fraction representations for the generating function of skew Ferrers diagrams. Firstly, from [1, Entry 6.4.1, p. 161, with $a = -w$, $b = -1$ and $\lambda = 0$] we obtain the representation

$$1 + G(q,w) = \frac{1}{1-qw} - \frac{q^2w}{1-q(1+qw)} - \frac{q^4w}{1-q^2(1+qw)} - \frac{q^6w}{1-q^3(1+qw)} - \cdots . \tag{11}$$

Secondly, by [1, Theorem 6.4.1, p. 160, again with $a = -w$, $b = -1$ and $\lambda = 0$] we find

$$1 + G(q,w) = \frac{1}{1} - \frac{qw}{1-q(1-w)} - \frac{qw}{1-q(q-w)} - \frac{qw}{1-q\,(q^2-w)} - \cdots . \tag{12}$$

References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's Lost Notebook, Part 1, Springer 2005

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[3] G. Bhatnagar, How to prove Ramanujan's q-continued fractions arXiv:1205.5455v3 [math.HO]

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