Multiples and Divisors

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Before discussing multiplication, let us speak about addition. The number $A(k)$ of distinct sums $i+j\leq k$ such that $1\leq i\leq k/2$, $1\leq j\leq k/2$ is clearly $2\lfloor k/2\rfloor-1$. Hence the number $A(2n)$ of distinct elements in the $n\times n$ addition table involving $\{1,2,\ldots,n\}$ satisfies $\lim_{n\to\infty}A(2n)/n=2$, as expected.

We turn to multiplication. Let $M(k)$ be the number of distinct products $ij\leq k$ such that $1\leq i\leq\sqrt{k}$, $1\leq j\leq\sqrt{k}$. One might expect that the number $M(n^2)$ of distinct elements in the $n\times n$ multiplication table to be approximately $n^2/2$; for example, $M(10^2)=42$. In a surprising result, Erdös [1, 2, 3] proved that $\lim_{n\to\infty}M(n^2)/n^2=0$. More precisely, we have [4]

$$\lim_{k\to\infty}\frac{\ln(M(k)/k)}{\ln(\ln(k))}=-\delta$$

where

$$\delta=1-\frac{1+\ln(\ln(2))}{\ln(2)}=0.0860713320...$$

In spite of good estimates for $M(k)$, an asymptotic formula for $M(k)$ as $k\to\infty$ remains unknown [5].

Given a positive integer n, define

$$\rho_1(n)=\max_{\substack{d|n,\\ d\leq\sqrt{n}}}d,\qquad \rho_2(n)=\min_{\substack{d|n,\\ d\geq\sqrt{n}}}d;$$

thus $\rho_1(n)$ and $\rho_2(n)$ are the two divisors of n closest to $\sqrt{n}$. Let

$$R_1(N)=\sum_{n=1}^{N}\rho_1(n),\qquad R_2(N)=\sum_{n=1}^{N}\rho_2(n).$$

It is not difficult to prove that

$$\lim_{N\to\infty}\frac{\ln(N)}{N^2}R_2(N)=\frac{\pi^2}{12}.$$

An analogous asymptotic expression for $R_1(N)$ is still open, but Tenenbaum [6, 7, 8] proved that

$$\lim_{N\to\infty}\frac{\ln(R_1(N)/N^{3/2})}{\ln(\ln(N))}=-\delta$$

where δ is exactly as before. It is curious that one limit is so much harder than the other, and that the same constant δ appears as with the multiplication table problem.

Erdös conjectured long ago that almost all integers n have two divisors d, d' such that $d < d' \leq 2d$. By "almost all", we mean all integers n in a sequence of asymptotic density 1, abbreviated as "p.p." Given n, select divisors $a_n < b_n$ for which b_n/a_n is minimal. To prove the conjecture, it is sufficient to show that $b_n/a_n \to 1^+$ as $n \to \infty$ p.p.; that is, $\ln(\ln(b_n/a_n)) \to -\infty$ p.p. Maier & Tenenbaum [9, 10, 11] succeeded in doing this and, further, demonstrated that

$$\lim_{n\to\infty}\frac{\ln(\ln(b_n/a_n))}{\ln(\ln(n))}=-(\ln(3)-1)=-0.0986122886\ldots\text{ p.p.}$$

Another way of viewing this problem is by counting those integers n up to N without such divisors d and d'. If $\varepsilon(N)$ is the number of these exceptional integers, then [4]

$$\lim_{N\to\infty}\frac{\ln(\varepsilon(N)/N)}{\ln(\ln(\ln(N)))}\leq-\beta$$

where

$$\beta=1-\frac{1+\ln(\ln(3))}{\ln(3)}=0.0041547514\ldots.$$

As the inequality suggests, we don't know if this constant is necessarily optimal.

Yet another way of viewing this problem is via the Hooley function

$$\Delta(n)=\max_{x\geq0}\sum_{\substack{d|n,\\ e^x<d\leq e^{x+1}}}1,$$

that is, the greatest number of divisors of n contained in any interval of logarithmic length 1. More interesting constants emerge here, but their optimality is questionable. In fact, it is conjectured [4] that $\Delta(n)/\ln(\ln(n))$ accumulates not at a single point, but over an entire subinterval $(u, v) \subseteq (0, \infty)$. Estimates of u and v would be good to see someday.

Ramanujan [12] studied the asymptotics of $\sum_{n=1}^{N} 1/d(n)$ as $N \to \infty$, where [13] $d(n)$ is the number of distinct divisors of n. See [14] for more details. This is a special case of a result in [4, 15], which is used to prove the following arcsine distributional law for random divisors d of n:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{P}\left(\frac{\ln(d)}{\ln(n)}<x\right)=\frac{2}{\pi}\arcsin\left(\sqrt{x}\right).$$

Consequently, an integer has (on average) many small divisors and many large divisors.

Sita Ramaiah & Suryanarayana [16] found a corresponding formula for $\sum_{n=1}^{N} 1/\sigma(n)$, where [17] $\sigma(n)$ is the sum of all divisors of n. DeKoninck & Ivić [18] had asserted that constants appearing in such a formula would be complicated; they were right! [14] It turns out that the Riemann hypothesis [19] is true if and only if [20, 21]

$$\sigma(n) < e^{\gamma} n \ln(\ln(n)) \qquad \text{for all sufficiently large } n,$$

where γ is the Euler-Mascheroni constant [22].

An integer n is **highly composite** if $d(m) < d(n)$ for all $m < n$. Let $Q(N)$ denote the number of highly composite integers $n \leq N$. It is known that [11, 23, 24, 25, 26]

$$1.136 \leq \liminf_{N\to\infty} \frac{\ln(Q(N))}{\ln(N)} \leq 1.44, \qquad \limsup_{N\to\infty} \frac{\ln(Q(N))}{\ln(N)} \leq 1.71,$$

based on Diophantine approximations of the quantity $\ln(3/2)/\ln(2) = 0.5849625007....$ It is conjectured that the limit exists and

$$\lim_{N\to\infty} \frac{\ln(Q(N))}{\ln(N)} = \frac{\ln(2)+\ln(3)+\ln(5)}{4\ln(2)} = 1.2267226489....$$

but this appears to be difficult.

Let us return to the constant δ, which appears in several other places in the literature [27, 28, 29, 30, 31, 32, 33]. We mention only three. With regard to Erdös' conjecture, Roesler [34] added a further constraint that $a_n b_n = n$ when minimizing b_n/a_n; he proved that

$$\lim_{N\to\infty} \frac{\ln\left(\frac{1}{N}\sum_{n=1}^{N} \frac{a_n}{b_n}\right)}{\ln(\ln(N))} = -\delta.$$

Hence the integers are fairly quadratic, in the sense that $b_n - a_n$ is quite small on average. We wonder what happens to the limiting ratio if a_n/b_n is replaced in the summation by b_n/a_n.

An odd prime p is said to be **symmetric** [35, 36] if there exists an odd prime q such that $|p-q| = \gcd(p-1, q-1)$. For example, any twin prime is symmetric. It is known that the reciprocal sum of symmetric primes is finite (like Brun's constant [37]). If the twin prime conjecture is true, then there are infinitely many symmetric primes. Let $S(n)$ denote the number of symmetric primes $\leq n$. It is conjectured that

$$\lim_{n\to\infty} \frac{\ln(S(n)/n)}{\ln(\ln(n))} = -1-\delta$$

and a heuristic argument supporting this formula appears in [35].

Finally, let $T(N)$ denote the number of integers $n \leq N$ satisfying the inequality $d(n) \geq \ln(N)$. Norton [38], responding to a question raised by Steinig, proved that there are positive constants $\xi < \eta$ with

$$\xi \leq \rho(N) = \frac{T(N)}{N\ln(N)^{-\delta}\ln(\ln(N))^{-1/2}} \leq \eta$$

for all large N. Balazard, Nicolas, Pomerance & Tenenbaum [39] proved that the ratio $\rho(N)$ does not tend to a limit as $N \to \infty$, and that

$$\rho(N) \sim f\left(\tfrac{\ln(\ln(N))}{\ln(2)}\right) \qquad \text{as } N \to \infty$$

where $f(x)$ is an explicit left-continuous function of period 1 with only countably many jump discontinuities. Deléglise & Nicolas [40] further computed that

$$\xi = \lim_{x \to 0^+} f(x) = 0.9382786811..., \qquad \eta = f(0) = 1.1481267734...$$

are the best possible asymptotic bounds on $\rho(N)$. We have seen such oscillatory functions on numerous occasions elsewhere in number theory and combinatorics [41, 42]. The quantities

$$\chi = \frac{1}{\Gamma(1+\lambda)} \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{\lambda} \left(1 + \frac{\lambda}{p}\right) = 0.3495143728...$$

$$\frac{\chi}{1-\ln(2)}\sqrt{\frac{\ln(2)}{2\pi}} = 0.3783186209... = \frac{\xi}{2.4801282017...} = \frac{\eta}{3.0348143331...}$$

also play an intermediate role [40], where $\lambda = \ln(2)^{-1}$.

In a late-breaking development, Ford [43] proved that there exist positive constants $c < C$ such that

$$c\frac{N}{\ln(N)^{\delta}\ln(\ln(N))^{3/2}} \leq M(N) \leq C\frac{N}{\ln(N)^{\delta}\ln(\ln(N))^{3/2}}$$

for large N, and positive constants $c' < C'$ such that

$$c'\frac{N^{3/2}}{\ln(N)^{\delta}\ln(\ln(N))^{3/2}} \leq R_1(N) \leq C'\frac{N^{3/2}}{\ln(N)^{\delta}\ln(\ln(N))^{3/2}}.$$

for large N. Thus, for the first time, the true order of magnitude of $M(N)$ and of $R_1(N)$ is known. See also [44] for an application to computer science.

0.1. Addendum. A famous result is [45, 46, 47, 48, 49]

$$\limsup_{n\to\infty} \ln(d(n))\frac{\ln\ln n}{\ln n} = \ln 2$$

but the analogous result for the iterated divisor

$$\begin{aligned}\limsup_{n\to\infty} \ln(d(d(n)))\frac{\ln\ln n}{\sqrt{\ln n}} &= \left(8\sum_{j=1}^{\infty}\ln\left(1+\frac{1}{j}\right)^2\right)^{1/2} \\ &= 2.7959802335...\end{aligned}$$

was proved only recently [50].

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