The Power of Permution

Basic Algorithms and Arithmetics of Permutations
Offer Easy Access into a Fascinating Realm

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Introduction

Addition, subtraction, multiplication and division are operations of arithmetic that most students master at an early age. The algorithms presented here are equally accessible. For all their simplicity, they reveal intriguing properties of permutations and produce interesting patterns that seem hitherto unexplored.

Permutations of 1,2,3… *n*

The set of integers from 1 to n may (by serial transpositions of adjacent terms) be permutated in n! ways. Thus, e.g., for $n = 4$, $4! = 24$ permutations (p) such as 4321, 2341 and so on are possible. (Note that for $n < 10$, it is convenient to represent p sans spaces and/or commas.) The set 1 to n in counting order is called the *identity* permutation (I). Then n is the *length* of I and all permutations thereof. The length n may also be represented as l_n.

An operation to be utilized here is called (to coin a term) *permution*. It is represented by the symbol ○. For p of equal length, the statement $p \circ p' = p''$ means that p' operates on p to generate p''. The operator ○ is designed so that the transposition sequence that turned I into p' now applies to p to turn it into p''. Some examples below illustrate how this happens.

Let, for instance, $p = 4321$ and $p' = 2341$. Parsing 2341, we see that it is simply $I = 1234$ cycled leftward one term. Thus, by ○, the effect of p' on p is simply to cycle it one term leftward as well: i.e., $4321 \circ 2341 = 3214$. Figure 1 below shows this process step by step.

$$\begin{array}{cllll} & 4 & 3 & 2 & 1 \\ \circ & 2 & 3 & 4 & 1 \\ \hline & 3 & & & \end{array} \qquad \begin{array}{cllll} & 4 & 3 & 2 & 1 \\ \circ & 2 & 3 & 4 & 1 \\ \hline & 3 & 2 & & \end{array} \qquad \begin{array}{cllll} & 4 & 3 & 2 & 1 \\ \circ & 2 & 3 & 4 & 1 \\ \hline & 3 & 2 & 1 & \end{array} \qquad \begin{array}{cllll} & 4 & 3 & 2 & 1 \\ \circ & 2 & 3 & 4 & 1 \\ \hline & 3 & 2 & 1 & 4 \end{array}$$

Figure 1: The four steps of the ○ operation on two l_4 numerical permutations

The first step in the permution process locates the first number on the second line in figure 1. The number in this example is 2, so we count over two digits at the top line to find the number in that space. It's the number 3, a number that is then placed in the third tier of the first column. Steps 2, 3 and 4 continue in this way…

Now switch terms so $p = 2341$ and $p' = 4321$. As 4321 is 1234 in reverse order, its effect is to reverse the order of 2341. (Since $4321 \circ 2341 \neq 2341 \circ 4321$, these p do not *commute*.)

$$\begin{array}{cllll} & 2 & 3 & 4 & 1 \\ \circ & 4 & 3 & 2 & 1 \\ \hline & 1 & & & \end{array} \qquad \begin{array}{cllll} & 2 & 3 & 4 & 1 \\ \circ & 4 & 3 & 2 & 1 \\ \hline & 1 & 4 & & \end{array} \qquad \begin{array}{cllll} & 2 & 3 & 4 & 1 \\ \circ & 4 & 3 & 2 & 1 \\ \hline & 1 & 4 & 3 & \end{array} \qquad \begin{array}{cllll} & 2 & 3 & 4 & 1 \\ \circ & 4 & 3 & 2 & 1 \\ \hline & 1 & 4 & 3 & 2 \end{array}$$

Figure 2: $p' = 4321$ is I reversed; hence it reverses the order of the terms in 2341

Where $I = 1,2,3\ldots n$ is the identity permutation, the two examples in figure 3 suffice to show that $I \circ p = p \circ I = p$ for all p and I of length n.

$$\begin{array}{cccc} & 2 & 1 & 4 & 3 \\ \circ & 1 & 2 & 3 & 4 \\ \hline & 2 & 1 & 4 & 3 \end{array} \qquad \begin{array}{cccc} & 1 & 2 & 3 & 4 \\ \circ & 2 & 1 & 4 & 3 \\ \hline & 2 & 1 & 4 & 3 \end{array}$$

Figure 3: For all p and I of equal length, $I \circ p = p \circ I = p$

Powers of *p*

Every p is a k^{th} root of identity. The superscripts called *exponents* in common algebra can serve a function here very similar to their more familiar role. Consistent with that, k is called p's *degree*.

Definition: The degree of p is the smallest $k > 0$ for which $p^k = p^0$.

The degree of p is an integer k indicating how many times p must permute itself until it becomes the identity, $p^0 = I$. As for other powers of p (always integral), it is sufficient for this treatise that a few basic laws of exponents apply. For example, $p^1 \circ p^{-1} = p^{-1} \circ p^1 = p^0$ as usual, where for all p of length n, $p^0 = I = 1,2,3\ldots n$. Then $p^1 = p$, $p \circ p = p^2$, $p \circ p^2 = p^3$ and so on.

A process that uses ○ to derive the degree of p is shown in figure 4 below. In the example, 43512 is permuted by itself until 12345 appears. Figure 4 shows that, in this case, $p^6 = p^0$.

$$\begin{array}{cccccc l} & 4 & 3 & 5 & 1 & 2 & \quad k = 1 \\ \circ & 4 & 3 & 5 & 1 & 2 & \\ \hline = & 1 & 5 & 2 & 4 & 3 & \quad k = 2 \\ \circ & 4 & 3 & 5 & 1 & 2 & \\ \hline = & 4 & 2 & 3 & 1 & 5 & \quad k = 3 \\ \circ & 4 & 3 & 5 & 1 & 2 & \\ \hline = & 1 & 3 & 5 & 4 & 2 & \quad k = 4 \\ \circ & 4 & 3 & 5 & 1 & 2 & \\ \hline = & 4 & 5 & 2 & 1 & 3 & \quad k = 5 \\ \circ & 4 & 3 & 5 & 1 & 2 & \\ \hline = & 1 & 2 & 3 & 4 & 5 & \quad k = 6 \end{array}$$

Figure 4: An algorithm that determines the degree of p

Figure 4 shows that 43512 is a 6^{th} root of identity; i.e., $43512^6 = 12345$. Hence, $k = 6$ is this p's degree. This particular 'degree of p' algorithm is easy to code, but inefficient, and as p and k grow larger it's too tedious to perform by hand. Since k of p is often useful information, a more direct way to derive it is needed. Internal cycles are the key, and a method for factoring permutations into cycles is now described. Our approach to identifying these cycles begins with a derivation of p^{-1}, the *inverse* of p.

Definition: Every $p = p^1$ has an inverse, p^{-1}, such that $p^1 \circ p^{-1} = p^{-1} \circ p^1 = p^0 = I$.

Given p, the algorithm below finds p^{-1}. An l_{12} example illustrates the process in figure 5:

7	9	11	4	8	3	2	5	1	12	6	10
9											

Figure 5: The first step in an algorithm that finds p^{-1}

A permutation of length n is entered at the top of a two-row, n-columns table; the lower line is empty. Begin at the first (leftmost) cell on the upper line and count over to the cell that holds the digit 1. Since in this example it is nine places over, the digit 9 is entered below the 7 in the first column. Next, count again from the first cell to find the digit 2. It is in the seventh column and so, as seen in figure 6 below, a 7 is entered under the 9 in the second column. To continue in this fashion generates the inverse permutation on the second line of the table in figure 6.

	7	9	11	4	8	3	2	5	1	12	6	10
○	9	7	6	4	8	11	1	5	2	12	3	10
	1	2	3	4	5	6	7	8	9	10	11	12

Figure 6: An l_{12} permutation permuted by its inverse

Finding p's inverse can, as in figure 6, help to reveal cycles therein. E.g., we see at once that 4 is a fixed term. Then double entries over ***I***-numbers 5,8 and 10,12 identify transposition pairs. The trio of 3,11,6 composes a 3-cycle, and the 4-cycle 1,7,2,9 accounts for the remaining terms.

The Degree of *p* as the LCM of its Cycles

Permuting p by its inverse helps to make the cycles stand out in this figure 6 example, but these short and easily discernable cycles make this a special case. A more general method for factoring p into cycles (a disjoint cycle notation) is presented in figures 7 and 8 below.

4	11	5	7	2	9	6	10	1	12	8	3	= (1,4,7,6,9)
1	2	3	4	5	6	7	8	9	10	11	12	

Figure 7: The first part of a cycle-extraction algorithm

The permutation to be factored is entered on the top row of an n column, 2-row table with I of length n on the row below. Terms of I count successive columns, showing the example in figure 7 to be of length 12. The first cycle to be identified always starts with 1. The number above 1 is 4, so 1,4 signifies that the cycle starts with a 4 in the first cell. A move to 4 on the I row finds 7 above it; then 6 above 7; 9 above 6 and 1 above 9. Thus the first cycle is 1,4,7,6,9.

4	11	5	7	2	9	6	10	1	12	8	3	= (1,4,7,6,9)(2,11,8,10,12,3,5)
1	2	3	4	5	6	7	8	9	10	11	12	

Figure 8: An l_{12} permutation factored into its cycles

Then the smallest number that does not appear in the first cycle initiates the second, and so on until all numbers are accounted for and the cycles closed. The l_{12} permutation in figure 8 factors into cycles (1,4,7,6,9) and (2,11,8,10,12,3,5). Its degree k is the lowest common multiple (*lcm*) of these cycles, and thus $k = 5 \cdot 7 = 35$. Another approach is to start with cycles to generate (within constraints of compatibility) a p of chosen length and degree, as seen in figure 9 below:

$$p = (1,7,8,5,2)(3,11,9)(4,10,6,12)$$

Figure 9: p with cycles 5, 3, 4 and $k = 60$ 'reconstitutes' to 7,1,11,10,2,12,8,5,3,6,9,4

Note that this notation makes it easy to find inverses. E.g.,

if $p = (1,4,7,6,9)(2,11,8,10,12,3,5)$
then $p^{-1} = (1,9,6,7,4)(2,5,3,12,10,8,11)$

if $p = (1,7,8,5,2)(3,11,9)(4,10,6,12)$
$p^{-1} = (1,2,5,8,7)(3,9,11)(4,12,6,10)$

Pisano Periods and Related Cycles

Define F_n as the n^{th} term of the *Fibonacci* (φ) *sequence*, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$. With this definition, we can put the various concepts and tools we've just developed into play. A recursive procedure generates a series of permutations with F_n as exponents.

$p^0 \circ p^1 = p^1$, $p^1 \circ p^1 = p^2$, $p^1 \circ p^2 = p^3$, $p^2 \circ p^3 = p^5$, $p^3 \circ p^5 = p^8$, $p^5 \circ p^8 = p^{13}$, $p^8 \circ p^{13} = p^{21}$...

In figure 10 below, entries for p^0 are 1,2; 1,2,3; 1,2,3…*j*; that is, successive identity elements of length *j*, as highlighted in blue. Entries for p^1 are 2,1; 2,3,1; 2,3,4…*j*,1; i.e., terms of *I* are shifted one place to the left to create a cyclical *p* of length *j*. Then $p^0 \circ p^1 = p^1$ kicks off the recursion:

n =	F_n	*j* =	2		3		4		5		6		7		8		9	
0	0		12	**1**	123	**1**	1234	**1**	12345	**1**	123456	**1**	1234567	**1**	12345678	**1**	123456789	**1**
1	1		21	2	231	**3**	2341	4	23451	**5**	234561	6	2345671	**7**	23456781	8	234567891	**9**
2	1		21	2	231	**3**	2341	4	23451	**5**	234561	6	2345671	**7**	23456781	8	234567891	**9**
3	2		12	**1**	312	**3**	3412	**2**	34512	**5**	345612	**3**	3456712	**7**	34567812	**4**	345678912	**9**
4	3				123	**1**	4123	4	45123	**5**	456123	2	4567123	**7**	45678123	8	456789123	**3**
5	5				312	**3**	2341	4	12345	**1**	612345	6	6712345	**7**	67812345	8	678912345	**9**
6	8				312	**3**	1234	**1**	45123	**5**	345612	**3**	2345671	**7**	12345678	**1**	912345678	**9**
7	13				231	**3**			45123	**5**	234561	6	7123456	**7**	67812345	8	567891234	**9**
8	21				123	**1**			23451	**5**	456123	2	1234567	**1**	67812345	8	456789123	**3**
9	34								51234	**5**	561234	**3**	7123456	**7**	34567812	**4**	891234567	**9**
10	55								12345	**1**	234561	6	7123456	**7**	81234567	8	234567891	**9**
11	89								51234	**5**	612345	6	6712345	**7**	23456781	8	912345678	**9**
12	144								51234	**5**	123456	**1**	5671234	**7**	12345678	**1**	123456789	**1**
13	233								45123	**5**	612345	6	3456712	**7**			912345678	**9**
14	377								34512	**5**	612345	6	7123456	**7**			912345678	**9**
15	610								12345	**1**	561234	**3**	2345671	**7**			891234567	**9**
16	987								34512	**5**	456123	2	1234567	**1**			789123456	**3**
17	1597								34512	**5**	234561	6					567891234	**9**
18	2584								51234	**5**	561234	**3**					234567891	**9**
19	4181								23451	**5**	612345	6					678912345	**9**
20	6765								12345	**1**	456123	2					789123456	**3**
21	10946										345612	**3**					345678912	**9**
22	17711										612345	6					912345678	**9**
23	28657										234561	6					234567891	**9**
24	46368										123456	**1**					123456789	**1**

Figure 10: The Pisano periods generated by cyclic *p* to powers F_n

Each number *j* across the top of the table in figure 10 heads a pair of adjacent columns: leftmost is the column of *p*s created by iteration; right-hand column numbers then give a *p*'s degree. Note of the p^1 in row #1 of the table that length = degree. Thus for length *j*, $p^{F_n} = p^0$ iff $F_n = 0 \pmod{j}$. I.e., when F_n is a multiple of the length, then the n^{th} term in the *p* sequence = *I*. For $a = 1,2,3\ldots$, F_{an} is a multiple of F_n and so there are regular cycles. E.g., let $n = 5$; then $F_5 = 5$, $F_{10} = 55$, $F_{15} = 610$ and so on. Thence, as seen in column $j = 5$, every fifth $p = I$. A period recurs when F_{an} is followed by an F_{an+1} such that $F_{an+1} = 1 \pmod{j}$, so that $p^{F_{an+1}} = 2,3,4\ldots j,1$.

(Numbers in the *j* columns in the tables are in bold and regular type. This indicates *p*'s *parity*, which is based on a count of the number (t) of transpositions required to return any *p* to *I*. E.g., 213 to 123 has $t = 1$; 2143 to 1234 has $t = 2$. An even *t* is indicated by bold type, regular means *t* is odd. This will be elaborated later.)

The full periods (including $j = 1$) 1, 3, 8, 6, 20, 24, 16, 12, 24… are in the Online Encyclopedia of Integer Sequences as the *Pisano periods* (OEIS [A001175](A001175)). These are the periods of Fibonacci sequence numbers mod n, $n = 1,2,3…$. Lengths of the *Pisano cycles* within the periods are listed as 1, 3, 4, 6, 5, 12, 8, 6, 12… (OEIS [A001177](A001177)). Cycle patterns for $n = 1…10$ are listed on Wikipedia's Pisano periods page; e.g., the cycles of $n = 5$ are 01123 03314 04432 02241. To recreate such cycles from the table in figure 10, take the first digit of the successive p in a column (for $j = 5$; 12234 14425…) and decrement each term by unity (1).

To continue this present exploration, the initial value in the φ-sequence is increased from 0 to 2. This puts *Lucas numbers* in the exponent column, as seen in figure 11 below.

n =	L_n	j = 2		3		4		5		6		7		8		9	
0	2	12	1	312	3	3412	2	34512	5	345612	3	3456712	7	34567812	4	345678912	9
1	1	21	2	231	3	2341	4	23451	5	234561	6	2345671	7	23456781	8	234567891	9
2	3	21	2	123	1	4123	4	45123	5	456123	2	4567123	7	45678123	8	456789123	3
3	4	12	1	231	3	1234	1	51234	5	561234	3	5671234	7	56781234	2	567891234	9
4	7			231	3	4123	4	34512	5	234561	6	1234567	1	81234567	8	891234567	9
5	11			312	3	4123	4			612345	6	5671234	7	45678123	8	345678912	9
6	18			123	1	3412	2			123456	1	5671234	7	34567812	4	123456789	1
7	29			312	3					612345	6	2345671	7	67812345	8	345678912	9
8	47			312	3					612345	6	6712345	7	81234567	8	345678912	9
9	76									561234	3	7123456	7	56781234	2	567891234	9
10	123									456123	2	5671234	7	45678123	8	789123456	3
11	199									234561	6	4567123	7	81234567	8	234567891	9
12	322									561234	3	1234567	1	34567812	4	891234567	9
13	521									612345	6	4567123	7			912345678	9
14	843									456123	2	4567123	7			789123456	3
15	1364									345612	3	7123456	7			678912345	9
16	2207									612345	6	3456712	7			345678912	9
17	3571									234561	6					891234567	9
18	5778									123456	1					123456789	1
19	9349									234561	6					891234567	9
20	15127									234561	6					891234567	9
21	24476									345612	3					678912345	9
22	39603									456123	2					456789123	3
23	64079									612345	6					912345678	9
24	103682									345612	3					345678912	9

Figure 11: Periods generated by $p = 2,3,4…j,1$ to the power of successive Lucas numbers

Lucas numbers (OEIS [A000032)](A000032) are represented as L_n, with $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-2} + L_{n-1}$. This change of the initial term has many surprising effects. The Lucas series, not as well-known as the φ-sequence, is equally amazing and they are so closely related as to be literally inseparable. (See '[Fibonacci Numbers and the Golden Section'](Fibonacci Numbers and the Golden Section) site.) But the focus now is on Lucas number divisibility and a unique perspective on this question that these p-sequences provide.

Lucas Number Divisibility as a Function of Certain Properties of Cyclical *p*

For a closer look at the effects of raising $F_0 = 0$ to $L_0 = 2$ on the Pisano periods, define P_{Fn} as F_n periods (figure 10), and P_{Ln} as L_n periods. One effect of the F_0 to L_0 shift is to scramble the internal cycles to various degrees; but the sole example of a changed period length (within the scope of the table) is limited to P_{L5}. (See [A106291](A106291).) Note that cycles in the degree columns, once their length is established, repeat monotonically. In the P_F to P_L transit they are typically shifted but unchanged. The exceptions are P_{L5} and P_{L8}. Taking them in order, we look into that now.

We define a *cyclical p* as a permutation created by cycling the terms of I. E.g., by this process, 12345 gives 23451, 34512, 45123, 51234 and itself. All five of these p appear in the $j = 5$ column of figure 10. Since each of these p begins with a different number, we will identify them by their first digit as the elements 1, 2, 3, 4 and 5, and note that 2,5 and 3,4 are inverse pairs.

To recap the events in the P_{F5} column of figure 10: 1 is the initial element and the first iteration, $1 \circ 2$, duplicates 2; $2^2 = 3$ and $2 \circ 3 = 3^{-1} = 4$. Thus the 4th iteration, $3 \circ 4 = 1$, completes the first 5-cycle within the period. On the next round, 1 is followed by two 4s, and the process continues, three more times, until 1 is followed by 2 again and the period is complete.

In the P_{L5} column in table 11, $2 \circ 3$ is reversed. Here, $3 \circ 2$ instigates a period where inverses never adjoin, but are always separated by another element. Hence, absent any $p^1 \circ p^{-1} = I$ opportunities, only four elements are in play. Without I in the mix, no ps are repeated, and the four terms in the order 3, 2, 4, 5 thus complete the period. Hence, the 20-term P_{F5} sequence in figure 10 is foreshortened in figure 11 by a factor of 5. Now, based on the absence of $1 = p^0 = I$ in this four-term period, we conjecture that no Lucas number is divisible by 5.

A stunted period length is an inheritable attribute. It is common, and by the same proportion, to all P_{Lj} with length $j = 0 \pmod 5$, at least to the extent of the entries in figure 12 below.

$j =$	P_{Lj}	1	3	8	6	4	24	16	12	24	12	10	24	28	48	8	24	36	24	18	12	16	30	48	24	20
1 - 25	P_{Fj}	1	3	8	6	20	24	16	12	24	60	10	24	28	48	40	24	36	24	18	60	16	30	48	24	100
26 - 50	P_{Lj}	84	72	48	14	24	30	48	40	36	16	24	76	18	56	12	40	48	88	30	24	48	32	24	112	60
	P_{Fj}	84	72	48	14	120	30	48	40	36	80	24	76	18	56	60	40	48	88	30	120	48	32	24	112	300

Figure 12: The first 50 Pisano periods (P_F) aligned below their Lucas sequence (P_L) counterparts.

Next at P_{L8}, note that the full cycle in the degree column has not merely shifted, but changed. The P_{F8} terms in figure 10 have degrees of 1,8,8,4,8.8; in figure 11, the P_{L8} degree cycle is 4,8,8,2,8.8. The cycle is complete and it repeats monotonically. No $p^0 = 1$ appears here, and so we conjecture that no Lucas number is divisible by 8.

In these examples and by these conjectures, the problem of Lucas number divisibility is reframed. It is equivalent to the question of which P_{Lj} cycles have I in their pattern; that is, which of those p-sequences that initiate from $p^2 \circ p^1$ will include or preclude the identity element $p^0 = I$?

To get a better sense of the problem, consider the OEIS sequence A064362. This entry lists the numbers 5, 8, 10, 12, 13, 15, 16, 17, 20, 21... as n such that no Lucas number is a multiple of n. These numbers are highlighted in the table in figure 13 below. Blue highlights identify terms in A124378, called the 'primitive elements' of A064362; i.e., the numbers in the latter set that are not divisible by any other numbers in the set.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

Figure 13: For $n = 1\ldots100$, the highlighted numbers do not divide any Lucas sequence terms

Each p-sequence generated thus far can be said to comprise a single permutation taken to various powers. Clearly, a cyclic sequence that initiates with $p^1 \circ p^2$ will eventually generate p^0 because, working backwards, $p^1 \circ p^1 = p^2$, and p^0 must precede that. But for $p^2 \circ p^1$, the likelihood of I appearing seems ever more remote as n grows larger. While the idea that p-sequence properties correlate to Lucas number divisibility is interesting of itself, it also suggests the more general investigation that follows.

An Extended Pallet of Permutation Pairs

The next step is to consider all of the $n!^2$ possible pairings (allowing duplicates) of permutations of length n. It is at $n = 3$ where we first find opportunity to get beyond the 'powers of p' patterns. The 3! permutations of 123 are listed below.

1	123	**1**	4	132	2
2	231	**3**	5	213	2
3	312	**3**	6	321	2

Figure 14: The six p of length 3

The six l_3 permutations in figure 14 are numbered for identification on the left. The degree of each p is to its right. The p numbered 1, 2, 3 are of even parity, indicated as usual by bold type, and 4, 5, 6 are odd. Parity will be of interest in some of what follows and, to clarify the concept, a basic method to qualify it is provided now.

Parity is based on a count of transpositions required to create/configure a permutation. One way to evaluate it is just to return each digit to its original position, counting, then summing, the number of 'jumps' (transpositions) along the way. Take the cyclical p = 23451 as an example. To return the digit 1 to its position in I = 12345 takes four jumps across 5, 4, 3 and 2 in succession; four is an even number, so p's parity is even. For p = 4123, 1423 requires one jump, 1243 another and 1234 another. The sum of 1+1+1 = 3, and the parity is odd.

There is more to it, but this serves for present purposes. It remains only to point out that parity is conserved in $p \circ p$ transactions in ways already familiar from ordinary numbers. For an example, take the integers, where odd + odd = even + even = even, and even + odd = odd + even = odd. These patterns are also found in the positive/negative signage of numbers as they are multiplied; $1 \cdot 1 = -1 \cdot -1 = 1$ and $-1 \cdot 1 = 1 \cdot -1 = -1$. Thus, in the recursive algorithm employed here, parity permits just two cyclic patterns: odd, odd, **even**… and **even**, **even**, **even**….

While the six l_3 p aren't much material to work with, there's enough to introduce and investigate sequences other than those based on the powers of a p. First though, is a review, in figure 15, of the way that the P_{F2} and P_{F3} sequences in table 10 were produced.

1	**1**	123	**1**	**1**	123	**1**	123	**1**
2	5	213	2	**2**	231	**3**	312	**3**
3	5	213	2	**2**	231	**3**	312	**3**
4	**1**	123	**1**	**3**	312	**2**	231	**3**

Figure 15: $p^0 \circ p^1$ gives the Pisano periods for length j = 2 and 3 as seen in table 10

In figure 15, l_3 permutations of degree 2 and 3 generate sequences of length equal to the second and third Pisano periods. The pattern on the left is $p^0 \circ p^1$, $p^1 \circ p^1$ and $p^1 \circ p^{-1}$. Thus the three 2nd degree p account for nine pairings. The eight pairs generated by the 3rd degree p in the table on the right exhaust those possibilities. Add in the 123 ○ 123 pairing for a total of 18 pairs. This covers all of the possible 'powers of p' sequences. The $p^0 \circ p^1$, $p^1 \circ p^1$ and $p^1 \circ p^{-1}$ pairings are excluded in figure 16 below.

1	4	132	4	132	6	321	2
2	5	213	6	321	4	132	2
3	**3**	312	**2**	231	**3**	312	**3**
4	6	321	5	213	5	213	2
5	5	213	6	321	4	132	2
6	**2**	231	**3**	312	**2**	231	**3**

Figure 16: For all non-‘powers of p’ combinations, $p \circ p'$ gives period 6

Permutations in figure 16 are highlighted in yellow, according to which terms are transposed. A *p*-sequence of period *n* includes *n* distinct pairings, and hence 18 more pairings are counted here. This accounts for all $3!^2 = 36$ possible pairs. To check our work from a combinatorial perspective, an online calculator takes $n = 5$ and $r = 2$ as arguments in ‘*n* choose *r*’ to show that *p* numbered 2, 3, 4, 5 and 6 in figure 14 can be paired in 20 ways. But inverses 2,3 and 3,2 get tossed. Thus combinatorics confirms that 18 pairs partitioned into three periods exhaust those possibilities.

Periods and Cycles Produced by l_4 Permutation Pairs

1	1234	**1**	**5**	1342	**3**	**9**	3124	**3**
2	2143	**2**	**6**	1423	**3**	**10**	3241	**3**
3	3412	**2**	**7**	2314	**3**	**11**	4132	**3**
4	4321	**2**	**8**	2431	**3**	**12**	4213	**3**
13	1243	2	17	3214	2	21	3142	4
14	1324	2	18	4231	2	22	3421	4
15	1432	2	19	2341	4	23	4123	4
16	2134	2	20	2413	4	24	4312	4

Figure 17: The 24 permutations of the terms of $I = 1234$

Note that the *p* in figure 17 are organized first by parity, then degree and finally numerical order. There’s a lot more to work with at this level; let’s see if we can keep track of all $4!^2 = 576$ distinct ways these permutations can pair off.

First is to tally all of the ways that these l_4 *p* can generate Pisano periods. There are nine degree-2 *p*, and these, as we have seen, generate period-3 sequences: 9 * 3 = 27 pairings. There are eight degree-3 *p* in the table, and each inverse pair generates a period-8 sequence. That’s 4 * 8 = 32 pairs. There are six degree-4 *p*; each p^2 pair gives a period-6 sequence and 6 * 6 = 36 pairs. Add 1234^2 to 27 + 32 + 36 for a sum of 96 pairs accounted for thus far.

We move on to non-cyclical pairings of low-degree in figure 18.

		2○2			**2○2**			2○**2**			2○**2**			2○2			2○2	
1	**2**	2143	**2**	**3**	3412	**2**	16	2134	2	13	1243	2	15	1432	2	17	3214	2
2	**3**	3412	**2**	**2**	2143	**2**	**2**	2143	**2**	**2**	2143	**2**	17	3214	2	15	1432	2
3	**4**	4321	**2**	**4**	4321	**2**	13	1243	2	16	2134	2	**3**	3412	**2**	**3**	3412	**2**
1	**2**	2143	**2**	**3**	3412	**2**	16	2134	2	13	1243	2	15	1432	2	17	3214	2
2	**3**	3412	**2**	**2**	2143	**2**	**2**	2143	**2**	**2**	2143	**2**	17	3214	2	15	1432	2

Figure 18: Period-3 sequences from degree-2 pairs

Headers over columns identify degrees of *p*s in the initial pairs. Permutations that are numbered according to figure 17 will be designated, when necessary, as p_7, p_{12}, and so on

Figure 18 has samples of degree-2 pairs that produce sequences of period 3. They are numbered, in the shaded column to the immediate left, per the table in figure 17. All of the *p* in figure 18 have degree 2 in common. The even parity *p*s, the **2**s, are self-inverse and thus any two of them combined by $\circ$ produce the third; so period 3. The six odd-parity 2s form three pairs with disjoint transpositions, the condition for 2 $\circ$ 2 = **2**. Take, for example, the pair 1243 (p_{13}) and 2134 (p_{16}) and note that the transpositions <u>21</u> and <u>43</u> don't directly interact. So $p_{13} \circ p_{16} = p_2$; but $p_{16} \circ p_2$ reverses that back to p_{13}, and so period 3. A list of period-3 sequences that exclude *I* is in figure 19 below.

1	**2**	**2**	**2**	13	16	15	17	14	18	2
2	**3**	**4**	**2**	16	13	17	15	18	14	2
3	**4**	**3**	**2**	**2**	**2**	**3**	**3**	**4**	**4**	**2**

Figure 19: Period-3 sequences that don't include *I*

The permutations in figure 19 are not displayed explicitly, but rather represented by the numbers assigned in figure 17. The count is 24 pairs in eight *I*-free period-3 sequences.

In degree-2 pairings where transpositions overlap, 2 $\circ$ 2 = **3** with period 6. Pairs 2 $\circ$ 4 = **2** and 4 $\circ$ 2 = **2** have the same period. Some examples are shown in figure 20.

		2○2			2○**3**			2○**2**			4○**2**	
1	16	2134	2	16	2134	2	14	1324	2	23	4123	4
2	14	1324	2	**7**	2314	**3**	**2**	2143	**2**	**2**	2143	**2**
3	**7**	2314	**3**	14	1324	2	21	3142	4	15	1432	2
4	17	3214	2	16	2134	2	18	4231	2	19	2341	4
5	14	1324	2	**9**	3124	**3**	**2**	2143	**2**	**4**	4321	**2**
6	**9**	3124	**3**	17	3214	2	20	2413	4	15	1432	2
1	16	2134	2	16	2134	2	14	1324	2	23	4123	4
2	14	1324	2	**7**	2314	**3**	**2**	2143	**2**	2	2143	**2**

Figure 20: Period-6 sequences from three kinds of combinations

1	13	14	15	13	17	18	14	16	17	15	16	18	2
2	14	15	13	17	18	13	16	17	14	16	18	15	2
3	**6**	**6**	**6**	**12**	**12**	**12**	**9**	**9**	**9**	**11**	**11**	**11**	**3**
4	15	13	14	18	13	17	17	14	16	18	15	16	2
5	14	15	13	17	18	13	16	17	14	16	18	15	2
6	**5**	**5**	**5**	**10**	**10**	**10**	**7**	**7**	**7**	**8**	**8**	**8**	**3**

Figure 21: All 72 period-6 2○2 = **3** combinations

As the sequences grow longer, and sets of sequences grow larger, different patterns emerge. E.g., the 12 sequences in figure 21 fall naturally into sets of three.

1	23	21	21	24	24	23	4	14	15	13	18	17	16	2	24	22	23	19	21	20	4
2	17	18	14	16	13	15	2	21	23	24	21	23	24	4	24	22	23	19	21	20	4
3	**2**	**2**	**3**	**3**	**4**	**4**	**2**	**2**	**2**	**3**	**3**	**4**	**4**	**2**	**2**	**2**	**3**	**3**	**4**	**4**	**2**
4	19	20	20	22	22	19	4	14	15	13	18	17	16	2	22	24	19	23	20	21	4
5	15	14	18	13	16	17	2	20	19	22	20	19	22	4	24	22	23	19	21	20	4
6	**2**	**2**	**3**	**3**	**4**	**4**	**2**	**3**	**4**	**4**	**2**	**2**	**3**	**2**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Figure 22: All possible 4 ○ 2 = **2** and 2 ○ 4 = **2** combinations, along with the three degree-4 inverse pairs

Figure 22 also includes, for the sake of comparison, six l_4 Pisano period generators. These have already been counted, so just the two sets of six period-6 sequences at the left account for another 72 pairs. So far then, our inventory comprises 24 + 72 + 72 + 96 = 264 pairs.

Next, **3**s and **2**s combine to generate period-16 sequences. Parity decrees that non-inverse **3** ○ **3** pairs must equal either **2** or **3**, and the cycles within the periods are uniformly **3**, **3**, **3** and **2**.

		3○2			**2○3**			**3○3**			**3○3**	
1	**7**	2314	**3**	**2**	2143	**2**	**7**	2314	**3**	**7**	2314	**3**
2	**2**	2143	**2**	**7**	2314	**3**	**2**	1342	**3**	**11**	4132	**3**
3	**10**	3241	**3**	**6**	1423	**3**	**10**	2143	**2**	**12**	4213	**3**
4	**11**	4132	**3**	**8**	2431	**3**	**11**	3124	**3**	**2**	2143	**2**
5	**5**	1342	**3**	**4**	4321	**2**	**5**	4213	**3**	**8**	2431	**3**
6	**4**	4321	**2**	**5**	1342	**3**	**4**	4132	**3**	**5**	1342	**3**
7	**8**	2431	**3**	**12**	4213	**3**	**8**	3412	**2**	**7**	2314	**3**
8	**9**	3124	**3**	**7**	2314	**3**	**9**	3241	**3**	**3**	3412	**2**
9	**10**	3241	**3**	**2**	2143	**2**	**10**	1423	**3**	**6**	1423	**3**
10	**2**	2143	**2**	**10**	3241	**3**	**2**	3124	**3**	**10**	3241	**3**
11	**7**	2314	**3**	**11**	4132	**3**	**7**	2143	**2**	**8**	2431	**3**
12	**6**	1423	**3**	**5**	1342	**3**	**6**	1342	**3**	**2**	2143	**2**
13	**8**	2431	**3**	**4**	4321	**2**	**8**	2431	**3**	**12**	4213	**3**
14	**4**	4321	**2**	**8**	2431	**3**	**4**	3241	**3**	**9**	3124	**3**
15	**5**	1342	**3**	**8**	3124	**3**	**5**	3412	**2**	**6**	1423	**3**
16	**12**	4213	**3**	**10**	3241	**3**	**12**	4132	**3**	**3**	3412	**2**
1	**7**	2314	**3**	**2**	2143	**2**	**7**	2314	**3**	**7**	2314	**3**
2	**2**	2143	**2**	**7**	2314	**3**	**2**	1342	**3**	**11**	4132	**3**

Figure 23: **3** ○ **3**, **3** ○ **2** and vice versa are all period 16

Something novel in figure 23 is the presence of so many inverted pairs in the same sequence. Two versions of the same table are given in figure 24, where the highlights on the rightmost call attention to the symmetries.

1	**5**	**5**	**5**	**6**	**6**	**6**	**3**	**5**	**5**	**5**	**6**	**6**	**6**
2	**7**	**10**	**11**	**8**	**9**	**12**	**3**	**7**	**10**	**11**	**8**	**9**	**12**
3	**3**	**4**	**2**	**4**	**2**	**3**	**2**	**3**	**4**	**2**	**4**	**2**	**3**
4	**6**	**6**	**6**	**5**	**5**	**5**	**3**	**6**	**6**	**6**	**5**	**5**	**5**
5	**10**	**11**	**7**	**12**	**8**	**9**	**3**	**10**	**11**	**7**	**12**	**8**	**9**
6	**8**	**9**	**12**	**7**	**10**	**11**	**3**	**8**	**9**	**12**	**7**	**10**	**11**
7	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**2**	**3**	**4**	**2**	**3**	**4**
8	**12**	**8**	**9**	**10**	**11**	**7**	**3**	**12**	**8**	**9**	**10**	**11**	**7**
9	**9**	**12**	**8**	**11**	**7**	**10**	**3**	**9**	**12**	**8**	**11**	**7**	**10**
10	**6**	**6**	**6**	**5**	**5**	**5**	**3**	**6**	**6**	**6**	**5**	**5**	**5**
11	**3**	**4**	**2**	**4**	**2**	**3**	**2**	**3**	**4**	**2**	**4**	**2**	**3**
12	**7**	**10**	**11**	**8**	**9**	**12**	**3**	**7**	**10**	**11**	**8**	**9**	**12**
13	**11**	**7**	**10**	**9**	**12**	**8**	**3**	**11**	**7**	**10**	**9**	**12**	**8**
14	**12**	**8**	**9**	**10**	**11**	**7**	**3**	**12**	**8**	**9**	**10**	**11**	**7**
15	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**2**	**3**	**4**	**2**	**3**	**4**
16	**8**	**9**	**12**	**7**	**10**	**11**	**3**	**8**	**9**	**12**	**7**	**10**	**11**

Figure 24: **2**,**3** and **3**,**3** combine in 96 different ways

Finally, at the close of this l_4 excursion, the highest degrees, **3** and 4, come together to create the longest period, that of 18 terms. The truncated table in figure 25 introduces the pairs.

		2○**3**			**3**○2			2○4			4○2			**3**○4			4○**3**			4○4			4○4	
1	13	1243	2	**7**	2314	**3**	16	2134	2	21	3142	4	**9**	3124	**3**	21	3142	4	19	2341	4	21	3142	4
2	**7**	2314	**3**	13	1243	2	21	3142	4	16	2134	2	21	3142	4	**9**	3124	**3**	21	3142	4	19	2341	4
3	20	2413	4	19	2341	4	**10**	3241	**3**	**5**	1342	**3**	19	2341	4	24	4312	4	**12**	4213	**3**	**6**	1423	**3**
4	22	3421	4	**8**	2431	**3**	23	4123	4	19	2341	4	**6**	1423	**3**	18	4231	2	16	2134	2	16	2134	2
5	**5**	1342	**3**	21	3142	4	14	1324	2	22	3421	4	16	2134	2	**7**	2314	**3**	20	2413	4	23	4123	4
6	17	3214	2	17	3214	2	**12**	4213	**3**	**11**	4132	**3**	23	4123	4	19	2341	4	**6**	1423	**3**	**12**	4213	**3**
7	24	4312	4	**11**	4132	**3**	24	4312	4	14	1324	2	**12**	4213	**3**	21	3142	4	19	2341	4	21	3142	4
		…			…			…			…			…			…			…			…	

Figure 25: Sequences from combinations of 2, **3** and 4 are period-18

The full sequences appear in numerical notation in figure 26.

1	19	21	24	19	20	22	19	24	20	19	22	21	4
2	20	22	23	21	24	23	22	21	23	24	20	23	4
3	**9**	**12**	**8**	**12**	**9**	**5**	**11**	**6**	**10**	**6**	**11**	**7**	**3**
4	13	15	14	16	15	18	14	17	16	18	17	13	2
5	21	24	19	20	22	19	24	20	19	22	21	19	4
6	**11**	**7**	**10**	**6**	**10**	**7**	**12**	**8**	**5**	**9**	**5**	**8**	**3**
7	19	21	24	19	20	22	19	24	20	19	22	21	4
8	14	13	15	18	16	15	16	14	17	13	18	17	2
9	**8**	**9**	**12**	**5**	**12**	**9**	**10**	**11**	**6**	**7**	**6**	**11**	**3**
10	22	23	20	24	23	21	21	23	22	20	23	24	4
11	21	24	19	20	22	19	24	20	19	22	21	19	4
12	**7**	**10**	**11**	**10**	**7**	**6**	**8**	**5**	**12**	**5**	**8**	**9**	**3**
13	15	14	13	15	18	16	17	16	14	17	13	18	2
14	20	22	23	21	24	23	22	21	23	24	20	23	4
15	**12**	**8**	**9**	**9**	**5**	**12**	**6**	**10**	**11**	**11**	**7**	**6**	**3**
16	22	23	20	24	23	21	21	23	22	20	23	24	4
17	14	13	15	18	16	15	16	14	17	13	18	17	2
18	**10**	**11**	**7**	**7**	**6**	**10**	**5**	**12**	**8**	**8**	**9**	**5**	**3**

Figure 26: *p* of degrees 2, **3** and 4 combine in 216 ways

4	19	22	21	23	24	20	19	24	20	23	22	21	19	21	24	23	20	22	19	20	22	23	21	24
4	20	21	22	24	23	19	21	20	24	22	23	19	22	24	21	20	23	19	24	22	20	21	23	19
2	13	15	15	14	14	13	16	15	15	18	18	16	14	17	17	16	16	14	18	17	17	13	13	18
4	21	20	24	22	19	23	20	21	22	24	19	23	24	22	20	21	19	23	22	24	21	20	19	23
4	19	22	21	23	24	20	19	24	20	23	22	21	19	21	24	23	20	22	19	20	22	23	21	24
2	14	14	13	13	15	15	18	18	16	16	15	15	16	16	14	14	17	17	13	13	18	18	17	17
3	**9**	**7**	**12**	**10**	**8**	**11**	**12**	**10**	**9**	**7**	**5**	**6**	**11**	**8**	**6**	**5**	**10**	**12**	**6**	**5**	**11**	**8**	**7**	**9**
3	**11**	**12**	**7**	**8**	**10**	**9**	**6**	**9**	**10**	**5**	**7**	**12**	**12**	**6**	**8**	**10**	**5**	**11**	**9**	**11**	**5**	**7**	**8**	**6**
3	**8**	**10**	**9**	**11**	**12**	**7**	**5**	**7**	**12**	**6**	**9**	**10**	**10**	**5**	**11**	**12**	**6**	**8**	**7**	**8**	**6**	**9**	**11**	**5**

Figure 27: Periods in figure 26 are divided into two cycles with the **3**s taken out

As we've seen, the larger is n in l_n, the more material to work with and periods and patterns grow longer and become more intricate accordingly. While sequences are characterized individually by their periods and cycles, the patterns they form when gathered into tables are another interesting attribute. Is there anything predictable here?

264 + 216 + 96 = 4!^2 = 576, so all possible l_4 pairings are accounted for. Next, a short survey of l_5 shows again how complications compound as the length of p increases. To what extent can the power of combinatorics help enumerate and catalogue these varieties of sequences?

Excerpts of l_5

The six ways that l_5 permutations can generate the P_{L5} pattern in figure 11 are listed below. They are color-coded for easy identification.

1	23451	23514	24153	24531	25134	25413	**5**
2	45123	54132	53412	35214	43521	34251	**5**
3	51234	41253	31524	51423	31452	41532	**5**
4	34512	35421	45231	43152	54213	53124	**5**

Figure 28: Six **5** ○ **5**, period-4 sequences

The colors offer some insight into what's going on in the various tables presented below.

1	23451	43152	41532	31524	35421	23451	35421	35214	54213	24153	**5**
2	25413	24153	54132	51234	35214	43152	43521	31524	51234	54132	**5**
3	31524	35421	23451	43152	41532	54213	24153	23451	35421	35214	**5**
4	42351	13542	41325	32541	13425	25341	32415	15243	24315	12453	**3**
5	21543	34125	52431	13254	45312	43215	14523	21354	52431	35142	**2**
6	24153	54132	51234	35214	25413	43521	31524	51234	54132	43152	**5**
7	14235	52314	15243	24315	52143	12534	51342	42135	13542	41325	**3**
8	25413	24153	54132	51234	35214	43152	43521	31524	51234	54132	**5**
9	45312	21543	34125	52431	13254	35142	43215	14523	21354	52431	**2**
10	13425	42351	13542	41325	32541	12453	25341	32415	15243	24315	**3**

Figure 29: Two **5** ○ **5**, period-50, cycle-10 tables

Every degree-5 p that appears in figure 29 is eventually echoed by its inverse. Yet they are never allowed so close to one another as to produce I. In the tables to follow, degree-5 p will at times be seen solo, but if/when other p of the same color appear, it's either p again or else p^{-1}.

There is an alternate and more direct way to generate inverse pairs; i.e., the generators depicted below. We could also call them recursive *dynamos*, because it seems they are producing a current of sorts, as the end product becomes input for the next cycle/iteration.

25413				54132				25413
24153				41532				24153
43152	○	23451	=	31524	○	51234	=	43152
43521				35214				43521
35421				54213				35421

		25413		31524				43152
		24153		35214				43521
23451	○	43152	=	54213	○	51234	=	35421
		43521		54132				25413
		35421		41532				24153

Figure 30: Alternating and rotating p-circuits

A p of the 2^{nd} degree produces smaller circuits below:

25413				41253				25413
53412	○	34125	=	41532	○	34125	=	53412
23451				45231				23451
23514				51234				23514

34125	○	23451	=	41253	○	34125	=	25413
		53412		51234				23514
		25413		45231				23451
		23514		41532				53412

Figure 31: Another system of ‘AC’ and ‘DC’ *p*-circuits

1	23154	54231	21453	34521	6
2	23451	24531	24153	34251	**5**
3	31542	43125	15234	52413	4
4	42153	35241	23415	14532	4
5	41325	15324	52314	51342	**3**
6	54123	31254	53421	21534	6
7	52413	31542	43125	15234	4
8	34251	23451	24531	24153	**5**
9	41235	15423	32514	53142	4
10	53421	21534	54123	31254	6
11	52314	51342	41325	15324	**3**
12	13452	42531	25143	34215	4
13	53142	41235	15423	32514	4
14	24153	34251	23451	24531	**5**
15	34521	23154	54231	21453	6
16	15342	42315	15342	42315	2
17	31524	53124	51234	51423	**5**
18	31254	53421	21534	54123	6
19	53142	41235	15423	32514	4
20	42351	25341	24315	14352	**3**
21	43125	15234	52413	31542	4
22	53421	21534	54123	31254	6
23	51234	51423	31524	53124	**5**
24	15342	42315	15342	42315	2

Figure 32: 6 ○ **5** yields period 96 with four cycles

1	21534	34152	53421	6
2	54231	21453	35124	6
3	43152	43521	41532	**5**
4	32514	54312	23415	4
5	13245	12543	15342	2
6	35214	54213	25413	**5**
7	25314	34215	52413	4
8	54231	21453	35124	6
9	41532	43152	43521	**5**
10	35124	54231	21453	6
11	52413	25314	34215	4
12	45231	41253	45123	**5**
13	13245	12543	15342	2
14	42531	41352	43125	4
15	43521	41532	43152	**5**
16	35124	54231	21453	6
17	51432	23154	34521	6
18	43215	42513	45312	**2**

1	23154	31254	6
2	53214	34152	6
3	41325	25341	**3**
4	15234	42153	4
5	45132	45213	6
6	34125	53241	**2**
7	13452	32514	4
8	31254	23154	6
9	41325	25341	**3**
10	53214	34152	6
11	53142	34215	4
12	42513	15432	**2**
13	43251	35124	6
14	15234	42153	4
15	41325	25341	**3**

Figure 33: 6 ○ 6 yields period 54 with three cycles (left) and period 30 with two

Period length 96 in figure 32 is the record so far, but with 14400 possible pairings, there's more to see here, and size can be a surprise. E.g., check out these period-14 runs in figure 34:

1	23451	43521	31524	43521	**5**
2	23514	24531	53124	23514	**5**
3	34125	32154	45312	35142	**2**
4	51234	54213	24153	54213	**5**
5	53412	45231	51423	24531	**5**
6	42351	13425	32541	41325	**3**
7	13425	42351	41325	32541	**3**

Figure 34: Two **5** ○ **5**, period-14, cycle-2 sequences

Obviously this exposition only scratches the surface of what could prove to be a very fertile field. The foregoing serves at least to show the power of this simple approach in opening new avenues of exploration. We've gotten a glimpse of the diversity these patterns present, and a sense of what lies ahead in the journey through permutations of ever-greater length. It suggests that even the simply stated questions below might offer a challenge.

- Given the set of permutations of length *n*, how many different periods does it generate?
- How many periods of each length are there, and what cycles do they contain?

Addition Modulo l_n

A new algorithm opens yet another dimension to this investigation. It's based on the idea of summing permutations of equal length. Take, e.g., permutations 3241 and 2413 and add them. Normal addition would give a sum of 5654; but with *modular addition* (♦), when the sum of two digits in *p* and *p'* exceeds *n* (where *n* = the length of *p*), we keep only the excess. Hence, in this case, the sum *p* ♦ *p'* = *p''* = 1214. While terms in *p''* are not (as required for ○) all distinct, this is not a problem, for operations such as 2413 ♦ 1214 = 3223 are valid too. Next is to apply ♦ in the familiar recursive process and see what comes of it.

The mutant *p*s that contain duplicate terms, we'll call *q*s. They often define cycles within periods, as in figure 35 below. Period lengths seem rarely to vary for a given *n*, but cycles tend to be more complicated. The *n* = 3 patterns below are straightforward though, with two 4-cycles in period 8.

1	123	**1**	213	2	213	2	231	**3**	123	**1**
2	321	2	231	**3**	132	2	312	**3**	231	**3**
3	111	3	111	3	312	**3**	213	2	321	2
4	132	2	312	**3**	111	3	222	6	222	6
5	213	2	123	**1**	123	**1**	132	2	213	2
6	312	**3**	132	2	231	**3**	321	2	132	2
7	222	6	222	6	321	2	123	**1**	312	**3**
8	231	**3**	321	2	222	6	111	3	111	3
1	123	**1**	213	2	213	2	231	**3**	123	**1**
2	321	2	231	**3**	132	2	312	**3**	231	**3**

Figure 35: ♦ (modular addition) creates a period-8 pattern for *n* = 3

The examples in figure 35 are chosen to provide a broad mixture of degrees. With ○, such mixing results in a multiplicity of periods, but for ♦, the period length is typically constant for a given *n*.

The *q*s in the tables above and below are highlighted in red. The property of degree is (at present) meaningless for *q*: the numbers to their immediate right give instead the sum of a *q*'s digits.

1	3142	4	1234	1	3142	4	2134	2	2314	**3**	2134	2	3142	4	2314	**3**
2	2413	4	2341	4	3124	**3**	1243	2	1432	2	3142	4	2341	4	3124	**3**
3	1111	4	3131	8	2222	8	3333	12	3342	12	1232	8	1443	12	1434	12
4	3124	**3**	1432	2	1342	**3**	4132	**3**	4334	14	4334	14	3344	14	4114	10
5	4231	2	4123	4	3124	**3**	3421	4	3232	10	1122	6	4343	14	1144	10
6	3311	8	1111	4	4422	12	3113	8	3122	8	1412	8	3243	12	1214	8
1	3142	4	1234	1	3142	4	2134	2	2314	**3**	2134	2	3142	4	2314	**3**
2	2413	4	2341	4	3124	**3**	1243	2	1432	2	3142	4	2341	4	3124	**3**

Figure 36: ♦ creates a 6-cycle for $n = 4$

We see in figure 36 that the degrees of the initial *p* are not the sole influence on the *q*/*p* ratio; e.g., the ratio is 1:2 in the first 4 ♦ 4 example, and 2:1 in the next. Note that *q*-sums here are not always a multiple of *n*. Again, period length is constant for all pairings, regardless of degree.

	1		2		3		4		5		6		7		8		9	
1	21534	6	23154	6	31524	**5**	52314	**3**	34521	6	14235	**3**	11111	5	11111	5	55555	25
2	54231	6	23451	**5**	35214	**5**	23154	6	14235	**3**	34521	6	23154	6	23451	**5**	12345	**1**
3	25215	15	41555	20	11233	10	25413	**5**	43251	6	43251	6	34215	4	34512	**5**	12345	**1**
4	24441	15	14451	15	41442	20	43512	6	52431	**2**	22222	10	52314	**3**	52413	4	24135	4
5	44151	15	55451	20	52125	15	13425	**3**	45132	6	15423	4	31524	**5**	31425	4	31425	4
6	13542	**3**	14352	**3**	43512	6	51432	6	42513	**2**	32145	2	33333	15	33333	15	55555	25
7	52143	**3**	14253	4	45132	6	14352	**3**	32145	2	42513	**2**	14352	**3**	14253	4	31425	4
8	15135	15	23555	20	33144	15	15234	4	24153	**5**	24153	**5**	42135	**3**	42531	4	31425	4
9	12223	10	32253	15	23221	10	24531	**5**	51243	4	11111	5	51432	6	51234	**5**	12345	**1**
10	22353	15	55253	20	51315	15	34215	4	25341	**3**	35214	**5**	43512	6	43215	**2**	43215	**2**
11	34521	6	32451	4	24531	**5**	53241	**2**	21534	6	41325	**3**	44444	20	44444	20	55555	25
12	51324	4	32154	**2**	25341	**3**	32451	4	41325	**3**	21534	6	32451	4	32154	**2**	43215	**2**
13	35345	20	14555	20	44322	15	35142	**2**	12354	2	12354	2	21345	2	21543	**2**	43215	**2**
14	31114	10	41154	15	14113	10	12543	2	53124	**5**	33333	15	53241	**2**	53142	4	31425	4
15	11454	15	55154	20	53435	20	42135	**3**	15423	4	45132	6	24531	**5**	24135	4	24135	4
16	42513	**2**	41253	**5**	12543	2	54123	6	13542	**3**	23415	4	22222	10	22222	10	55555	25
17	53412	**5**	41352	4	15423	4	41253	**5**	23415	4	13542	**3**	41253	**5**	41352	4	24135	4
18	45425	20	32555	20	22411	10	45321	4	31452	**5**	31452	**5**	13425	**3**	13524	4	24135	4
19	43332	15	23352	15	32334	15	31524	**5**	54312	4	44444	20	54123	6	54321	**2**	43215	**2**
20	33252	15	55352	20	54245	20	21345	2	35214	**5**	25341	**3**	12543	2	12345	**1**	12345	**1**
1	21534	6	23154	6	31524	**5**	52314	**3**	34521	6	14235	**3**	11111	5	11111	5	55555	25
2	54231	6	23451	**5**	35214	**5**	23154	6	14235	**3**	34521	6	23154	6	23154	6	12345	**1**

Figure 37: Modular addition creates period 20 for $n = 5$

Figure 37 shows three *q*/*p* patterns within the period-20 sequences. Maybe more patterns will emerge in a broader sampling of the $5!^2$ possible permutation pairs. Elements commute over ♦, but the order of pairs is important in the recursive context. E.g., see columns 5 and 6 where a *p* ♦ *p'* to *p'* ♦ *p* inversion gives a markedly different pattern.

Patterns in columns 1–3 have a *q*/*p* ratio of 3:2 with *q*-sums a multiple of 5, but 3 ♦ 6 and 6 ♦ 3 pairings in columns 4 and 5 produce periods that are *q*-free. (Why?) As noted, inverting the initial *p*s in column 5 gives, in column 6, a sequence where monotonic *q*s appear. Columns 7 and 8 lead off with 11111, and for all random *p* chosen so far, the same *q*s define the cycles that recur within the period. Note in column 9 that 55555 functions as the identity element for ♦.

So l_3 gives period 8, l_4 is period 6 and l_5 has period 20… and if it seems there is something familiar about these numbers, the table in figure 38 surely will bring it to mind.

♦	j=	2		3		4		5		6		7		8		9	
0		22	4	333	9	4444	16	55555	25	666666	36	7777777	49	88888888	64	999999999	81
1		12	**1**	123	**1**	1234	**1**	12345	**1**	123456	**1**	1234567	**1**	12345678	**1**	123456789	**1**
2		12	**1**	123	**1**	1234	**1**	12345	**1**	123456	**1**	1234567	**1**	12345678	**1**	123456789	**1**
3		22	4	213	2	2424	12	24135	4	246246	24	2461357	**3**	24682468	40	246813579	**6**
4				333	9	3214	2	31425	4	363636	27	3625147	6	36147258	2	369369369	54
5				213	2	1234	**1**	55555	25	543216	**2**	5316427	6	52741638	**2**	516273849	**6**
6				213	2	4444	16	31425	4	246246	24	1234567	**1**	88888888	64	876543219	**2**
7				123	**1**			31425	4	123456	**1**	6543217	2	52741638	**2**	483726159	**3**
8				333	9			12345	**1**	363636	27	7777777	49	52741638	**2**	369369369	54
9								43215	**2**	426426	24	6543217	2	24682468	40	753186429	**3**
10								55555	25	123456	**1**	6543217	2	76543218	2	123456789	**1**
11								43215	**2**	543216	**2**	5316427	6	12345678	**1**	876543219	**2**
12								43215	**2**	666666	36	4152637	**3**	88888888	64	999999999	81
13								31425	4	543216	**2**	2461357	**3**			876543219	**2**
14								24135	4	543216	**2**	6543217	2			876543219	**2**
15								55555	25	426426	24	1234567	**1**			753186429	**3**
16								24135	4	363636	27	7777777	49			639639639	54
17								24135	4	123456	**1**					483726159	**3**
18								43215	**2**	426426	24					123456789	**1**
19								12345	**1**	543216	**2**					516273849	**6**
20								55555	25	363636	27					639639639	54
21										246246	24					246813579	**6**
22										543216	**2**					876543219	**2**
23										123456	**1**					123456789	**1**
24										666666	36					999999999	81

Figure 38: Pisano periods from modular addition

The table's motif is similar to that of figure 10, the better to highlight the amazing accord of their periods and cycles. We can also generate an analog to the table in figure 11.

♦	j=	2		3		4		5		6		7		8		9	
0		22	4	213	2	2424	12	24135	4	246246	24	2461357	**3**	24682468	40	246813579	**6**
1		12	**1**	123	**1**	1234	**1**	12345	**1**	123456	**1**	1234567	**1**	12345678	**1**	123456789	**1**
2		12	**1**	333	9	3214	2	31425	4	363636	27	3625147	**4**	36147258	2	369369369	54
3		22	4	123	**1**	4444	16	43215	**2**	426426	24	4152637	**3**	48484848	48	483726159	**3**
4				123	**1**	3214	2	24135	4	123456	**1**	7777777	49	76543218	2	753186429	**3**
5				213	2	3214	2			543216	**2**	4152637	**3**	36147258	2	246813579	**6**
6				333	9	2424	12			666666	36	4152637	**3**	24682468	40	999999999	81
7				213	2					543216	**2**	1234567	**1**	52741638	**2**	246813579	**6**
8				213	2					543216	**2**	5316427	6	76543218	2	246813579	**6**
9										426426	24	6543217	2	48484848	48	483726159	**3**
10										363636	**27**	4152637	**3**	36147258	2	639639639	54
11										123456	**1**	3625147	**4**	76543218	2	123456789	**1**
12										426426	24	7777777	49	24682468	40	753186429	**3**
13										543216	**2**	3625147	**4**			876543219	**2**
14										363636	27	3625147	**4**			639639639	54
15										246246	24	6543217	2			516273849	**6**
16										…		2461357	**3**			…	

Figure 39: Modular addition gives periods of the Lucas sequence mod n

One way to describe the process that created the P_{Lj} table in figure 11 is to say that permutations on the 3rd row of the P_{Fj} table in 10 replaced those on row zero. The same process turns figure 38 into figure 39, and we have again the periods of the Lucas sequence mod n as in A001175.

Let the ps and the q in the P_{L5} column in figure 39 be designated by their initial digits, and note that 1 ♦ 4 = 2 ♦ 3 = 5. Then, the same as with figure 11, the 2,3 to 3,2 transposition assures that inverses can't interact, thus contracting the 20-term period to period 4.

Here's a curiosity: 1,2,3,4 and 5 form a group under ♦ and, likewise, 1,2,3,4 under ○. Moreover, 31425 ♦ 43215 = 31425 ○ 43215 = 24135: how often does this kind of correlation occur?

Note that the pattern of qs in columns 6, 7 and 8, figure 37, is identical, difference-wise, to the P_{L5} pattern 2,4,5,3. I.e., 2, 1, –2, –1.

Some questions asked of ○'s effect on S_n (the set of all permutations of length n) apply to ♦:

- Given S_n, how many different periods does ♦ generate?
- How many of each length are there, and what cycles do they contain?

Higher Dimensions

Any of these sequences will create a lattice in 3-space. An example is P_{L5} (represented as 3, 2, 4, 5). To begin, we put 3,2,4,5 in a column, set 2,4,5,3 next to it and do the math.

3	2	4	5
2	4	5	3
4	5	3	2
5	3	2	4

Figure 40: A square based on P_{L5}

2	4	5	3
4	5	3	2
5	3	2	4
3	2	4	5

4	5	3	2
5	3	2	4
3	2	4	5
2	4	5	3

5	3	2	4
3	2	4	5
2	4	5	3
4	5	3	2

Figure 41: Three more squares make a cube

Next, figure 41 squares are stacked on the first in order to create a cube. Extending this cube fills space with a lattice that is in a sense directional, but the main 3D diagonal runs the opposite way.

We can use ○ to permute one sequence (s) by another (s'). In the example below, s is from the 1st column in figure 21 and s' is from the 2nd. Then $s \circ s'$ creates an array.

13	14	**6**	15	14	**5**	13	14	**6**	15	14	**5**	13	14	**6**	15	14	**5**	13	14	**6**	…
14	15	**6**	13	15	**5**	14	15	**6**	13	15	**5**	14	15	**6**	13	15	**5**	14	15	**6**	…
6	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	…
15	13	**6**	14	13	**5**	15	13	**6**	14	13	**5**	15	13	**6**	14	13	**5**	15	13	**6**	…
14	15	**6**	13	15	**5**	14	15	**6**	13	15	**5**	14	15	**6**	13	15	**5**	14	15	**6**	…
5	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	…

Figure 42: The second column of figure 21 permutes the first to initiate an array

Starting from the upper-left corner, the row, column and diagonal are all proper sequences. All of the six rows in figure 42 are proper of course, but most columns are not. The internal periods combine for period-24 overall.

Any columns could be chosen from figure 21, not just two adjacent. Indeed, it's not necessary that *s* and *s′* should be taken from the same table, as even *s* of different periods can be juxtaposed. For example, a period-8 repeated three times fits next to four period-6s. But, to keep it simple, the first two sequences in period-16 figure 24 are aligned at the left below,

5	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**…**
7	**10**	**5**	**3**	**12**	**8**	**7**	**4**	**11**	**6**	**12**	**3**	**5**	**9**	**11**	**4**	**7**	**10**	**5**	**3**	**12**	**…**
3	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**…**
6	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**…**
10	**11**	**5**	**4**	**8**	**9**	**10**	**2**	**7**	**6**	**8**	**4**	**5**	**12**	**7**	**2**	**10**	**11**	**5**	**4**	**8**	**…**
8	**9**	**10**	**2**	**7**	**6**	**8**	**4**	**5**	**12**	**7**	**2**	**10**	**11**	**5**	**4**	**8**	**9**	**10**	**2**	**7**	**…**
2	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**…**
12	**8**	**7**	**4**	**11**	**6**	**12**	**3**	**5**	**9**	**11**	**4**	**7**	**10**	**5**	**3**	**12**	**8**	**7**	**4**	**11**	**…**
9	**12**	**11**	**3**	**10**	**6**	**9**	**2**	**5**	**8**	**10**	**3**	**11**	**7**	**5**	**2**	**9**	**12**	**11**	**3**	**10**	**…**
6	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**5**	**6**	**1**	**6**	**6**	**5**	**1**	**5**	**…**
3	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**…**
7	**10**	**5**	**3**	**12**	**8**	**7**	**4**	**11**	**6**	**12**	**3**	**5**	**9**	**11**	**4**	**7**	**10**	**5**	**3**	**12**	**…**
11	**7**	**5**	**2**	**9**	**12**	**11**	**3**	**10**	**6**	**9**	**2**	**5**	**8**	**10**	**3**	**11**	**7**	**5**	**2**	**9**	**…**
12	**8**	**7**	**4**	**11**	**6**	**12**	**3**	**5**	**9**	**11**	**4**	**7**	**10**	**5**	**3**	**12**	**8**	**7**	**4**	**11**	**…**
2	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**2**	**3**	**4**	**…**
8	**9**	**10**	**2**	**7**	**6**	**8**	**4**	**5**	**12**	**7**	**2**	**10**	**11**	**5**	**4**	**8**	**9**	**10**	**2**	**7**	**…**

Figure 43: The second column of figure 24 permutes the first

The period works out as 16 * 3 = 48. Strangely, the 1^{st} and 7^{th} columns are identical. Moreover, rows 2 and 12 are the same as rows 8 and 14, shifted four columns out of phase. This same offset relates rows 6 and 16 to 5 and 9 to 13. Thus, in the nine period-16 rows in the array, there are just three distinct sequences: those that start with the number 6 in figure 24.

The six sequences of figure 24 may be paired (allowing repetition as usual) in 36 ways. But one of either *s* or *s′* in a pair can be cycled, thus creating 16 sequences per pair. That means 216 tables similar to those just above. Since we allow *s* = *s′*, one of those tables will be, for each sequence, a square akin to the one in figure 40. Will any of the other 210 arrays be so distinctive, or perhaps as remarkable in some other way?

Of the 12 sequences in figure 26, there are (excluding reversing the direction of an *s*) 18 ways to set *s* and *s′* side by side. Hence they combine to create 2592 arrays. Anything new and interesting going on with these?

ADDENDA

Some Different Styles of Representation for Groups

A modified notation provides another way to look at groups. To the left in figure 44 is table 1, a minimalist version of a typical group table. Now, the numbering system that denotes the l_3 *p* in figure 14 is applied to construct the two group tables to the right

Table 1

	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

Table 2

○	**1**	**2**	**3**	
1	1	2	3	**1**
2	2	3	1	**3**
3	3	1	2	**3**
	1	**3**	**3**	

Table 3

○	4	5	6	
1	1	2	3	**1**
3	3	1	2	**3**
2	2	3	1	**3**
	2	2	2	

Figure 44: C_3 to the left and middle, S_3/D_3 to the far right

In the field of table 1, numbers 1, 2 and 3 are seen as separate elements of the cyclic group C_3. In the field of table 2, the numbers are now said to compose l_3 permutations, both horizontally and vertically. Numbers in the row and column headers then correspond (as indicated by the smaller font) to the labeling scheme that identifies the *p* in table 14. Numbers appended in the new column and row on the right and below give the degrees of permutations they adjoin.

In table 3, the bottom two rows of table 2 are transposed. The effect is to change the degree of the vertical *p*s from three to two and parity from even to odd. Odd *p* are now orthogonal to even *p* and, after adjusting the numbering and degree notation at the top and bottom edges accordingly, we find the symmetric group S_3 and the isomorphic dihedral group D_3 compacted into a 3x3 table.

This idea of permuting rows is taken to the next level based on the patterns above and the labeling scheme for l_4 permutations in figure 17. The first step is to construct C_4, as seen in the upper-left table in figure 45 below. The bottom three rows of C_4 are then permuted in the remaining 3! – 1 = 5 possible ways.

	1	19	**3**	23	
1	1	2	3	4	**1**
19	2	3	4	1	4
3	3	4	1	2	**2**
23	4	1	2	3	4
	1	4	**2**	4	

	5	20	**9**	18	
1	1	2	3	4	1
3	3	4	1	2	**2**
23	4	1	2	3	4
19	2	3	4	1	4
	3	4	**3**	2	

	6	16	**10**	24	
1	1	2	3	4	**1**
23	4	1	2	3	4
19	2	3	4	1	4
3	3	4	1	2	**2**
	3	2	**3**	4	

	13	**7**	22	**11**	
1	1	2	3	4	1
19	2	3	4	1	4
23	4	1	2	3	4
3	3	4	1	2	**2**
	2	**3**	4	**3**	

	14	**8**	21	**12**	
1	1	2	3	4	**1**
3	3	4	1	2	**2**
19	2	3	4	1	4
23	4	1	2	3	4
	2	**3**	4	**3**	

	15	**2**	17	**4**	
1	1	2	3	4	**1**
23	4	1	2	3	4
3	3	4	1	2	**2**
19	2	3	4	1	4
	2	**2**	2	**2**	

Figure 45: The 24 l_4 permutations partitioned into six four-element sets

Note that the 24 columns of the tables in figure 45 are all distinct; they contain all 24 *p* from the list of l_4 permutations in figure 17. Here's where it gets a little tricky. The next step is to pick three of these columns and transform them to rows. These three rows are now matched with the remaining three columns in three 4x4 squares. This can be accomplished in numerous ways. One option appears in figure 46.

	1	23	3	19	
15	1	4	3	2	2
2	2	1	4	3	2
17	3	2	1	4	2
4	4	3	2	1	2
	1	4	2	4	

	13	11	7	22	
6	1	4	2	3	3
16	2	1	3	4	2
24	4	3	1	2	4
10	3	2	4	1	3
	2	3	3	4	

	14	21	12	8	
5	1	3	4	2	3
9	3	1	2	4	3
20	2	4	1	3	4
18	4	2	3	1	2
	2	4	3	3	

Figure 46: S_4 represented as six sets of order-4 permutations in three 4x4 squares

Note that at the left in figure 46 is the group D_4, which always appears when the order of the last three rows of a cyclic group is reversed. This configuration shows the essence of the idea, and of course other choices create other patterns. With this established, we go on to Klein's four-group, seen at the upper left in figure 47 below.

	1	2	3	4	
1	1	2	3	4	1
2	2	1	4	3	2
3	3	4	1	2	2
4	4	3	2	1	2
	1	2	2	2	

	5	8	9	12	
1	1	2	3	4	1
3	3	4	1	2	2
4	4	3	2	1	2
2	2	1	4	3	2
	3	3	3	3	

	6	7	10	11	
1	1	2	3	4	1
4	4	3	2	1	2
2	2	1	4	3	2
3	3	4	1	2	2
	3	3	3	3	

	13	16	22	24	
1	1	2	3	4	1
2	2	1	4	3	2
4	4	3	2	1	2
3	3	4	1	2	2
	2	2	4	4	

	14	20	21	18	
1	1	2	3	4	1
3	3	4	1	2	2
2	2	1	4	3	2
4	4	3	2	1	2
	2	4	4	2	

	15	19	17	23	
1	1	2	3	4	1
4	4	3	2	1	2
3	3	4	1	2	2
2	2	1	4	3	2
	2	4	2	4	

Figure 47: Permuting the rows of the Klein group creates a partition of S_4

Columns of the tables in figure 47 offer a new set of options for expressing S_4 in the context of three 4x4 squares. Next, three columns are converted to rows and again fitted with the remaining columns to make squares. Klein group elements are all of even parity, a uniformity that carries over into the partitions. This property stands out in the following three figures.

	1	2	4	3	
13	1	2	4	3	2
16	2	1	3	4	2
22	3	4	2	1	4
24	4	3	1	2	4
	1	2	2	2	

	6	7	11	10	
14	1	3	2	4	2
18	4	2	3	1	2
20	2	4	1	3	4
21	3	1	4	2	4
	3	3	3	3	

	5	8	9	12	
15	1	4	3	2	2
17	3	2	1	4	2
23	4	1	2	3	4
19	2	3	4	1	4
	3	3	3	3	

Figure 48: Another way to represent S_4 as three 4x4 squares

	1	2	4	3	
13	1	2	4	3	2
16	2	1	3	4	4
22	3	4	2	1	4
24	4	3	1	2	2
	1	2	2	2	

	6	7	10	11	
5	1	3	4	2	3
8	4	2	1	3	3
12	2	4	3	1	3
9	3	1	2	4	3
	3	3	3	3	

	14	18	21	20	
15	1	4	3	2	2
17	3	2	1	4	2
19	2	3	4	1	4
23	4	1	2	3	4
	2	2	4	4	

Figure 49: Another option

	1	4	2	3	
6	1	4	2	3	3
7	2	3	1	4	3
10	3	2	4	1	3
11	4	1	3	2	3
	1	2	2	2	

	13	22	24	16	
5	1	3	4	2	3
8	2	4	3	1	3
12	4	2	1	3	3
9	3	1	2	4	3
	2	4	4	2	

	14	18	21	20	
15	1	4	3	2	2
17	3	2	1	4	2
19	2	3	4	1	4
23	4	1	2	3	4
	2	2	4	4	

Figure 50: And another

Now that the process of constructing these squares is familiar, it need not be restricted to the partitions created in processes above. Permutations can be drawn from the list in figure 17 and then assembled into squares ‘freehand’. Here is an example of that:

	1	23	3	19	
15	1	4	3	2	2
2	2	1	4	3	2
17	3	2	1	4	2
4	4	3	2	1	2
	1	4	2	4	

	6	7	10	11	
5	1	3	4	2	3
8	4	2	1	3	3
12	2	4	3	1	3
9	3	1	2	4	3
	3	3	3	3	

	13	22	16	24	
14	1	3	2	4	2
20	2	4	1	3	4
18	4	2	3	1	2
21	3	1	4	2	4
	2	4	4	2	

Figure 51: A custom construction of S_4 in three squares

There are a lot of ways then of creating these kinds of tables. (How many different combinations are possible?) Perhaps if they are extended to larger l_n and subjected to closer scrutiny, some interesting patterns will emerge. It is sort of cool, for example, that a degree-3 p intersects its inverse on the main diagonal in figure 51.

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