Derivation of explicit formulas

a(n): Number of ways of making change for n cents using coins of $1=c(1) < c(2) < \ldots < c(k)$ cents.

General recurrence (explanation in A187243):
$f(n,1) = 1$
$1 < j \le k$: $f(0,j) = 1$ or, for $n>0$:
(a) $f(n,j) = \sum_{m=0}^{n \text{ div } c(j)} f(n - m * c(j), j-1)$
(b) $\quad = \sum_{m=0}^{n \text{ div } c(j)} f(n \bmod c(j) + m * c(j), j-1)$, transformation m→n div c(j) - m
$a(n) = f(n,k)$
Integer division: a div b = (a – a mod b)/b = floor(a/b).

A187243 , c(1) = 1, c(2) = 5, c(3) =10:

$a(n) = (q+1)\cdot(q+1+s)$ with q= n div 10 and s = (n mod 10) div 5

Derivation:
Let q = n div 10 and r = n mod 10, i.e. $n = 10q + r$ with $0 \le r \le 9$.
$f(n,3) = \sum_{m=0}^{q} f(r+10m, 2)$, see recurrence (b)
$\quad = \sum_{m=0}^{q} f(r' + 5 * (2m+s), 2)$ with s = r div 5 and r' = r mod 5

$f(5\cdot(r' + 2m + s), 2) = \sum_{\mu=0}^{2m+s} f(r' + 5\mu + s, 1) = \sum_{\mu=0}^{2m+s} 1 = 2m + 1 + s$
$a(n) = f(n,3) = \sum_{m=0}^{q} (2m+1+s) = q(q+1) + (q+1)(1+s) = (q+1)(q+1+s)$, q.e.d.
Another formula:
$a(n) = f(n,3) = floor\left(\frac{(x+10)^2}{100}\right)$ with x = n – n mod 5

A001299, c(1) = 1, c(2) = 5, c(3) =10, c(4) = 25

$a(n) = round((100x^3 + 135 x^2 + 53x)/6) + 1$ with x= floor(n/5)/10

Derivation:
1) n = 50x (note that 50 = LCM(5,10,25)):
$f(n,3) = (q+1)\cdot(q+1+s)$ with q= n div 10 and s = (n mod 10) div 5
$f(n,4) = \sum_{m=0}^{n \text{ div } 25} f(25m, 3)$
a) $m=2\mu$, $f(50\mu) = (5\mu+1)^2$
b) $m=2\mu+1$, $f(25+50\mu) = (5\mu+3)(5\mu+4)$
$f(n,4) = \sum_{\mu=0}^{x} f(50\mu, 3) + \sum_{\mu=0}^{x-1} f(25+50\mu, 3)$
$= \sum_{\mu=0}^{x} (25\mu^2 + 10\mu + 1) + \sum_{\mu=0}^{x-1} (25\mu^2 + 35\mu + 12)$
$= \sum_{\mu=0}^{x-1} (50\mu^2 + 45\mu + 13) + 25x^2 + 10x + 1$
$= \frac{50}{6} x(x-1)(2x-1) + \frac{45}{2} x(x-1) + 13x + 25x^2 + 10x + 1$
$= \frac{1}{6}(100x^3 + 135x^2 + 53x + 6)$ (1)
2) Other cases:
Obviously: $a(n_1) = a(n_2)$ when n_1 div 5 = n_2 div 5 because GCD(5,10,25) = 5.

Or let n=5y+z with z= n mod 5. When the change for 5y cents is done the change for the remaining z cents is fixed.

As a special result, formula (1) holds for $n = 50x + r$ with $0 \le r \le 4$:

$a(n) = \frac{1}{6}(100x^3 + 135x^2 + 53x + 6)$ with x= floor(n/5)/10 instead of x=n/50 (2)

3) Remaining cases:

$n = 50x + 5y,\ 1 \le y \le 9$

The analysis of these cases is straightforward but arduous. The deviations from formula (2) are small and can be compensated by the round-function.

A000008 , $c(1) = 1, c(2) = 2, c(3) = 5, c(4) = 10$.

$$a(n) = (q+1)\left(round\left(\frac{(n+4)^2}{20}\right) - \frac{1}{6}q(3n - 10q + 7)\right)$$

$f(n, 3) = round\left(\frac{(n+4)^2}{20}\right)$, see A000115

$f(n, 4) = \sum_{m=0}^{q} f(n - 10m, 3)$, q= n div 10 , see recurrence (a)

$$f(n-10m, 3) = round(\frac{(n+4-10m)^2}{20}) = f(n, 3) - (n+4)m + 5m^2$$

$f(n, 4) = (q+1)\cdot f(n, 3) - \frac{n+4}{2}q(q+1) + \frac{5}{6}q(q+1)(2q+1)$

$f(n, 4) = (q+1)\left(round\left(\frac{(n+4)^2}{20}\right) - \frac{1}{6}q(3n - 10q + 7)\right)$