GALLERY OF WALKS ON THE SQUARE LATTICE BY A TURING PLOTTER FOR BINARY SEQUENCES

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ABSTRACT. We illustrate infinite walks on the square lattice by interpretation of binary sequences of zeros and ones as directives to a single pen plotter. Each digit of the sequence tells the plotter to walk one edge of the lattice, and to change its direction to the right or left thereafter if the digit was a zero.

1. Introduction

Any binary stream of zeros and ones defines a walk on a simple square lattice by rules that could be used to illustrate Lindenmayer systems (supposed the digits were composed by formal rules):

- Start at the position (0,0) and point into the direction (1,0)=R.
- Read the next digit from the binary stream, from left to right. If the digit is $b_i = 1$, move ahead by one unit into the pointing direction. Digits in the binary stream are enumerated by $i \geq 1$.
- If the next digit in the stream is $b_i = 0$, move ahead in the pointing direction by one unit. If i was odd, change the pointing direction by turning left—from R(ight) to U(p), from U(p) to L(eft), from L(eft) to D(own), or from D(own) to R(ight). If i was even, change the pointing direction by a right turn—from R(ight) to D(own), from D(own) to L(eft), from L(eft) to U(p), or from U(p) to R(ight).
- Read the next digit b_{i+1} from the stream, and update the position and pointing direction again according to the rules of the second and third bullet.

The rules define a "Turing machine plotter" with a state held in three internal registers: the two Cartesian components of the current position and one of four possible pointing directions $\{RULD\}$. The binary sequences serve as programs that feed the plotter.

2. Periodic Binaries and Patterns

The simplest programs are the sequences

$$000000000000000000000000000000\ldots = \bar{0} = 0, \tag{1}$$

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which represent a left turn, right turn, left-turn, right-turn... staircase walk along a diagonal and [6, A000012]

$$111111111111111111111111111111\ldots \quad = \quad \bar{1} \quad = \quad 1 \tag{2}$$

which walks straight along the horizontal axis to infinity. The bar over one or more binary symbols denotes periodic repetition of the digit in base 2.

Placing an implicit dot at the very start of the sequence, the binary sequence represents an associated constant between zero and one,

$$c = \sum_{i\ge 1} \frac{b_i}{2^i}; \quad b_i = \lfloor c2^i \rfloor \mod 2. \tag{3}$$

If the number has a binary representation which is periodic, $b_{k+p} = b_k$ with some period length p, the constant c is a rational number deduced by a geometric series:

$$\begin{aligned}\overline{b_1 b_2 \ldots b_p} \rightsquigarrow c = (b_1 + \frac{b_2}{2} + \frac{b_3}{2^2} + \cdots \frac{b_p}{2^{p-1}})\frac{1}{2} \\ + (b_1 + \frac{b_2}{2} + \frac{b_3}{2^2} + \cdots \frac{b_p}{2^{p-1}})\frac{1}{2^{p+1}} + (b_1 + \frac{b_2}{2} + \frac{b_3}{2^2} + \cdots \frac{b_p}{2^{p-1}})\frac{1}{2^{2p+1}} + \cdots \\ = (b_1 + \frac{b_2}{2} + \frac{b_3}{2^2} + \cdots \frac{b_p}{2^{p-1}})\frac{2^{p-1}}{2^p - 1}\end{aligned} \tag{4}$$

Note that there is no 1-to-1 correspondence between the numbers c and walks on the square lattice. One cannot, for example, walk counter-clockwise around a unit square, which would require the steps turn left, turn left, turn left,... because the rules for the plotter do not permit two consecutive steps of turns in the same sense (since this requires adjacent zeros of the same parity i of their indices which cannot be established).

That is not necessarily a constraint to the artist, because one could generate a figure twice as large by insertion of 1's, so the walk around a 2×2 square could be generated by the decoding of [6, A000035]

$$1010101010101010\cdots \quad = \quad \overline{10} \quad \rightsquigarrow \quad c \quad = \quad 1/3. \tag{5}$$

Insertion of another pair of 1's in regular intervals generates a walk around a 4×4 square via

$$11101110111011101110\ldots \quad = \quad \overline{1110} \quad \rightsquigarrow \quad c \quad = \quad 14/15. \tag{6}$$

If there is the same number of left and right turns and if the period length is even, the figure may return to the horizontal line and look like an alley. Such an alley generated by a sequence of period $p = 20$ is shown in Figure 1, and by another sequence of period $p = 3$ in figure 2.

An example with a period length of $p = 15$ is Figure 3, drawn with sequence A011659 of the OEIS [6].

If the sequence has a period of even length and does not have matching numbers of 1's that let the graph return to the horizontal line, the pattern repeats but moves away from the horizontal line at some average angle (Figure 4).

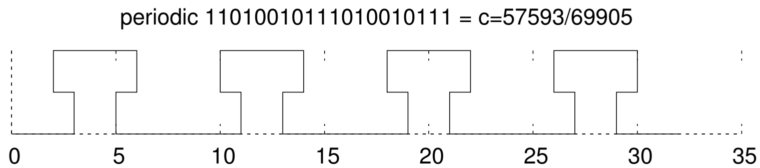

FIGURE 1. An infinite series of T-shapes from a periodic binary with essentially two left turns, four right turns and two left turns, where marked by the 8 zeros. The first four periods are shown.

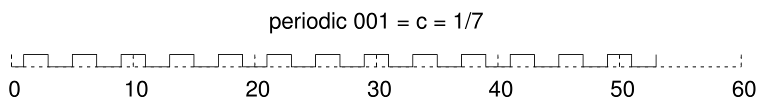

FIGURE 2. An infinite series of pulses from a binary with period 001, sequence A079978 [6].

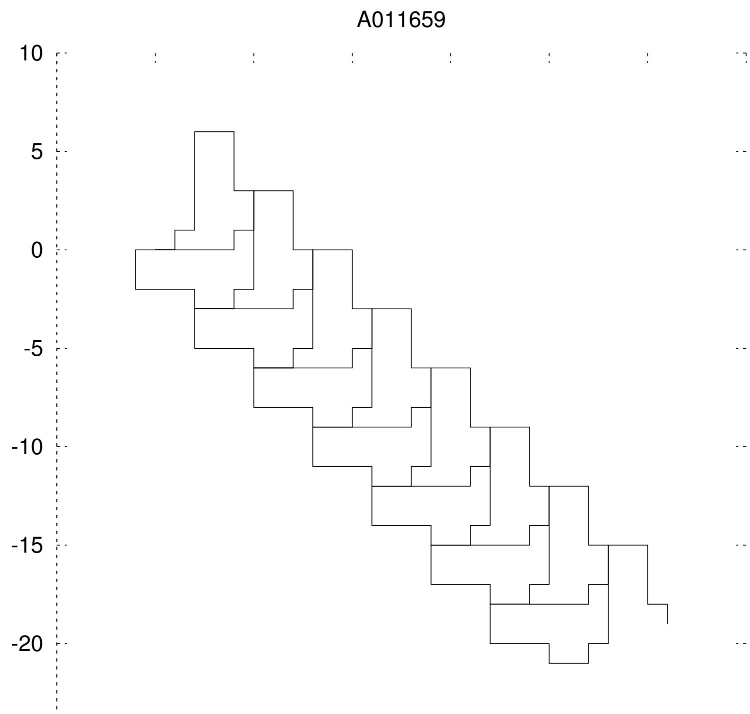

FIGURE 3. A shape generated from the periodic $\overline{000111101011001}$.

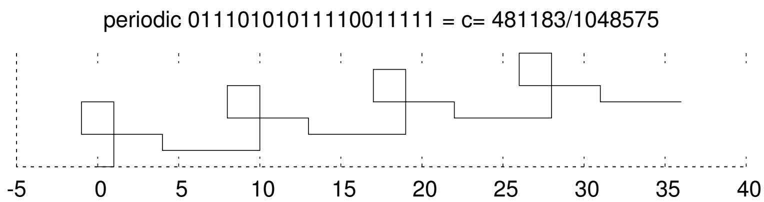

FIGURE 4. This pattern from a sequence with period $p = 20$ repeats with a sloping base.

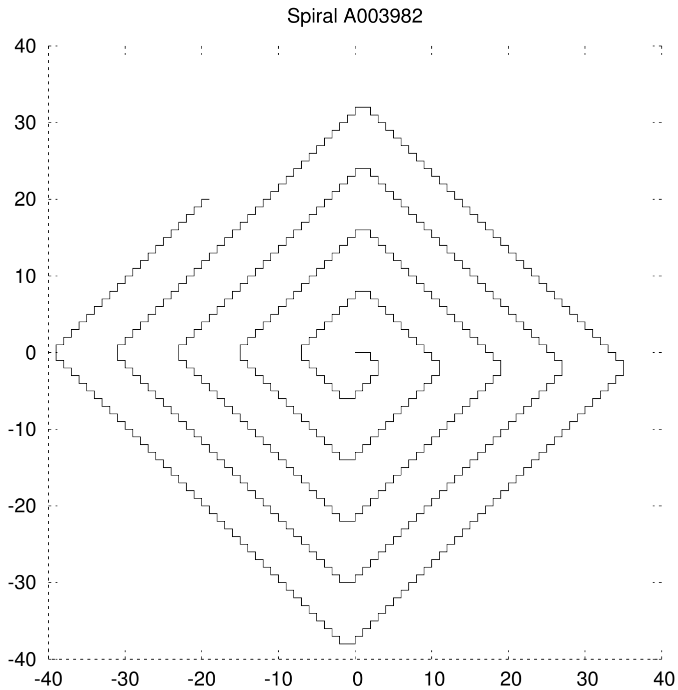

FIGURE 5. The figure from 10001000000010000000000... [6, A003982]

3. SPIRALS

Walks along spirals are generated by aperiodic binary sequences with runs of 1's or 0's which become longer after some regular number of turns. If the distance between the spiral arms increases at a constant rate, the pivotal 0's or 1's between these runs at the turns must have indices i that are generated by second order polynomials.

Figure 5 shows a spiral walk defined by a binary sequence where 1's settle only at positions $i = 1, 5, 13, \ldots$ as listed in [6, A001844], computed by the polynomial $i = 2n(n + 1) + 1$ at $n = 0, 1, 2, \ldots$. The associated constant is [6, A190406]

$$c = \sum_{n\geq 0} 1/2^{2n(n+1)+1} = \frac{1}{2}\sum_{n\geq 0} 1/4^{n(n+1)} = \frac{1}{4}\frac{1}{(1/4)^{1/4}} 2(1/4)^{1/4} \sum_{n\geq 0} (\frac{1}{4})^{n(n+1)}$$
$$= \frac{1}{2^{3/2}}\vartheta_2(0, 1/4) \approx \frac{1}{2^{3/2}} \times 1.502947261299 \approx 0.531372100115277\ldots \tag{7}$$

in terms of a Jacobi Theta Functions [1, 16.27.2].

The spiral in Figure 6 is created by a binary sequence with zeros only at positions $b_i = 0$ where $i = 1, 3, 5, 9, 13, 19, \ldots$ are the elements of the OEIS sequence A080827 [6].

There are two subsequences of i, the one is $i = 1, 5, 13\ldots$ already mentioned in the previous example. The other subsequence is $i = 3, 9, 19, \ldots$ shown in [6,

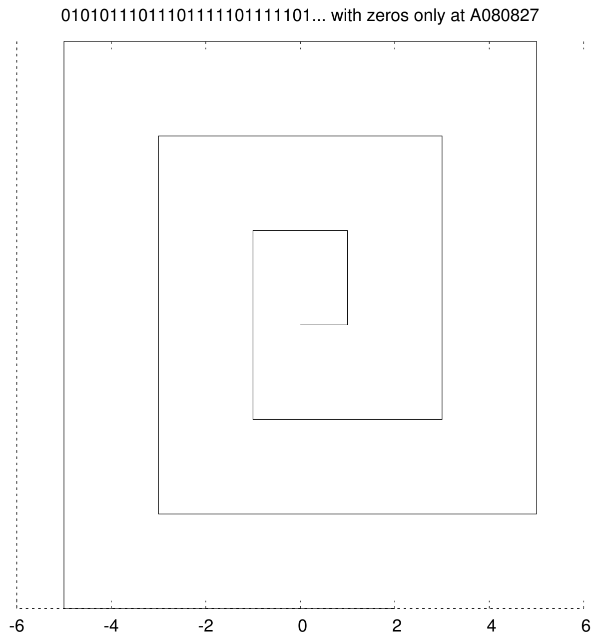

FIGURE 6. A spiral from a sequence with positions of zeros at 1, 3, 5, 9, 13, 19,... with first order differences increasing by 2 at regular intervals.

A058331] which defines the constant [1, 16.27.3]

$$\frac{1}{2^3}+\frac{1}{2^9}+\cdots=\sum_{n\ge 1}\frac{1}{2^{2n^2+1}}=\frac{1}{2}\sum_{n\ge 1}\frac{1}{4^{n^2}}=\frac{1}{4}2\sum_{n\ge 1}\frac{1}{4^{n^2}}=\frac{1}{4}[\vartheta_3(0,1/4)-1] \approx \frac{1}{4}[1.50782012986019431366500 2\ldots-1]. \tag{8}$$

Combined with (7), the constant associated with Figure 6 is

$$c=1-(\frac{1}{2^1}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^9}+\ldots)=1-\left\{\frac{1}{2^{3/2}}\vartheta_2(0,1/4)+\frac{1}{4}[\vartheta_3(0,1/4)-1]\right\} \approx 0.341672867419674286103869 59\ldots \tag{9}$$

"Avoided" spirals appear if the turns are alternatingly right and left for a quadratic binary. An example is Figure 7 from the sequence 1001000010000001000000001 0... of the characteristic function of squares [6, A010052].

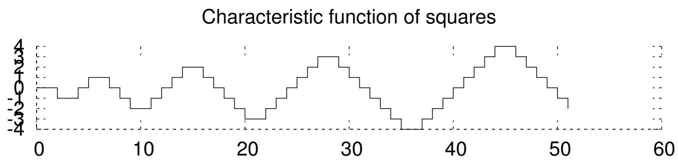

FIGURE 7. Walk with $b_i = 1$ if i is a square, $b_i = 0$ otherwise.

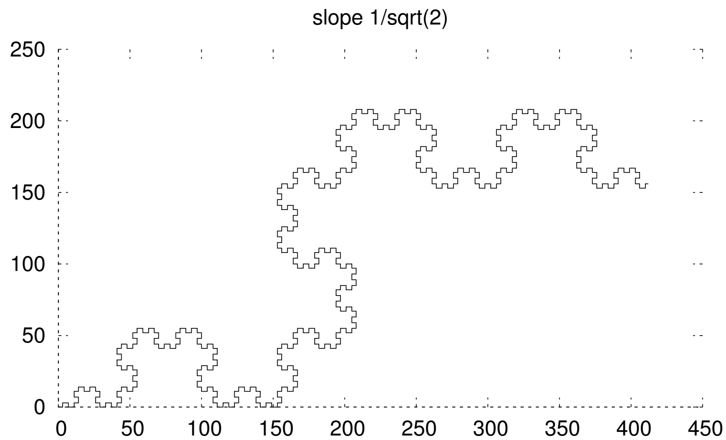

FIGURE 8. The figure from A080764, slope of $1/\sqrt{2}$.

4. First differences of Beatty sequences

Define Beatty sequences of irrational numbers α by

$$B(n) \equiv \lfloor n\alpha \rfloor, \quad n \geq 0, \tag{10}$$

which have average slope α as a function of n. For $\alpha < 1$, the first differences form a binary sequence (called the characteristic sequence of slope α [5]),

$$b_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor. \tag{11}$$

For some inverse square roots α, the walks in that family of b_i are displayed in Figures 8–17.

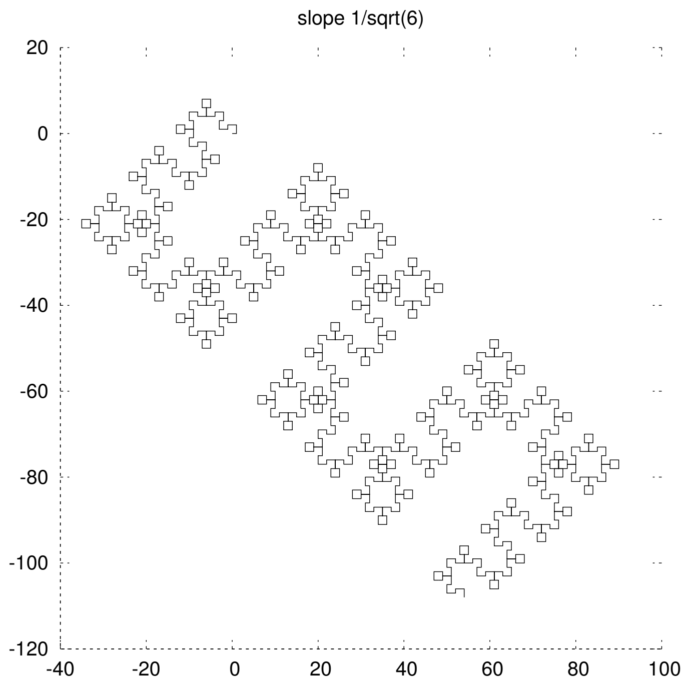

FIGURE 9. Slope of $\alpha = 1/\sqrt{6}$ in (11).

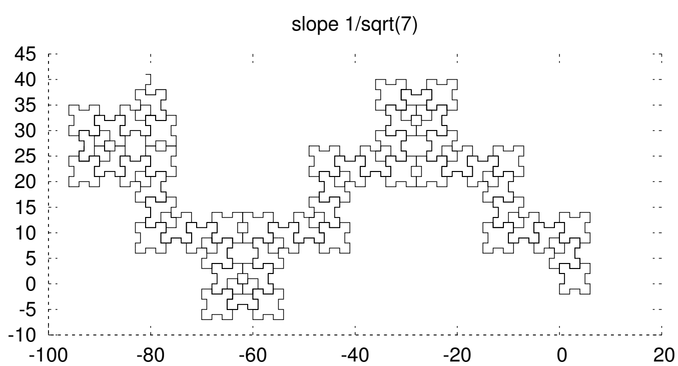

FIGURE 10. Slope of $\alpha = 1/\sqrt{7}$ in (11).

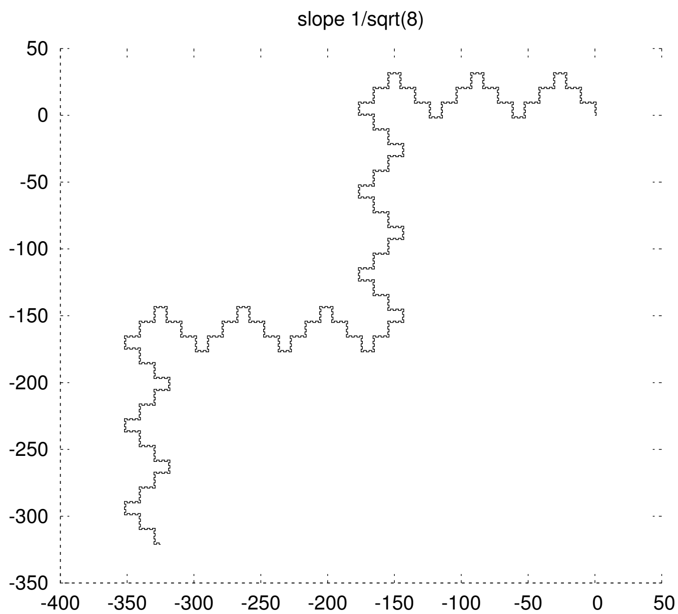

FIGURE 11. Slope of $\alpha = 1/\sqrt{8}$ in (11).

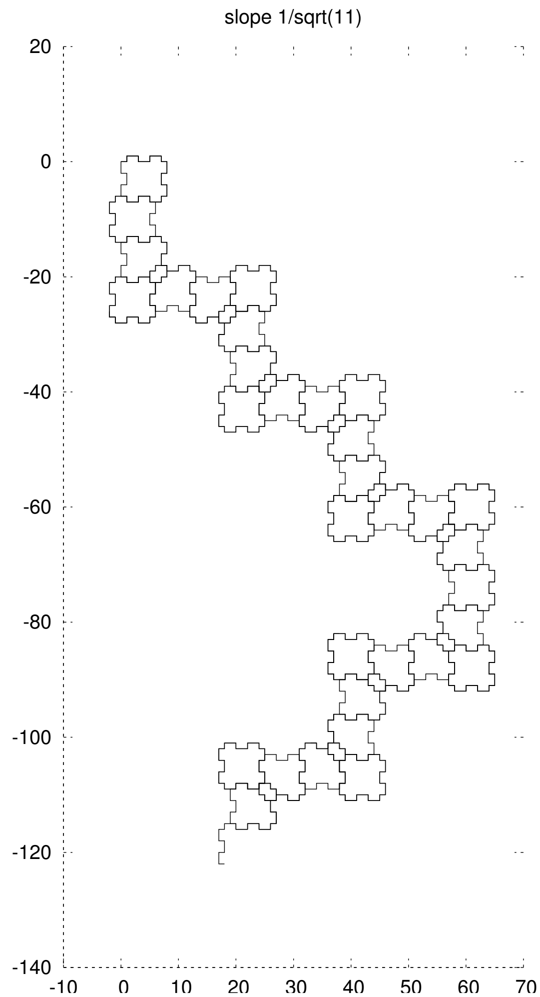

FIGURE 12. Slope of $\alpha = 1/\sqrt{11}$ in (11).

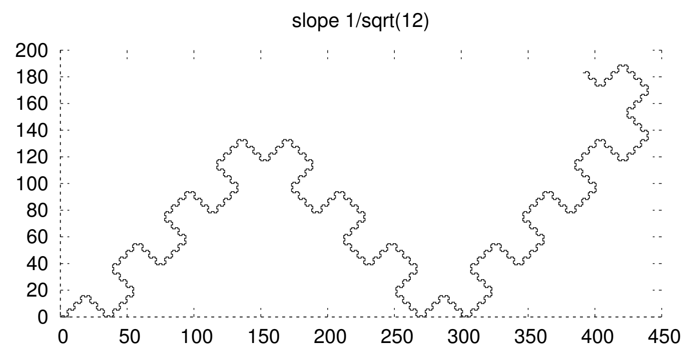

FIGURE 13. Slope of $\alpha = 1/\sqrt{12}$ in (11).

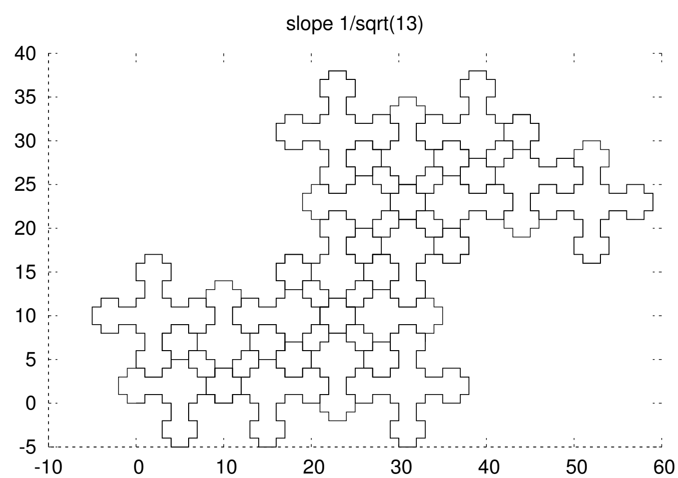

FIGURE 14. Slope of $\alpha = 1/\sqrt{13}$ in (11).

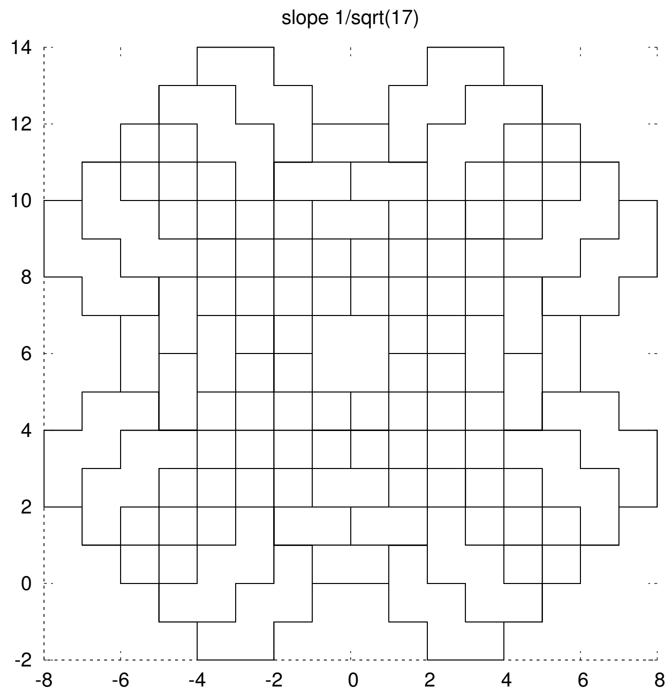

FIGURE 15. Slope of $\alpha = 1/\sqrt{17}$ in (11).

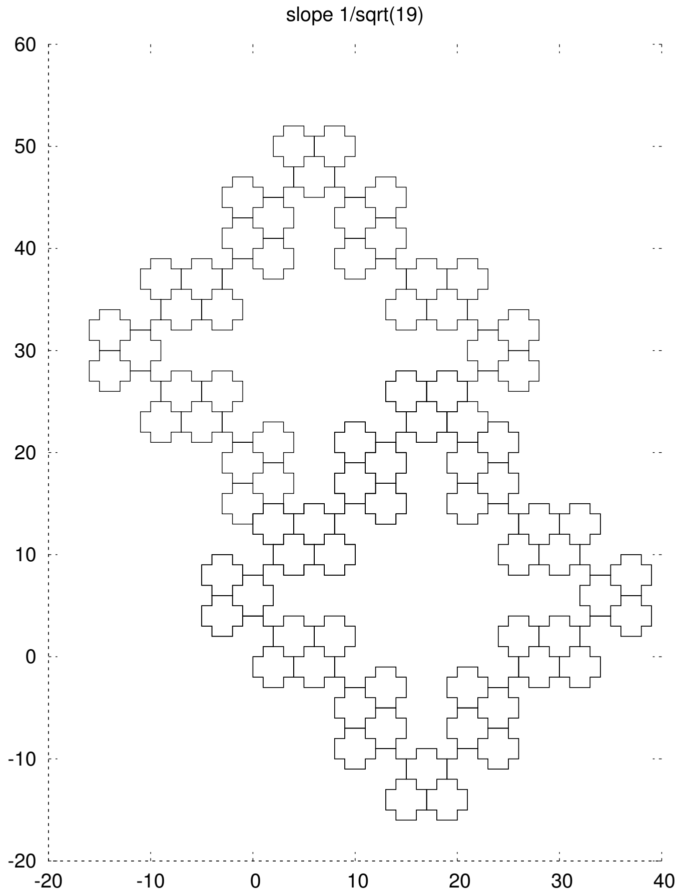

FIGURE 16. Slope of $\alpha = 1/\sqrt{19}$ in (11).

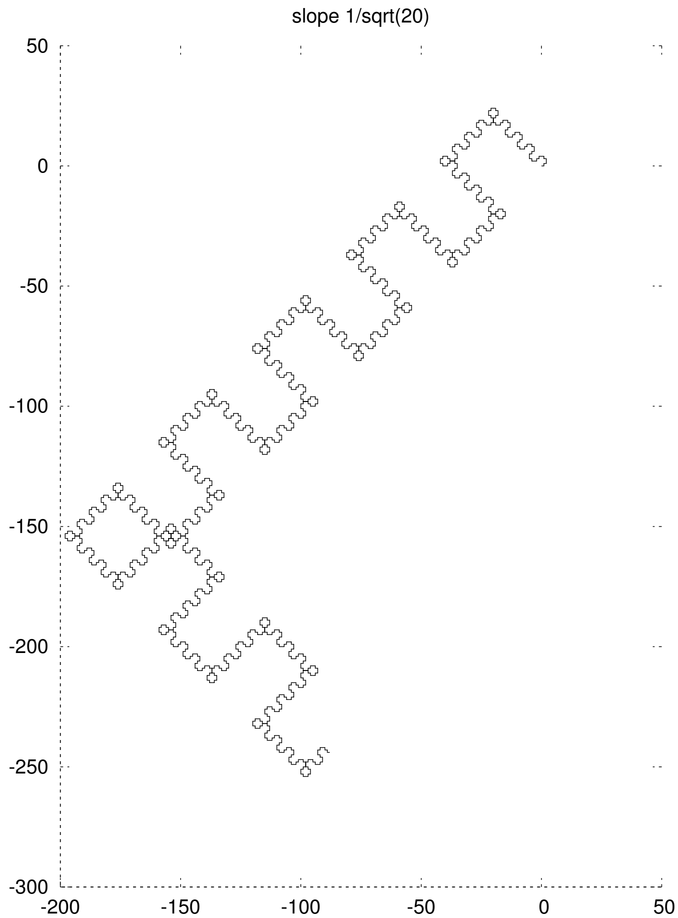

FIGURE 17. Slope of $\alpha = 1/\sqrt{20}$ in (11).

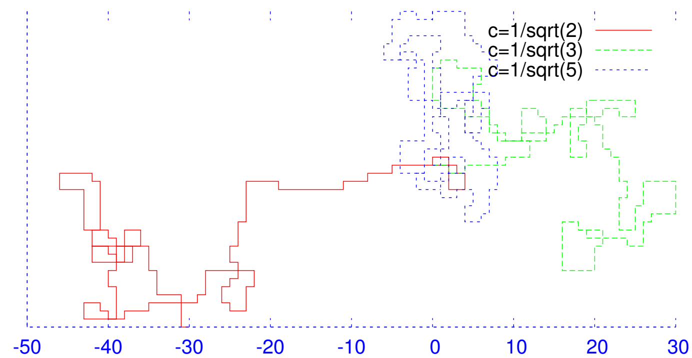

FIGURE 18. The sequences of binary expansions of $1/\sqrt{n}$ for $n = 2, 3$ and 5. They start 10110101000001... ([6, A004539]), 100100111100110100111... and 01110010011111001001...

The purpose of Figures 18–20 is to show that these square roots themselves do not generate regular patterns. They are nice proposals for building baseplates in the Bauhaus style.

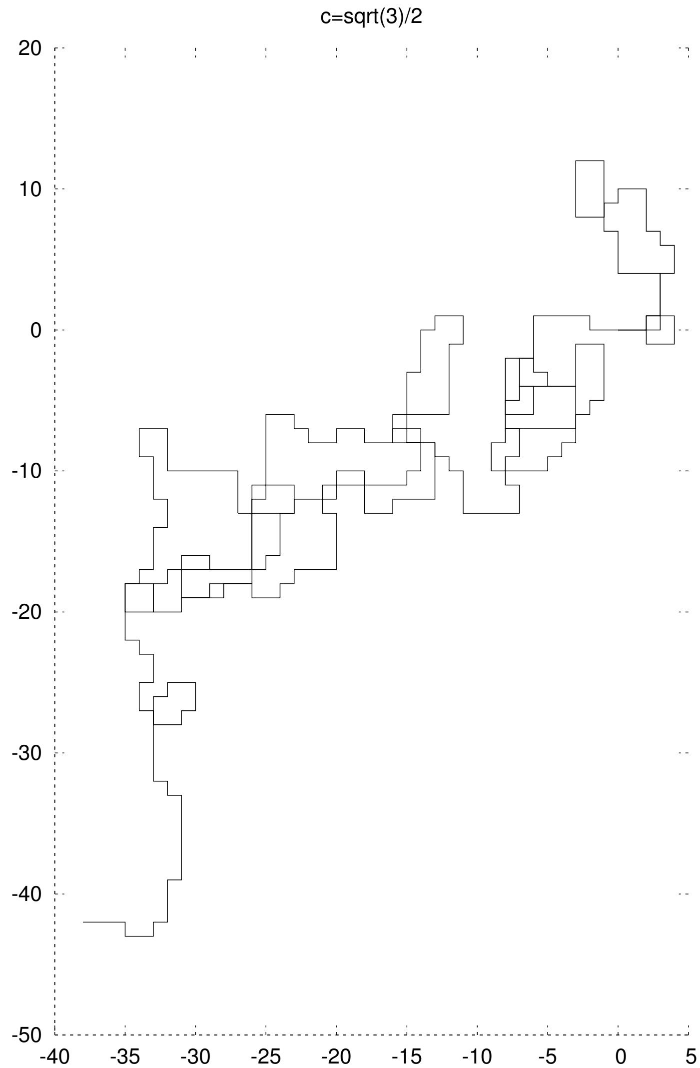

FIGURE 19. The sequence of the binary expansion of $\sqrt{3}/2$ 11011101101100111101 01... [6, A004547].

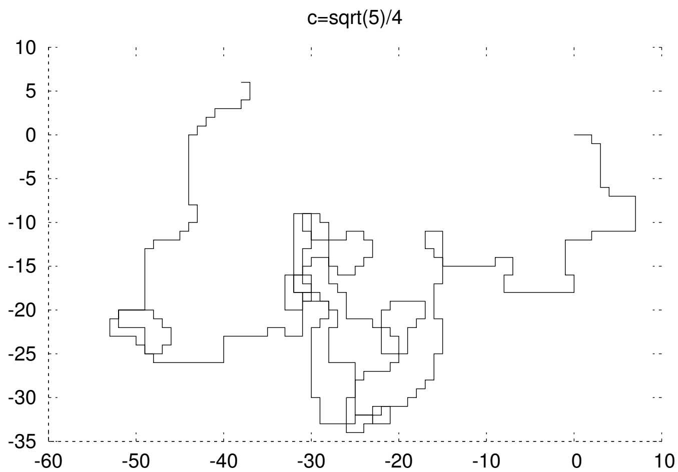

FIGURE 20. The sequence of the binary expansion of $\sqrt{5}/4$ 10001111000110111101111... [6, A004555].

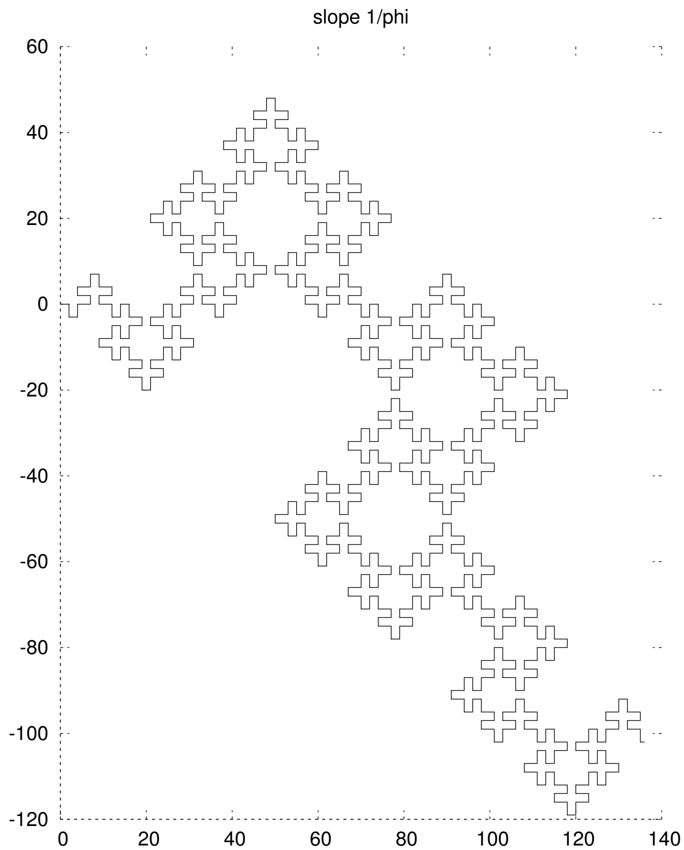

FIGURE 21. Slope of $1/\varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

Figure 21 can be constructed by insertion of the inverse golden ratio into (11), or by using the binary digits of the Rabbit Constant [6, A014565,A005614][3, §10.10][4]

$$1011010110110101101101\ldots \rightsquigarrow c \approx 0.709803442861291314641 78\ldots \tag{12}$$

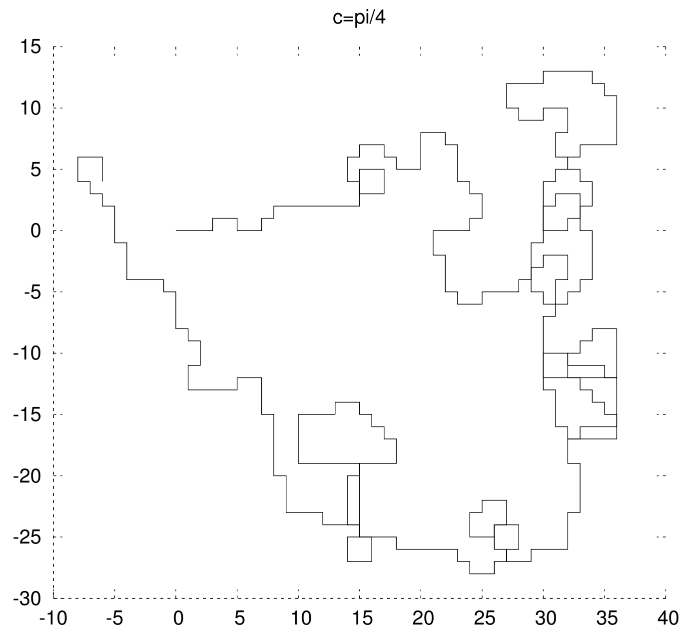

FIGURE 22. Walk of the sequence of the binary expansion of $c = \pi/4$, 1100100100001111110... [6, A004601]

5. Important irrationals

Curiosity about the walks generated by the binary expansions of π and $\log 2$ arises due to some regularity in their expansion in base 16 [2]. Figures 22 and 23, however, do not display any striking features.

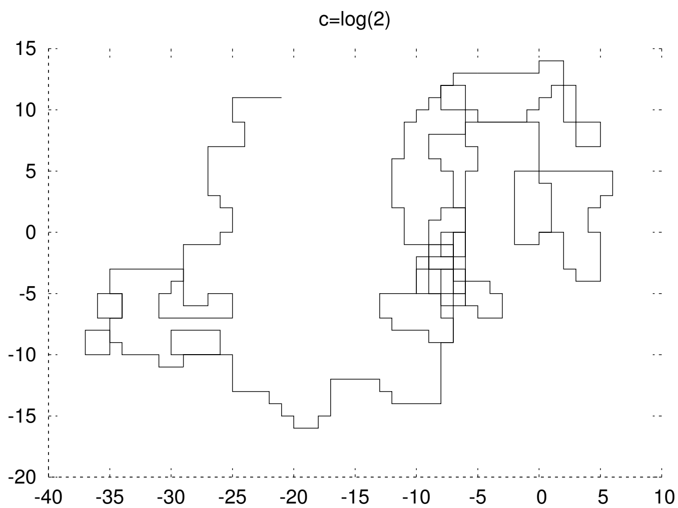

FIGURE 23. Walk of the sequence of the binary expansion of $c = \ln 2$, 101100010111001000. . . [6, A068426]

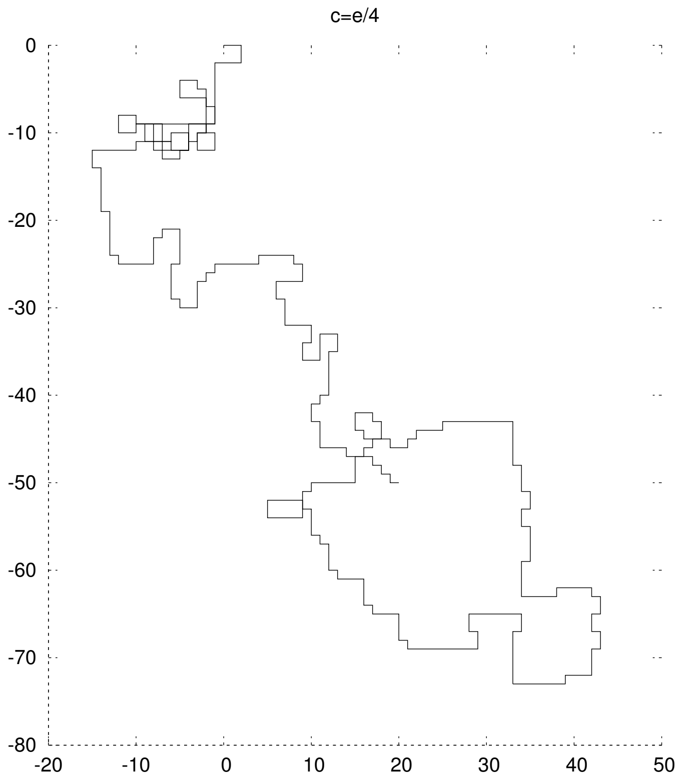

FIGURE 24. From the sequence of the binary expansion of $c = e/4$, 101011011111100001010... [6, A004593]

Not much is expected from the binary expansions of constants like the base of the natural logarithm (Figure 24) or the Euler-Mascheroni constant (Figure 25).

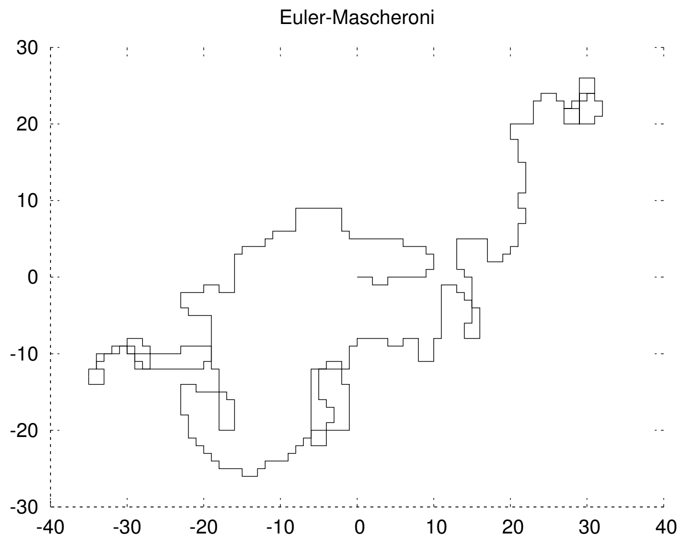

FIGURE 25. From the sequence of the binary expansion of $c = \gamma$, 1001001111000100011001111. . . [6, A104015]

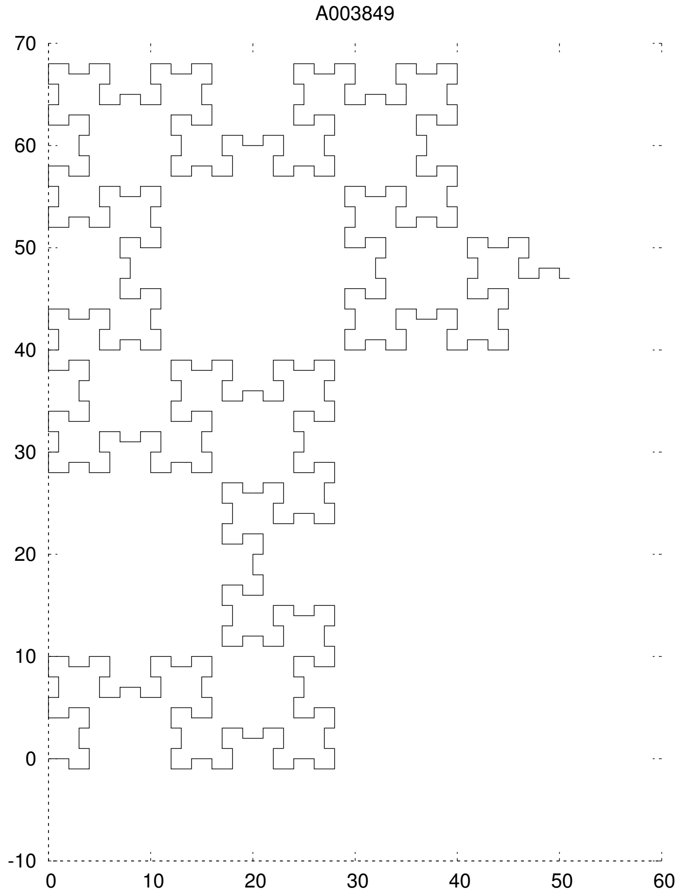

FIGURE 26. The Fibonacci Word defined in [6, A003849].

6. Fibonacci Words

Self-similarity is basically enforced by binary sequences generated by morphisms and Fibonacci words. An illustration of this kind is Figure 26 with a sequence

$$0100101001001010010\ldots;\, c \approx 0.2901965571387086853582 \tag{13}$$

define by applying ad infinitum the morphism $0 \to 01$ and $1 \to 0$ starting from a single 0.

Another example is Figure 27 with much richer over-plotting, representing $c \approx 0.58266130570818070796381$.

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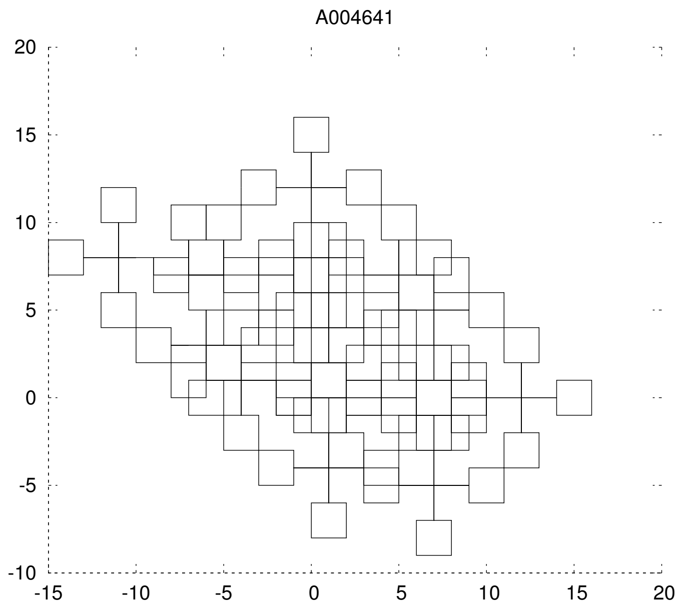

FIGURE 27. Walk by the binary sequence of the fixed point of $0 \to 10$, $1 \to 100$, 1001010100101001010... [6, A004641]

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