Expected Lifetimes and Inradii

Steven Finch

February 23, 2005

In earlier essays [1, 2], we examined 1-dimensional Brownian motion starting at 0; here, we generalize. A d-dimensional stochastic process $\{W_t : t \geq 0\}$ is a **Brownian motion** with *arbitrary* starting point W_0 if the component processes

$$W_{t,1} - W_{0,1},\ W_{t,2} - W_{0,2},\ \ldots,\ W_{t,d} - W_{0,d}$$

are independent 1-dimensional Brownian motions starting at 0 and, further, are independent of $W_{0,1}$, $W_{0,2}$, $\ldots$, $W_{0,d}$.

It is remarkable that d-dimensional Brownian motion can be used to represent the solution of the heat PDE [3, 4]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u, & t \geq 0,\ \xi \in \mathbb{R}^d, \\ u(0,\xi) = f(\xi), & f : \mathbb{R}^d \to \mathbb{R} \text{ piecewise continuous} \end{cases}$$

in the following sense:

$$\begin{aligned} u(t,\xi) &= \mathrm{E}\left(f(W_t) \mid W_0 = \xi\right) \\ &= \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(\omega) \exp\left(-\frac{|\xi - \omega|^2}{2t}\right) d\omega. \end{aligned}$$

As a corollary, if f is the Dirac impulse at 0, then u simplifies to

$$u(t,\xi) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\xi|^2}{2t}\right);$$

that is, the *heat kernel* coincides with the *Brownian transition density* starting at 0.

Also, let D denote an open, simply connected domain in $\mathbb{R}^d$ with piecewise smooth, closed, orientable boundary C. The solution of the Laplace PDE (Dirichlet boundary value problem):

$$\begin{cases} \triangle v = 0, & \xi \in D, \\ v(\xi) = g(\xi), & \xi \in C,\ g : C \to \mathbb{R} \text{ piecewise continuous} \end{cases}$$

can be written as

$$v(\xi) = \mathrm{E}\left(g(W_\tau) \,|\, W_0 = \xi\right)$$

where τ is the **lifetime** or **first exit time** of Brownian motion in D:

$$\tau = \inf\left\{t > 0 : W_t \notin D\right\}.$$

Consequently, if $C = C_0 \cup C_1$, $C_0 \cap C_1 = \emptyset$ and $g(\xi) = k$ for $\xi \in C_k$, then $v(\xi)$ is the probability that a Brownian particle which starts at $\xi \in D$ stops at some point $\eta \in C_1$.

These two examples are special cases of a more general principle that solutions of any parabolic or elliptic PDE can be represented as expectations of certain stochastic functionals. (A hyperbolic PDE such as the wave equation $\partial^2 u/dt^2 = (1/2)\triangle u$ apparently cannot be solved in this manner.)

So far we have seen how probability is a servant of analysis. An example of how analysis serves probability is that the expected lifetime $v(\xi) = \mathrm{E}\left(\tau \,|\, W_0 = \xi\right)$ satisfies the Poisson PDE

$$\begin{cases} \triangle v = -2, & \xi \in D, \\ v(\xi) = 0, & \xi \in C. \end{cases}$$

For instance, if D is the ball of radius r in $\mathbb{R}^d$ centered at 0, then $v_D(\xi) = (r^2 - |\xi|^2)/d$. In the remainder of this essay, let $d = 2$. If T is the equilateral triangular region in $\mathbb{R}^2$ with vertices $(0, 2a/3)$, $(\pm a/\sqrt{3}, -a/3)$, then

$$v_T(x,y) = \frac{1}{2a}\left(y - \sqrt{3}x - \frac{2}{3}a\right)\left(y + \sqrt{3}x - \frac{2}{3}a\right)\left(y + \frac{1}{3}a\right).$$

If S is the square region in $\mathbb{R}^2$ with vertices $(\pm b, \pm b)$, then [5]

$$v_S(x,y) = \frac{32b^2}{\pi^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\left[1 - \operatorname{sech}\left(\frac{(2k+1)\pi}{2}\right)\cosh\left(\frac{(2k+1)\pi y}{2b}\right)\right]\cos\left(\frac{(2k+1)\pi x}{2b}\right).$$

The lifetime functions $v_D(x,y)$, $v_T(x,y)$ and $v_S(x,y)$ are each maximized when $x = y = 0$. Define, for $b = 1/2$,

$$\gamma = v_S(0,0) = \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\left[1 - \operatorname{sech}\left(\frac{(2k+1)\pi}{2}\right)\right] = 0.1473427065....$$

This constant will be useful in the following; we wonder whether it has a closed-form expression.

When $r = 1/\sqrt{\pi}$, $a = \sqrt[4]{3}$ and $b = 1/2$, each of D, T and S have area 1 and

$$v_D(0,0) = \tfrac{1}{2\pi} = 0.159... > v_S(0,0) = \gamma = 0.147... > v_T(0,0) = \tfrac{2\sqrt{3}}{27} = 0.128....$$

In fact, among all planar regions of fixed area, the disk possesses the longest lifetime [6]. No such region with shortest lifetime exists, for consider the $c \times (1/c)$ finite strip as $c \to \infty$.

When $r = 1$, $a = 3$ and $b = 1$, each of D, T and S have inradius 1 (meaning the radius of the largest inscribed disk is unity) and

$$v_D(0,0) = \tfrac{1}{2} = 0.5 < v_S(0,0) = 4\gamma = 0.589... < v_T(0,0) = \tfrac{2}{3} = 0.666....$$

Clearly, among all planar regions of fixed inradius, the disk possesses the shortest lifetime. By way of contrast with the preceding, finding such a region with longest lifetime is an unsolved problem. Let

$$K = \sup_{D} \sup_{(x,y)\in D} \mathrm{E}\left(\tau \,|\, W_0 = (x,y)\right),$$

where the outer supremum is over all simply connected domains D in $\mathbb{R}^2$ of unit inradius; thus $K \geq 2/3$. The $2 \times \infty$ infinite strip improves this inequality to $K \geq 1$ and is the best such convex domain [7, 8]. Bañuelos & Carroll [9, 10] demonstrated that $1.584 < K < 3.228$; they speculated that the associated nonconvex domain D is extremal for certain other optimization problems as well.

0.1. Fundamental Drum Frequency. The bass tone of a kettledrum, whose head shape is a simply connected domain D in $\mathbb{R}^2$, is the square root of the smallest eigenvalue λ of [11, 12]

$$\begin{cases} \triangle u = -\lambda\, u, & \xi \in D, \\ u(\xi) = 0, & \xi \in C. \end{cases}$$

For instance, if D is the disk of radius r centered at $(0,0)$, then the first eigenfunction/eigenvalue pair is

$$u_D(x,y) = J_0\left(\frac{j_0\sqrt{x^2+y^2}}{r}\right), \qquad \lambda_D = \left(\frac{j_0}{r}\right)^2$$

where $J_0(z)$ is the zeroth Bessel function of the first kind and $j_0 = 2.4048255576...$ is its smallest positive zero. If T is the equilateral triangular region of height a centered at $(0, a/6)$, then [13, 14]

$$u_T(x,y) = \sin\left(\frac{\pi}{a}\left(y - \sqrt{3}x - \frac{2}{3}a\right)\right) + \sin\left(\frac{\pi}{a}\left(y + \sqrt{3}x - \frac{2}{3}a\right)\right) - \sin\left(\frac{2\pi}{a}\left(y + \frac{1}{3}a\right)\right),$$

$$\lambda_T = \frac{4\pi^2}{a^2}.$$

If S is the square region of side $2b$ centered at $(0,0)$, then

$$u_S(x,y)=\cos\left(\frac{\pi x}{2b}\right)\cos\left(\frac{\pi y}{2b}\right), \qquad \lambda_S=\frac{\pi^2}{2b^2}.$$

When D, T and S each have area 1,

$$\lambda_D=\pi j_0^2=18.168...<\lambda_S=2\pi^2=19.739...<\lambda_T=\tfrac{4\pi^2}{\sqrt{3}}=22.792....$$

The Faber-Krahn inequality states that, among all planar regions of fixed area, the disk possesses the lowest bass tone. No such region with highest bass tone exists, for consider the $c\times(1/c)$ finite strip as $c\to\infty$.

When D, T and S each have inradius 1,

$$\lambda_D=j_0^2=5.783...>\lambda_S=\tfrac{\pi^2}{2}=4.934...>\lambda_T=\tfrac{4\pi^2}{9}=4.386....$$

Clearly, among all planar regions of fixed inradius, the disk possesses the highest bass tone. Finding such a region with lowest bass tone is an unsolved problem. Let

$$\Lambda=\inf_D\ \lambda_D$$

where the infimum is over all simply connected domains D in $\mathbb{R}^2$ of unit inradius; thus $\Lambda\leq 4\pi^2/9$. The $2\times\infty$ infinite strip improves this inequality to $\Lambda\leq\pi^2/4=2.467...$ and is the best such convex domain [15, 16, 17]. In the other direction, Makai [18, 19, 20, 21, 22] proved that $\Lambda\geq 1/4$. The best bounds currently known [9] are $0.6197<\Lambda<2.1292$ and the associated nonconvex domain D is conjectured to be the same as before.

What does this have to do with Brownian motion? We give just one (of several) formulas [10, 23]:

$$\Lambda_D=2\ \sup\left\{c\geq 0:\sup_{(x,y)\in D}\mathrm{E}\left(e^{c\tau}\,|\,W_0=(x,y)\right)<\infty\right\}$$

for bounded, simply connected D. In words, the fact that $\lambda_D\geq\Lambda/\rho^2>0$ for D of inradius ρ means that if a drum produces an arbitrarily low bass tone, then it must contain an arbitrarily large circular subdrum.

0.2. Torsional Rigidity. Let us return to the expected lifetime function $v(x,y)$ and evaluate not its maximum value in the domain D, but rather twice its average value

$$\mu=\frac{2}{\mathrm{area}(D)}\int\limits_D\mathrm{E}\left(\tau\,|\,W_0=(x,y)\right)dx\,dy.$$

For instance, if D is the disk of radius r centered at $(0,0)$, then $\mu_D = r^2/2$. If T is the equilateral triangular region of height a centered at $(0, a/6)$, then $\mu_T = a^2/15$. If S is the square region of side $2b$ centered at $(0,0)$, then [5]

$$\begin{aligned}\mu_S &= \frac{4b^2}{3}\left[1 - \frac{192}{\pi^5}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^5}\tanh\left(\frac{(2k+1)\pi}{2}\right)\right] \\ &= \frac{1}{4}b^2(2.2492322392...) = b^2(0.5623080598...) = 4b^2(0.1405770149...).\end{aligned}$$

Again, we wonder about the possibility of closed-form evaluation.

When $r = 1/\sqrt{\pi}$, $a = \sqrt[4]{3}$ and $b = 1/2$,

$$\mu_D = \tfrac{1}{2\pi} = 0.159... > \mu_S = 0.140... > \mu_T = \tfrac{\sqrt{3}}{15} = 0.115....$$

This can be expressed in the language of elasticity theory. Pólya [24, 25, 26, 27] proved Saint Venant's conjecture that, among all cylindrical beams of prescribed cross-sectional area, the circular beam has the highest *torsional rigidity*. No such beam with lowest torsional rigidity exists, for consider the $c \times (1/c)$ rectangle as $c \to \infty$.

When $r = 1$, $a = 3$ and $b = 1$,

$$\mu_D = \tfrac{1}{2} = 0.5 < \mu_S = 0.562... < \mu_T = \tfrac{3}{5} = 0.6.$$

Among all cylindrical beams of prescribed cross-sectional inradius, the circular beam has the lowest normalized torsional rigidity (normalized by area, as defined earlier). Finding such a beam with highest normalized torsional rigidity is an unsolved problem. Let

$$M = \sup_D \mu_D$$

where the supremum is over all simply connected domains D in $\mathbb{R}^2$ of unit inradius; thus $M \geq 3/5$. The $2 \times c$ rectangle improves this inequality, as $c \to \infty$, to $M \geq 4/3$ and is the best such convex domain [28]. For nonconvex domains, we have the upper bound 6.456 [9], but little else is known about this problem.

0.3. Conformal Mapping. If E is an open, simply connected region in $\mathbb{C}$, define $\rho(E)$ to be the inradius of E. The **univalent Bloch-Landau constant** Θ is given by [29]

$$\Theta = \inf_f \rho(f(D))$$

where the infimum is over all one-to-one analytic functions f defined on the open unit disk D satisfying $f(0) = 1$, $f'(0) = 1$. Let g denote the conformal mapping of

D onto the infinite strip $-\pi/4 < \text{Im}(z) < \pi/4$:

$$g(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1},$$

hence $\Theta \geq \pi/4$. Szegö [30, 31] further proved that, if $f(D)$ is convex, then $\rho(f(D)) \leq \rho(g(D))$. For the nonconvex scenario, the best bounds currently known [9, 32, 33] are $0.57088 < \Theta < 0.65642$ and the associated nonconvex region $f(D)$ is conjectured to be the same as the nonconvex domain for the constants K and Λ.

0.4. Addendum. The constant γ indeed has a closed-form expression [34, 35]:

$$\gamma = 4 \frac{{}_4F_3\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}; \frac{5}{4}, \frac{5}{4}, 1; 1\right)}{B\left(\frac{1}{4}, \frac{1}{2}\right)^2} = 0.1473427065... = \frac{1}{2}(0.2946854131...)$$

where ${}_pF_q$ is the generalized hypergeometric function [36] and B is the Euler beta function ($B(x,y) = I(1,x,y)$ in [37]). An interesting double series representation:

$$\gamma = \frac{32}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1)\left[(2m-1)^2 + (2n-1)^2\right]}$$

follows from a formula in [38] which, in turn, was corrected in [39]. See also [40].

Both λ and μ can be defined via the calculus of variations [26]. It is more customary to take area$(D)\mu$ as torsional rigidity and this is equal to [41, 42]

$$\frac{1}{12} - \frac{16}{\pi^5} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} \coth\left(\frac{(2k+1)\pi}{2}\right) = 0.0260896517...$$

for an isosceles right triangle with sides 1, 1, $\sqrt{2}$ and [43, 44]

$$\begin{aligned} & 9\left[\frac{17\sqrt{3}}{192} - \frac{1}{\pi^5} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} \left\{2\tanh\left(\frac{(2k+1)\pi\sqrt{3}}{2}\right) - 9\tanh\left(\frac{(2k+1)\pi}{2\sqrt{3}}\right) + \right.\right. \\ & \left.\left. (-1)^k 9\sqrt{3}\,\text{sech}\left(\frac{(2k+1)\pi}{2\sqrt{3}}\right) + 27\sqrt{3}\sin\left(\frac{(2k+1)\pi}{3}\right)\right\}\right] \\ = \; & 0.0044516625... = \frac{9}{16}(0.0079140667...) \end{aligned}$$

for a 30°-60°-90° triangle with sides 1/2, $\sqrt{3}/2$ and 1. The corresponding value for a regular hexagon of unit side has attracted considerable attention [45, 46, 47, 48] – see history in [42] – a complicated formula in [49] gives ≈ 1.035459, as reported in [50], and verifies an unpublished calculation in [51].

References

[1] S. R. Finch, Ornstein-Uhlenbeck process, unpublished note (2004).

[2] S. R. Finch, Variants of Brownian motion, unpublished note (2004).

[3] R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth, 1984, pp. 36–39, 219–270; MR0750829 (87a:60054).

[4] R. Durrett, *Stochastic Calculus: A Practical Introduction*, CRC Press, 1996, pp. 125–166; MR1398879 (97k:60148).

[5] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed, McGraw-Hill, 1970, pp. 297–301, 309–315; MR0045547 (13,599i).

[6] R. Bañuelos, Queen Dido's property for eigenvalues and Brownian motion, unpublished lecture (2001).

[7] L. E. Payne, Bounds for the maximum stress in the Saint Venant torsion problem, *Indian J. Mech. Math.,* special issue (1968/69), part I, 51–59; MR0351225 (50 #3714).

[8] R. P. Sperb, *Maximum Principles and their Applications*, Academic Press, 1981, pp. 85–89; MR0615561 (84a:35033).

[9] R. Bañuelos and T. Carroll, Brownian motion and the fundamental frequency of a drum, *Duke Math. J.* 75 (1994) 575–602; addendum 82 (1996) 227; MR1291697 (96m:31003) and MR1387227 (97f:31004).

[10] T. Carroll, Old and new on the bass note, the torsion function and the hyperbolic metric, *Irish Math. Soc. Bull.* 47 (2001) 41–65; MR1880191 (2002k:58072).

[11] S. R. Finch, Sobolev isoperimetric constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 219–225.

[12] S. R. Finch, Bessel function zeroes, unpublished note (2003).

[13] M. A. Pinsky, The eigenvalues of an equilateral triangle, *SIAM J. Math. Anal.* 11 (1980) 819-827; 16 (1985) 848-851; MR0586910 (82d:35077) and MR0793926 (86k:35115).

[14] L. M. Cureton and J. R. Kuttler, Eigenvalues of the Laplacian on regular polygons and polygons resulting from their dissection, *J. Sound Vibration* 220 (1999) 83-98; MR1672258 (99j:35025).

[15] J. Hersch, Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum, *Z. Angew. Math. Phys.* 11 (1960) 387–413; MR0125319 (23 #A2622).

[16] R. Bañuelos, R. Latala and P. J. Méndez-Hernández, A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes, *Proc. Amer. Math. Soc.* 129 (2001) 2997–3008; MR1840105 (2002c:60125).

[17] P. J. Méndez-Hernández, Brascamp-Lieb-Luttinger inequalities for convex domains of finite inradius, *Duke Math. J.* 113 (2002) 93–131; MR1905393 (2003b:31005).

[18] E. Makai, A lower estimation of the principal frequencies of simply connected membranes, *Acta Math. Acad. Sci. Hungar.* 16 (1965) 319–323; MR0185263 (32 #2732).

[19] W. K. Hayman, Some bounds for principal frequency, *Applicable Anal.* 7 (1977/78) 247–254; MR0492339 (58 #11468).

[20] R. Osserman, A note on Hayman's theorem on the bass note of a drum, *Comment. Math. Helv.* 52 (1977) 545–555; MR0459099 (56 #17297).

[21] M. E. Taylor, Estimate on the fundamental frequency of a drum, *Duke Math. J.* 46 (1979) 447–453; MR0534061 (81g:35097).

[22] C. B. Croke, The first eigenvalue of the Laplacian for plane domains, *Proc. Amer. Math. Soc.* 81 (1981) 304–305; MR0593476 (82e:35061).

[23] S. E. Graversen and M. Rao, Brownian motion and eigenvalues for the Dirichlet Laplacian, *Math. Z.* 203 (1990) 699–708; MR1044073 (91f:35197).

[24] G. Pólya, Torsional rigidity, principal frequency, electrostatic capacity and symmetrization, *Quart. Appl. Math.* 6 (1948) 267–277; MR0026817 (10,206b).

[25] G. Pólya and A. Weinstein, On the torsional rigidity of multiply connected cross-sections, *Annals of Math.* 52 (1950) 154–163; MR0040159 (12,652e).

[26] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951, pp. 87–100, 121, 251–261; MR0043486 (13,270d).

[27] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, 1980, pp. 63–67, 158–159; MR0572958 (81e:35095).

[28] E. Makai, On the principal frequency of a membrane and the torsional rigidity of a beam, *Studies in Mathematical Analysis and Related Topics: Essays in Honor of George Pólya,* ed. G. Szegö, C. Loewner, S. Bergman, M. M. Schiffer, J. Neyman, D. Gilbarg, and H. Solomon, Stanford Univ. Press, 1962, pp. 227–231; MR0167004 (29 #4277).

[29] S. R. Finch, Bloch-Landau constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 456–459.

[30] G. Szegö, Über eine Extremalaufgabe aus der Theorie der schlichten Abbildungen, *Sitzungsberichte der Berliner Mathematische Gesellschaft* 22 (1923) 38–47; corrigendum 23 (1924) 64; *Collected Papers*, v. 1, ed. R. Askey, Birkhäuser Verlag, 1982, pp. 607-618.

[31] D. Minda, Lower bounds for the hyperbolic metric in convex regions, *Rocky Mountain J. Math.* 13 (1983) 61–69; MR0692577 (84j:30039).

[32] E. Beller and J. A. Hummel, On the univalent Bloch constant, *Complex Variables Theory Appl.* 4 (1985) 243-252; MR0801641 (86i:30005).

[33] S. Y. Zhang, On the schlicht Bloch constant, *Beijing Daxue Xuebao* 25 (1989) 537-540 (in Chinese); MR1045870 (91a:30014).

[34] G. Markowsky, On the expected exit time of planar Brownian motion from simply connected domains, *Elec. Commun. Probab.* 16 (2011) 652–663; arXiv:1108.1188; MR2853103 (2012j:60228).

[35] G. Markowsky, A method for deriving hypergeometric and related identities from the H^2 Hardy norm of conformal maps, *Integral Transforms Spec. Funct.* 24 (2013) 302–313; arXiv:1205.2458; MR3037253.

[36] S. R. Finch, Random triangles III, unpublished note (2010).

[37] S. R. Finch, Gambler's ruin, unpublished note (2008).

[38] F. B. Knight, *Essentials of Brownian Motion and Diffusion*, Amer. Math. Soc. 1981, pp. 70–71; MR0613983 (82m:60098).

[39] K. Helmes, S. Röhl and R. H. Stockbridge, Computing moments of the exit time distribution for Markov processes by linear programming, *Oper. Res.* 49 (2001) 516–530; MR1852358.

[40] M. W. Coffey, Expected exit times of Brownian motion from planar domains: Complements to a paper of Markowsky, arXiv:1203.5142.

[41] B. G. Galerkin, Torsion d'un prisme triangulaire, *Bulletin de l'Académie des Sciences de Russie* 13 (1919) 111–118.

[42] T. J. Higgins, A comprehensive review of Saint-Venant's torsion problem, *Amer. J. Phys.* 10 (1942) 248–259.

[43] G. E. Hay, The method of images applied to the problem of torsion, *Proc. London Math. Soc.* 45 (1939) 382–397; MR0000116 (1,21c).

[44] L.-C. Pan, Equilibrium, buckling and vibration of a 30°-60°-90°-triangular plate simply supported at the edges, *Sci. Sinica* 6 (1957) 347–379; MR0091635 (19,998c).

[45] T. Tsumura, On the torsion of prismatic cylinder with a regular polygonal section, *Journal of the Society of Mechanical Engineers (Tokyo)* 36 (1933) 60–63.

[46] J. L. Synge, *The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems*, Cambridge Univ. Press, 1957, pp. 214–270; MR0097605 (20 #4073).

[47] E. Romanelli, M. J. Maurizi and L. A. Patricio, Torsion of bars of regular polygonal cross section, *Journal of the Engineering Mechanics Division (ASCE)* 98 (1972) 415–424.

[48] W. C. Hassenpflug, Torsion of uniform bars with polygon cross-section, *Comput. Math. Appl.* 46 (2003) 313–392; MR2015439 (2004i:74069).

[49] B. R. Seth. Torsion of beams whose cross-section is a regular polygon of n sides, *Proc. Camb. Phil. Soc.* 30 (1934) 139–149.

[50] E. E. Jones, The torsional rigidity of anisotropic prisms, *J. Engineering Math.* 9 (1975) 39–51; MR0411328 (53 #15064).

[51] M. I. Aissen, *On the Estimation and Computation of Torsional Rigidity*, Ph.D. thesis, Stanford Univ., 1952.