Proof: A241008 ∪ A241010 == A174905

Hartmut F. W. Höft 2015-07-01

Definition:

Let s|n and t|n with s < t be a pair of divisors of n

(1) (s, t) is called a *2-pair* if $s < t < 2 \times s$.

(2) (s, t) is called a *minimal 2-pair* if it is a 2-pair and minimal in the lexicographic ordering.
[i.e. $(a, b) <_L (x, y)$ if (i) $a < x$ or (ii) $a = x$ and $b < y$]

Simple computations establish the following two observations:

(1) For every minimal 2-pair (s, t), gcd(s, t) = 1, 1 < s, t is odd and the inequality
$t \le \left\lfloor \left(\sqrt{8 \times n + 1} - 1\right) \big/ 2 \right\rfloor = r_n$ holds.

(2) Let $n \in \mathbb{N}$ such that $n = 2^m \times q$ with $m \ge 0$ and $q \ge 1$ odd. The following two statements are equivalent:
  (a) No pair of divisors s < t of n is a 2-pair.
  (b) Every pair of odd divisors s < t of n satifies $2^{m+1} \times s < t$.

Theorem

A174905 = A241008 ∪ A241010

Proof of A174905 ⊆ A241008 ∪ A241010:

Let n ∈ A174905. The link in A238443 shows a proof that $n = 2^k \in$ A241008, k ≥ 0. Therefore, we may assume that $n = 2^m \times q$ with $m \ge 0$ and $q > 1$ odd. Let $1 = s_0 < s_1 < s_2 < \ldots < s_k \le n$, $k \ge 1$, be all odd divisors of n. By Observation (2), $2^{m+1} \times s_i < s_{i+1}$ holds for all $0 \le i < k$. This implies that the entries of 1's in the n-th row of triangle A237048 alternate between odd and even indices:

$$1 = s_0 < 2^{m+1} < s_1 < 2^{m+1} \times s_1 < s_2 < \ldots < s_i \ (< 2^{m+1} \times s_i) \le r_n,$$

up through an appropriate index i. Therefore, the widths of the regions for the symmetric representation of $\sigma(n)$, n-th row in triangle A249223, alternate between 1 and 0, starting with 1. The number of regions is even if in the n-th row of triangle A237048 the largest index with a 1 has the form $2^{m+1} \times s_i$ so that n ∈ A241008, and odd if its has the form s_i so that n ∈ A241010.

Proof of A241008 ∪ A241010 ⊆ A174905:

We prove the contrapositive; so let n ∉ A174905 with $n = 2^m \times q$, $m \ge 0$, $q > 1$ odd, and let (s, t) be a minimal 2-pair for n. By Observation (1), gcd(s, t) = 1, t is odd, $t \le r_n$, and $1 < s$. If s = 2 then t = 3 and the n-th row of triangle A 237048 starts 1 0 1 … so that the width of the first region of the symmetric representation of $\sigma(n)$ will be 2 for its third leg, that is, n ∉ A241008 ∪ A241010. Therefore, s ≥ 3. Let $1 \le x < y \le s$ be any pair of odd divisors of n. If $y < 2^{m+1} \times x$ then there is $0 \le j \le m$ such that $2^j \times x < y < 2^{j+1} \times x$, i.e. $(2^j \times x, y)$ is a 2-pair, contradicting minimality of (s, t). Let $1 = s_0 < \ldots < s_i < s$, for suitable i ≥ 1, be the odd divisors of n less than s, then:

$$1 = s_0 < 2^{m+1} < s_1 < 2^{m+1} \times s_1 < \ldots < s_i < 2^{m+1} \times s_i < s < t < 2 \times s \qquad \text{if } s \text{ is odd, and}$$

$$1 = s_0 < 2^{m+1} < s_1 < 2^{m+1} \times s_1 < \ldots < s_i < s = 2^k \times u < t < 2 \times s = 2^{k+1} \times u \le 2^{m+1} \times u$$

if $s = 2^k \times u$ with $0 < k \le m$ and $u > 1$ odd. That is, odd and even indices with entries of 1's in the n-th row of triangle A237048 alternate between 1 and 0 starting with 1, until s when it is odd, and until s_i

when s is even. In either case, there are two successive 1's at odd indices s & t and s_i & t, respectively, since even indices with entries of 1 have the form $2^{m+1} \times x$ for some odd x. That means that the entry at index t in the n-th row of triangle A249223 will have a value of 2, so that $n \notin$ A241008 $\cup$ A241010.