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(with an extra leading column)

Faulhaber's Triangle

Mohammad Torabi-Dashti

December 22, 2011

Like Pascal's triangle, Faulhaber's triangle is easy to draw: all you need is pen, paper and a little recursion. The rows of Faulhaber's triangle are the coefficients of polynomials that represent sums of integer powers. Such polynomials are often called *Faulhaber formulae* [2], after Johann Faulhaber (1580–1635); hence we dub the triangle Faulhaber's triangle.

Constructing Faulhaber's triangle

Draw a right triangle, similar to the one shown in Figure 1. Number the rows, starting with row 0; number the columns from left to right, starting with column 1. The numbers on row i are found using the following recursive rules:

- The leftmost element of each row is chosen such that the row sums to 1. In particular, the only number on row 0 is 1.
- The element at row i and column j $(1 < j \leq i+1)$ is found by multiplying the number directly above and to the left by $\frac{i}{j}$.

Sums of integer powers

The sum of integer powers $1^p + 2^p + \cdots + n^p$, with integers $n, p \geq 0$, is a polynomial in n of degree $p+1$. That is $f_p(n) = a_{p+1}n^{p+1} + a_p n^p + \cdots + a_1 n + a_0$. Taking $n = 0$, it follows immediately that $a_0 = 0$. In order to find the coefficients of the polynomial, we draw Faulhaber's triangle. Row p of the triangle gives the coefficients $a_1, \cdots, a_{p+1}$.

row 0	1						
row 1	$\frac{1}{2}$	$\frac{1}{2}$					
row 2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$				
row 3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$			
row 4	$\frac{-1}{30}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$		
row 5	0	$\frac{-1}{12}$	0	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{6}$	
row 6	$\frac{1}{42}$	0	$\frac{-1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{7}$
...							

Figure 1: Faulhaber's triangle

For instance, to find $f_4(n)$ we use row 4 of Figure 1: $a_1 = -\frac{1}{30}$, $a_2 = 0$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{2}$ and $a_5 = \frac{1}{5}$. That is

$$f_4(n) = \sum_{i=1}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

We now observe that $f_p(n)$ is always of the shape $\frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + a_{p-1}n^{p-1} + a_{p-3}n^{p-3} + \ldots$, with all coefficients $a_{p-2k} = 0$ for $k > 0$. We also note that the numbers appearing on the vertical leg (leftmost column) of Faulhaber's triangle are the *Bernoulli numbers*, namely $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, etc. This is due to the well-know *Bernoulli formula* stating $f_p(n) = \frac{1}{p+1}\sum_{i=0}^{p}\binom{p+1}{i}B_i n^{p+1-i}$.

Why it works

Suppose the coefficient of n^a is α in $f_b(n)$, for some $1 < a \leq b+1$, and the coefficient of n^{a-1} in $f_{b-1}(n)$ is β. It can be shown that $\alpha = \frac{b}{a}\beta$, cf. [1, 3]. In Faulhaber's triangle, this corresponds to row $b-1$ containing β at column $a-1$, and row b containing α at column a. Note that our construction of Faulhaber's triangle ensures $\alpha = \frac{b}{a}\beta$.

Next, observe that $f_p(1) = a_{p+1} + \cdots + a_1 = 1$, for all p, so that $a_1 = 1 - (a_{p+1} + \cdots + a_2)$. This is the reason the leftmost element of each row is chosen such that the values on the row sum up to 1.

Now, by a straightforward induction, if the numbers on row p are the coefficients of $f_p(n)$, then the numbers on row $p+1$ are the coefficients of $f_{p+1}(n)$. The base case is immediate, as $f_0(n) = n$.

References

[1] David M. Bloom. An old algorithm for the sums of integer powers. *Mathematics Magazine*, 66(5):304–305, 1993.

[2] Donald Knuth. Johann Faulhaber and sums of powers. *Math. Comp.*, 61(203):277–294, 1993.

[3] Howard Sherwood. Sums of powers of integers and Bernoulli numbers. *Math. Gaz.*, 54(389):272–274, 1970.