Karlsruhe
April 08 2011

A187360. Minimal Polynomials of $2\cos\left(\frac{\pi}{n}\right)$

Wolfdieter L a n g [1]

The minimal polynomial of an algebraic number α of degree d_α is the monic, minimal degree rational polynomial which has as root, or as one of its roots, α. This minimal degree d_α is 1 iff α is rational, and the minimal polynomial in this case is $p(x) = x - \alpha$. For the notion 'minimal polynomial of an algebraic number' see, *e.g.*, [2], p. 28.

For the algebraic number $2\cos\left(\frac{\pi}{n}\right)$, for $n \in \mathbb{N}$, the degree (called here $\delta(n)$) is $\delta(1) = 1$, and $\delta(n) = \frac{\varphi(2\,n)}{2}$, with *Euler*'s totient function $\varphi(n) =$ A000010(n). This is the sequence A055034. These polynomials can be obtained from the ones of $\cos\left(\frac{2\,\pi}{n}\right)$ which are found, *e.g.*, under A181875/ A181876, and they have been called there $\Psi(n, x)$. See also [1], and [2], Theorem 3.9, p. 37, for the degree of these polynomials, from which the $\delta(n)$ formula above follows. From the trivial identity $\cos\left(\frac{\pi}{n}\right) = \cos\left(\frac{2\,\pi}{2\,n}\right)$ one finds the minimal polynomial of $2\cos\left(\frac{\pi}{n}\right)$, called here $C(n, x)$, from

$$C(n, x) = 2^{d(2\,n)}\,\Psi\left(2\,n, \frac{x}{2}\right)\ . \tag{1}$$

It turns out that these polynomials are in fact integer (not only rational) polynomials. The interest in these polynomials derives from the fact that $\rho(n) := 2\,cos\left(\frac{\pi}{n}\right)$ is the ratio of lengths between the largest diagonal and the side of the regular $n-$gon. The side length of a regular $n-$gon inscribed in the unit circle is $s(n) := 2\sin\left(\frac{\pi}{n}\right)$. See the W. Lang paper "The field $\mathbb{Q}(2\cos\left(\frac{\pi}{n}\right))$ and length ratios in the regular n-gon" given in a link under A187360.

References

[1] D. H. Lehmer, A Note on Trigonometric Algebraic Numbers, Am. Math. Monthly 40,8 (1933) 165-6.

[2] I. Niven, *Irrational Numbers*, The Math. Assoc, of America, second printing, distributed by John Wiley and Sons, 1963.

[3] William Watkins and Joel Zeitlin, The Minimal Polynomial of $\cos(2\pi/n)$, Am. Math. Monthly 100,5 (1993) 471-4.

---

[1] wolfdieter.lang@kit.edu, http://www-itp.particle.uni-karlsruhe.de/~wl

Table : Minimal polynomials of $2\cos\left(\frac{\pi}{n}\right)$ for $n = 1, 2, ..., 30$.

n	$C(n, x)$
1	$x+2$
2	x
3	$x-1$
4	x^2-2
5	x^2-x-1
6	x^2-3
7	$x^3-x^2-2\,x+1$
8	$x^4-4\,x^2+2$
9	x^3-3x-1
10	$x^4-5\,x^2+5$
11	$x^5-x^4-4\,x^3+3\,x^2+3\,x-1$
12	$x^4-4\,x^2+1$
13	$x^6-x^5-5\,x^4+4\,x^3+6\,x^2-3\,x-1$
14	$x^6-7\,x^4+14\,x^2-7$
15	$x^4+x^3-4\,x^2-4\,x+1$
16	$x^8-8\,x^6+20\,x^4-16\,x^2+2$
17	$x^8-x^7-7\,x^6+6\,x^5+15\,x^4-10\,x^3-10\,x^2+4\,x+1$
18	$x^6-6\,x^4+9\,x^2-3$
19	$x^9-x^8-8\,x^7+7\,x^6+21\,x^5-15\,x^4-20\,x^3+10\,x^2+5\,x-1$
20	$x^8-8\,x^6+19\,x^4-12\,x^2+1$
21	$x^6+x^5-6\,x^4-6\,x^3+8\,x^2+8\,x+1$
22	$x^{10}-11\,x^8+44\,x^6-77\,x^4+55\,x^2-11$
23	$x^{11}-x^{10}-10\,x^9+9\,x^8+36\,x^7-28\,x^6-56\,x^5+35\,x^4+$ $35\,x^3-15\,x^2-6\,x+1,$
24	$x^8-8\,x^6+20\,x^4-16\,x^2+1$
25	$x^{10}-10\,x^8+35\,x^6-x^5-50\,x^4+5\,x^3+25\,x^2-5\,x-1$
26	$x^{12}-13\,x^{10}+65\,x^8-156\,x^6+182\,x^4-91\,x^2+13$
27	$x^9-9\,x^7+27\,x^5-30\,x^3+9\,x-1$
28	$x^{12}-12\,x^{10}+53\,x^8-104\,x^6+86\,x^4-24\,x^2+1$
29	$x^{14}-x^{13}-13\,x^{12}+12\,x^{11}+66\,x^{10}-55\,x^9-165\,x^8+$ $120\,x^7+210\,x^6-126\,x^5-126\,x^4+56\,x^3+28\,x^2-7\,x-1$
30	$x^8-7\,x^6+14\,x^4-8\,x^2+1$
⋮	

Table 2: A187360(n, m) coefficient array of minimal polynomials of $2\cos\left(\frac{\pi}{n}\right)$, rising powers

n/m	0	1	2	3	4	5	6	...
1	2	1						
2	0	1						
3	-1	1						
4	-2	0	1					
5	-1	-1	1					
6	-3	0	1					
7	1	-2	-1	1				
8	2	0	-4	0	1			
9	-1	-3	0	1				
10	5	0	-5	0	1			
11	-1	3	3	-4	-1	1		
12	1	0	-4	0	1			
13	-1	-3	6	4	-5	-1	1	
14	-7	0	14	0	-7	0	1	
15	1	-4	-4	1	1			
⋮								