Notes on the number of $m \times n$ binary arrays with all 1's connected and a path of 1's from top row to bottom row.

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Inspired and egged on by Stan Wagon and POW 1296

We consider the sequences A163029 ($n = 3$) through A163036 ($n = 10$) in the OEIS, which count the number of $m \times n$ binary arrays with all 1's connected and a path of 1's from top row to bottom row, giving a general, if increasingly intractable, method of constructing closed formulas and recurrence relations for arbitrary n. (The Problem of the Week #1296 asked for the number of 12×2 arrays of this kind, rotated 90°— there are 47321.)

We first define a map ψ taking each binary array to a certain integer array; we first assign numbers to the components of the 1's in the bottom row; the first component on the left is numbered ± 1, the next ± 2, etc, $(+)$ if the component is connected to the top row, and $(-)$ if not. (A given component may meet the bottom row in more than one range.) We then replace each 1 with the number assigned to its component.

In the same manner, in any given row, we assign values to the components of the subarray formed by deleting all lower rows, and replace the 1's with the values of those components. Note that ψ is one-to-one and well-defined — our conventions produce exactly one integer array from a given binary array.

Here is an example (writing $\overline{n}$ for $-n$ to preserve formatting).

$$\begin{array}{cccccccc} 1&1&1&0&0&1&0&1\\ 1&0&0&1&0&1&0&1\\ 0&1&0&1&1&1&1&1\\ 0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&1\\ 0&1&1&1&0&1&1&1 \end{array} \quad \begin{array}{c} \\ \\ \psi \\ \mapsto \\ \\ \\ \end{array} \quad \begin{array}{cccccccc} 1&1&1&0&0&2&0&3\\ 1&0&0&\overline{2}&0&3&0&4\\ 0&\overline{1}&0&2&2&2&2&2\\ 0&\overline{1}&0&2&0&0&0&2\\ 0&\overline{1}&0&2&0&\overline{3}&0&2\\ 0&1&1&1&0&1&1&1 \end{array}$$

On the first row of the binary array, we have three components, which ψ numbers $1, 2, 3$.

Four components reach the bottom row of the subarray $\begin{array}{cccccccc} 1&1&1&0&0&1&0&1\\ 1&0&0&1&0&1&0&1 \end{array}$; which is now numbered $\begin{array}{cccccccc} 1&0&0&-2&0&3&0&4 \end{array}$ — the second component valued -2 does not meet the top row.

The bottom row of the next subarray $\begin{array}{cccccccc} 1&1&1&0&0&1&0&1\\ 1&0&0&1&0&1&0&1\\ 0&1&0&1&1&1&1&1 \end{array}$ meets only two components – one has disappeared, a new component has been introduced and three have been connected, and ϕ assigns the values $\begin{array}{cccccccc} 0&-1&0&2&2&2&2&2 \end{array}$. And so on.

We'll call an integer array a **word**, and its rows **letters**, read top to bottom. **Well-formed words** will be those produced by ψ from binary arrays where there is a path of 1's from top to bottom. **Connected well-formed** words will be well-formed words arising from binary arrays with a single component of 1's and a path of 1's from top to bottom.

A letter $(p_1, \ldots p_n)$ in a well-formed $m \times n$ word must satisfy:

- For some integer $k > 0$, $\{|p_1|, \ldots, |p_n|\} = \{0, 1 \ldots, k\}$
- For some $1 \leq i \leq n$, $p_i > 0$
- For each pair i, j, $p_i \neq -p_j$, unless $p_i = p_j = 0$
- For each $1 \leq i < n$, if $p_i \neq p_{i+1}$, then one or the other of p_i, p_{i+1} equals 0.
- For any $l > 1$, if $|p_i| = l$, then for some $1 \leq j < i$, $0 < |p_j| < |p_i|$.

These conditions simply ensure that our numbering is consistent with the description of ψ as a well-defined map.

Examples of well-formed letters are thus:

$$(0,1,1,0,2) \quad (0,-1,-1,0,2,2) \quad (1,1,0,0,-2,0,3) \quad (1,1,0,-2,0,1,1) \quad (1,0,2,0,1,0,2)$$

(Though the last letter can never appear in a word for topological reasons; we could add a condition to rule this out, but won't need to.)

Some tuples that are *not* well-formed letters include:

$$(0,2,2,0,1) \quad (1,2) \quad (1,0,-1) \quad (0,0,0) \quad (1,0,3)$$

The **values** of a letter $(p_1, \ldots p_n)$ are $\{p_1, \ldots, p_n\} \setminus \{0\}$, and each value has **runs** of indices. So for example, the letter $(1,1,1,0,2,2,0,1)$ has two values, 1 with runs $\{1,2,3\}, \{8\}$, and 2 with one run $\{5,6\}$.

(Note that a value may have more than one run, if a component of 1's in the original binary array meets a row in more than one way.)

An **initial letter** is a letter with only positive values, each with one run. Any well-formed word must begin with an initial letter — each component meeting the top row of an integer array meets the top row in one run.

A letter is a **terminal letter** if and only if 1 is its only value. Any connected well-formed word must end in a terminal letter — exactly one component of 1's meets the bottom row of the corresponding binary array.

We next give conditions on pairs of letters in well-formed words. First we define:

In a pair of letters L and R in a word, a value c of L **meets** a value d of R iff some run of c intersects a run of d.

Two runs of a value of R are **linked** if they each intersect the runs of some value of L. A value is **fully linked** iff there is a sequence of its runs, each linked to the next.

So in the following example, the value 1 of R meets values $1, 2$ and 3 of L, and is fully linked:

$$\begin{array}{cc} L & (1\ \ 0\ \ 2\ \ 2\ \ 2\ \ 0\ \ 3\ \ 3\ \ 3\ \ 0\ \ 1) \\ R & (1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1) \end{array}$$

(That last 1 in R is linked to the first run of 1's through the value 1 of L.)

Lemma A word is well-formed if and only if it begins with an initial letter, each row is a letter, and each pair of adjacent letters L and R satisfy

- Any value c of L meets exactly one value d of R, and if c has positive value, so too does d.
- Any positive value of R must meet positive value in L.
- Any value of R must be fully linked.

These conditions, essentially, ensure that the numbering of each row is consistent with the conventions of ψ: if a integer array is produced by ψ it must satisfy these conditions; if it satisfies these conditions, it can be produced by ψ. The proof is self-explanatory.

We now calculate:

The following method of counting words in a regular language is well known (see, for example, §4.7 of Richard P. Stanley's *Enumerative Combinatorics*).

Fixing n, let the well-formed letters be enumerated $l_1, \ldots, l_k$. Let s be the binary row vector where the ith entry is 1 if and only if l_i is an initial letter. Let e be the binary row vector where the ith entry is 1 if and only if l_i is a terminal letter. Let T be the binary $k \times k$ matrix with the entry on the ith row and jth column 1 if and only if the letter l_i may be followed by l_j in a well-formed word.

Then the number a_m of complete well-formed words of length m is

$$\mathsf{s} \cdot \mathsf{T}^m \cdot \mathsf{e} \tag{1}$$

This sequence has generating function $g(x) = \mathsf{s}(\sum \mathsf{T}^m x^m)\mathsf{e} = \mathsf{s}(I - \mathsf{T}x)^{-1}\mathsf{e}$. In practice, for $n = 7$ at least, rather than taking this inverse, it is more efficient to solve $(I - \mathsf{T}x)X = \mathsf{e}$ and take $g(x) = \mathsf{s}.X$.

This $g(x)$ will be a rational function; the denominator will be a factor of the characteristic polynomial of T; the denominator of $g(1/x)$ will be the characteristic polynomial of the recurrence. The numerator encodes the initial conditions, but for now we calculate these directly. The closed form of a_m is a linear combination of powers of the eigenvalues (or if any occur with multiplicity, the usual stuff).

We can also guess the characteristic polynomial of the recurrence: it will be a factor of the characteristic polynomial of the matrix, and in each case so far, it has been the largest factor irreducible in $\mathbb{Z}$. The guess can be checked by hand.

We therefore have:

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n=2

Letters: $01, 10, 11$, with $\mathsf{s} = 111, \mathsf{T} = \begin{matrix} 101 \\ 011 \\ 111 \end{matrix}$ and $\mathsf{e} = 111$. The characteristic polynomial of T is $(x - 1)(x^2 - 2x - 1)$. The eigenvector of 1 is $10\overline{1}$, and the factor $(x - 1)$ is not needed. The recurrence is $a_m - 2a_{m-1} - a_{m-2} = 0$. The initial terms are calculated from (1), and we have

$$a_m = ((1 - \sqrt{2})^{n+1} + (1 + \sqrt{2})^{n+1})/2$$

Answering POW 1296, we have the sequence

$$3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, \text{etc.}$$

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n=3, o.e.i.s. A163029

There are ten letters: $001, 010, 011, 100, 101, 102, 10\overline{2}, \overline{1}02, 110, 111$; with $\mathsf{s} = 1111010011, \mathsf{e} = 1111100011$

$$\text{and } \mathsf{T} = \begin{array}{l} 1010000101 \\ 0110000011 \\ 1110000111 \\ 0001001011 \\ 1011100011 \\ 0000010001 \\ 0000001001 \\ 0000000101 \\ 0111001011 \\ 1111100011 \end{array}$$

The characteristic polynomial is $1 - 8x + 6x^2 + 22x^3 - 41x^4 + 16x^5 + 31x^6 - 52x^7 + 34x^8 - 10x^9 + x^{10}$, but the factor with useful eigenvalues is $(-1 + 7x - x^2 - 6x^3 + 11x^4 - 7x^5 + x^6)$, giving the recurrence relation. Applying (1) we calculate initial terms, and a closed form for the terms a_m. The largest root is $\sim 5.05662...$ so this grows fairly quickly.

A recurrence relation for the sequence is $a_n = 7\ a_{(n-1)} - 11\ a_{(n-2)} + 6\ a_{(n-3)} + 1\ a_{(n-4)} - 7\ a_{(n-5)} - a_{(n-6)}$

with initial terms

$$\begin{aligned} a_1 &= 6 \\ a_2 &= 28 \\ a_3 &= 144 \\ a_4 &= 730 \\ a_5 &= 3692 \\ a_6 &= 18666 \end{aligned}$$

continuing:

94384, 477264, 2413346, 12203374, 61707810, 312032874, 1577831334, 7978491800, 40344192708, 204005208738, 1031576601204, 5216289773894, 26376789637884, 133377373911160..

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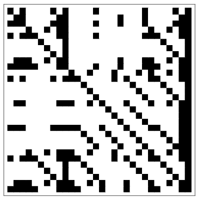
n=4, o.e.i.s. A163031

There are thirty letters:

$0001, 0010, 0011, 0100, 0101, 0102, 010\overline{2}, 0\overline{1}02, 0110, 0111, 1000, 1001, 1002, 100\overline{2}, \overline{1}002,$
$1010, 1020, 10\overline{2}0, \overline{1}020, 1011, 1022, 10\overline{22}, \overline{1}022, 1100, 1101, 1102, 110\overline{2}, \overline{11}02, 1110, 1111.$

The transition matrix can be drawn as

 and s and e are easily written down. From the sum of the entries of $(I - \mathsf{T}x)^{-1}$ we obtain the characteristic polynomial for the recurrence: $a_n = 15\ a_{(n-1)} - 59\ a_{(n-2)} + 97\ a_{(n-3)} - 19\ a_{(n-4)} - 210\ a_{(n-5)} + 222\ a_{(n-6)} + 22\ a_{(n-7)} - 113\ a_{(n-8)} + 7\ a_{(n-9)} - 71\ a_{(n-10)} 13\ a_{(n-11)} + a_{(n-12)}$

with initial values 15, 245, 5191, 104989, 2075271, 40792921, 801218515, 15736428305, 309080891641, 6070750256417, 119237718452471, 2341990743046197

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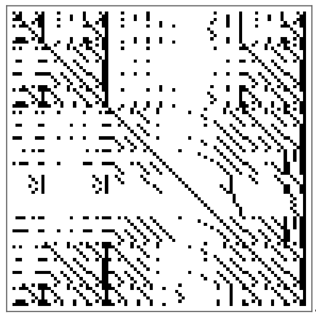
n=5, o.e.i.s. A163032

There are 91 letters, and transition matrix . It is nearly intractable to calculate the characteristic polynomial of this recurrence by taking the sum of the entries of $(I - \mathsf{T}x)^{-1}$, but it is still possible:

$a_n = 39\ a_{(n-1)} - 547\ a_{(n-2)} + 4079\ a_{(n-3)} - 17927\ a_{(n-4)} + 44032\ a_{(n-5)} - 29646\ a_{(n-6)} - 161261\ a_{(n-7)} + 507392\ a_{(n-8)} - 444090\ a_{(n-9)} - 488756\ a_{(n-10)} + 980592\ a_{(n-11)} - 101650\ a_{(n-12)} + 1438895\ a_{(n-13)} - 4174679\ a_{(n-14)} - 1246016\ a_{(n-15)} + 8663468\ a_{(n-16)} - 3414959\ a_{(n-17)} + 510459\ a_{(n-18)} - 1716806\ a_{(n-19)} - 2980202\ a_{(n-20)} + 5525320\ a_{(n-21)} + 331308\ a_{(n-22)} - 2267390\ a_{(n-23)} + 612301\ a_{(n-24)} + 136350\ a_{(n-25)} - 115804\ a_{(n-26)} + 34459\ a_{(n-27)} - 4461\ a_{(n-28)} - 99\ a_{(n-29)} 49\ a_{(n-30)} - a_{(n-31)}$

with initial terms
$$\begin{array}{rcl}
a_1 &=& 15\\
a_2 &=& 245\\
a_3 &=& 5191\\
a_4 &=& 104989\\
a_5 &=& 2075271\\
a_6 &=& 40792921\\
a_7 &=& 801218515\\
a_8 &=& 15736428305\\
a_9 &=& 309080891641\\
a_{10} &=& 6070750256417\\
a_{11} &=& 119237718452471\\
a_{12} &=& 2341990743046197\\
a_{13} &=& 45999883370408813\\
a_{14} &=& 903500281246849523\\
a_{15} &=& 17745974522766912147\\
a_{16} &=& 348555078644003475079\\
a_{17} &=& 6846095866027124445869\\
a_{18} &=& 134466635189696499098689\\
a_{19} &=& 2641107622912159591737683\\
a_{20} &=& 51874946271576878120170843\\
a_{21} &=& 1018894507491127148730135725\\
a_{22} &=& 20012474074877644930113119211\\
a_{23} &=& 393072212729683754846248880235\\
a_{24} &=& 7720472933137610629388374092373\\
a_{25} &=& 151640590153599847125992969616893\\
a_{26} &=& 2978427459208370993425603576852157\\
a_{27} &=& 58500366694568952264173125591847767\\
a_{28} &=& 1149026776804104294029040140287666277\\
a_{29} &=& 22568448856157327769860569990286441673\\
a_{30} &=& 443275034189934001489479158801663255597\\
a_{31} &=& 8706524634832323977386720525565332228553
\end{array}$$

$a_n = 30\ a_{(n-1)} - 376\ a_{(n-2)} + 2650\ a_{(n-3)} - 11652\ a_{(n-4)} + 32450\ a_{(n-5)} - 50810\ a_{(n-6)} + 8706\ a_{(n-7)} + 149633\ a_{(n-8)} - 316800\ a_{(n-9)} + 178009\ a_{(n-10)} + 369410\ a_{(n-11)} - 737149\ a_{(n-12)} + 227248\ a_{(n-13)} + 671802\ a_{(n-14)} - 669620\ a_{(n-15)} - 180974\ a_{(n-16)} + 480124\ a_{(n-17)} - 22597\ a_{(n-18)} - 149542\ a_{(n-19)} - 24135\ a_{(n-20)} + 3856\ a_{(n-21)} + 42193\ a_{(n-22)} + 18994\ a_{(n-23)} - 14562\ a_{(n-24)} - 8202\ a_{(n-25)} + 596\ a_{(n-26)} + 714\ a_{(n-27)} + 24\ a_{(n-28)} - 18\ a_{(n-29)} + a_{(n-30)}$

The initial values are 15, 245, 5191, 104989, 2075271, 40792921, 801218515, 15736428305,
309080891641,
6070750256417,
119237718452471,
2341990743046197,
45999883370408813,
903500281246849523,
17745974522766912147,
348555078644003475079,
6846095866027124445869,
134466635189696499098689,

2641107622912159591737683,
51874946271576878120170843,
1018894507491127148730135725,
20012474074877644930113119211,
393072212729683754846248880235,
7720472933137610629388374092373,
151640590153599847125992969616893,
2978427459208370993425603576852157,
58500366694568952264173125591847767,
1149026776804104294029040140287666277,
22568448856157327769860569990286441673,
443275034189934001489479158801663255597,
8706524634832323977386720525565332228553

The largest eigenvalue is $\sim 19.6414...$ so these grow quickly!

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n=6, o.e.i.s. A163033

There are 280 words, and the asymptotic growth rate of the sequence is about 38.091983254166494728...

A recurrence relation for the sequence is $a_n = 87\ a_{(n-1)} - 2935\ a_{(n-2)} + 54509\ a_{(n-3)} - 630711\ a_{(n-4)} + 4725868\ a_{(n-5)} - 21912987\ a_{(n-6)} + 44795035\ a_{(n-7)} + 131635076\ a_{(n-8)} - 1231668519\ a_{(n-9)} + 3347087887\ a_{(n-10)} - 125874878\ a_{(n-11)} - 24928496115\ a_{(n-12)} + 81563842866\ a_{(n-13)} - 160982299967\ a_{(n-14)} + 120747452752\ a_{(n-15)} + 1147666495639\ a_{(n-16)} - 5508899106335\ a_{(n-17)} + 8203457847402\ a_{(n-18)} + 4625801184826\ a_{(n-19)} - 31599876220425\ a_{(n-\ldots)}$ $81813168238665\ a_{(n-21)} - 194605304006872\ a_{(n-22)} + 63184048989135\ a_{(n-23)} + 716710633162786\ a_{(n-24)} - 856486093445099\ a_{(n-25)} - 362354581913702\ a_{(n-26)} - 393977298044241\ a_{(n-27)} - 690509074112335\ a_{(n-28)} + 8731108200795906\ a_{(n-29)} - 241373669285457\ a_{(n-30)} - 20193700110690433\ a_{(n-31)} - 524705751265555\ a_{(n-32)} + 14594940387158102\ a_{(n-33)} + 25147421555620198\ a_{(n-34)} + 10385936492683462\ a_{(n-35)} - 75970214134729814\ a_{(n-36)} - 45849742966958689\ a_{(n-37)} + 111968910976124101\ a_{(n-38)} + 76277143787941271\ a_{(n-39)} - 99439767991128856\ a_{(n-40)} - 85597630328042471\ a_{(n-41)} + 83507383293716287\ a_{(n-42)} + 55942147693581038\ a_{(n-43)} - 63204746282995117\ a_{(n-44)} - 12017304273499040\ a_{(n-45)} + 28043935132260787\ a_{(n-46)} - 4642597797504952\ a_{(n-47)} - 5205634425378807\ a_{(n-48)} + 2469318311108919\ a_{(n-49)} + 71128200728386\ a_{(n-50)} - 274518443415476\ a_{(n-51)} - 12445225264285\ a_{(n-52)} + 28443838816133\ a_{(n-53)} + 17158104144070\ a_{(n-54)} - 11581617961905\ a_{(n-55)} + 68101531730\ a_{(n-56)} + 1377416319025\ a_{(n-57)} - 317506957168\ a_{(n-58)} - 28351204351\ a_{(n-59)} + 15129231663\ a_{(n-60)} + 267751156\ a_{(n-61)} - 269802619\ a_{(n-62)} - 28597095\ a_{(n-63)} + 712820\ a_{(n-64)} + 674723\ a_{(n-65)} + 60461\ a_{(n-66)} + 1255\ a_{(n-67)} - 109\ a_{(n-68)} - a_{(n-69)}$

with initial terms

$$
\begin{aligned}
a_{1} &= 21 \\
a_{2} &= 639 \\
a_{3} &= 27651 \\
a_{4} &= 1111283 \\
a_{5} &= 42972329 \\
a_{6} &= 1642690309 \\
a_{7} &= 62618577481 \\
a_{8} &= 2385542862643 \\
a_{9} &= 90870669971589 \\
a_{10} &= 3461426734215747 \\
a_{11} &= 131852221160935917 \\
a_{12} &= 5022507165784282263 \\
a_{13} &= 191317198008071782069 \\
a_{14} &= 7287650933960562176059 \\
a_{15} &= 277601073047629123048799 \\
a_{16} &= 10574375404737019213183521 \\
a_{17} &= 402798930889412898222764663 \\
a_{18} &= 15343410133313800454710973205 \\
a_{19} &= 584460921912061546176406216583 \\
a_{20} &= 22263275650838141179724286364441 \\
a_{21} &= 848052323281305819580940493471077 \\
a_{22} &= 32303994897145741396178747897157361 \\
a_{23} &= 1230523232665129434509377260190063765 \\
a_{24} &= 46873070372543977015678470469723062507 \\
a_{25} &= 1785488211702293722203272324977040198157 \\
a_{26} &= 68012787060674991334640888919112168789525 \\
a_{27} &= 2590741945784416808967114734931519727303573 \\
a_{28} &= 98686498814686650717886857030406906215771051 \\
a_{29} &= 3759164460261364816554215585561584001394650349 \\
a_{30} &= 143194029669933732611244617603249680145599703043 \\
a_{31} &= 5454544580283735912161638356910086054657935281685 \\
a_{32} &= 207774420811272679883117058934357755586902251031943 \\
a_{33} &= 7914539758187141365741520388400745561700886580737773 \\
a_{34} &= 301480515933299527507104219394948523210083970246744185 \\
a_{35} &= 11483990764388720699981330605006682122231801136651949801 \\
a_{36} &= 437447983888097832374001922579555859976422869453615609965 \\
a_{37} &= 16663261276834317223746000512050367624700669571120400624981 \\
a_{38} &= 634736669516973814969149946740134652016884137842475372944539 \\
a_{39} &= 24178378586045979137527617346975559593733578729349297134150255 \\
a_{40} &= 921002392212561207940091050380382355965078895963537682293975243 \\
a_{41} &= 35082807701208163583823605830130619841396570258129252840215838155 \\
a_{42} &= 1336373723463564705318932322625278275126879887073496655714417649173 \\
a_{43} &= 50905125495482232813373329909756523324341940298555813599931042488905 \\
a_{44} &= 1939077187925153099990429171019836859839176759183154178149893225553435 \\
a_{45} &= 73863295770981189018651789725214718195826537055785717256262619992572369 \\
a_{46} &= 2813599425605762320565327579931843086496793538710561853543438236326246869 \\
a_{47} &= 107175582204107166591155869804691508142039810428803739486301914842943690473 \\
a_{48} &= 4082530482574394769188921140494434924090802478756050440996648040510215418943 \\
a_{49} &= 155511682776848104158231286663114358598976764008568993812749593901671511440057 \\
a_{50} &= 5923748416162950077778085555361842545067354135658183098822727446512670058411931 \\
a_{51} &= 225647325470374390177587141774462933488081384276155250443623937419316440591719437 \\
a_{52} &= 8595354143164958033769875430071977551373544948850818326352719304563287747767620857 \\
a_{53} &= 327414086085070181129691523019667369539498200690771644859462956111646521080014671607 \\
a_{54} &= 12471851884330720478121132740315715430868195270610052787316055812004706560591725639233 \\
a_{55} &= 475077573126370647033981352598673539853584986445727536303852461859488152118201400119931 \\
a_{56} &= 18096646959959769023798753234323860420671708032056863908723287979239392675224544283005215 \\
a_{57} &= 689337172955350526476479687654494676339402626914702610334559061919281883484239668838863793 \\
a_{58} &= 26258220048689684949160097260864360151807771844949536082389164237467511213643259380119638761 \\
a_{59} &= 1000227678378906398923340654186099913119151881590470235869554543862109835699132326030631187233 \\
a_{60} &= 38100655975163133049590275978383264003075500010806917480868833018254795961147197969884072029595 \\
a_{61} &= 1451329549378672662385675833916181378017389120103377467110955859777880890336037320508200139985855 \\
a_{62} &= 55284020891209403878497443968511773211301914734463115392538297319337803274395920766646007454967513 \\
a_{63} &= 2105877998010939266352553782363792616443541386818522961345789417539839930782528650128278874596971129 \\
a_{64} &= 80217069435550361426601286685019137874307296332358019452570817954886391414466837099750661687639717583 \\
a_{65} &= 3055627265637295318869850814043643877231726666381223313086616719855110835683243416854348729411941150549 \\
a_{66} &= 116394902633630408753846704256946658458772773706037419823145129697536217842621938228981702385576463535409 \\
a_{67} &= 4433712681990589165098597013252997656202038379768940613991689208054967821531324696689652654463593468607741 \\
a_{68} &= 168888909236171139648302947954975137370570038534366063387145418951942908030267942228789851064575587551522555 \\
a_{69} &= 6433313502438676095519326515454620192131080807242850217068895750986650257553268258876493483103461354537826393
\end{aligned}
$$

For $n = 7$ the recurrence has 178 terms. For $n = 8$ and beyond, the question is intractable using Mathematica.