Notes on 2-periodic continued fractions and Lehmer sequences

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We show that the sequence of numerators and the sequence of denominators of a 2-periodic continued fraction are (essentially) Lehmer sequences. Examples include A022998 and A026741, A203976 and A005013, A243470 and A243469.

Lehmer sequences

Let R and Q be nonzero integers such that $R-4Q \neq 0$ and let α, β be the distinct roots of the quadratic equation $x^2 - \sqrt{R}x + Q = 0$. The Lehmer sequence $(L_n)_{n>1}$ is an integer sequence defined by

$$L_n \equiv L_n(R, Q) = \begin{cases} (\alpha^n - \beta^n) / (\alpha - \beta) & \text{for } n \text{ odd} \\ (\alpha^n - \beta^n) / (\alpha^2 - \beta^2) & \text{for } n \text{ even.} \end{cases}$$
(1)

Lehmer sequences were introduced in [1] as a generalization of Lucas sequences [5]. Lehmer was mainly interested in the arithmetical properties of his sequences and restricted his attention to the case where both α and β were real, that is, he supposed both R and R-4Q were positive integers. For our limited purposes in this note we don't assume these conditions hold.

The Lehmer sequence $L_n(R,Q)$ begins

$$[1, 1, R - Q, R - 2Q, R^2 - 3RQ + Q^2, (R - Q)(R - 3Q), ...]$$
.

The sequence satisfies the pair of second-order linear recurrence equations: $L_0=0, L_1=1$ and for $n\geq 1$

$$L_{2n} = L_{2n-1} - QL_{2n-2}$$

$$L_{2n+1} = RL_{2n} - QL_{2n-1}.$$
(2)

The ordinary generating function for the Lehmer sequence is easily calculated from (1) as

$$\sum_{n\geq 1} L_n x^n = \frac{x(1+x+Qx^2)}{1-(R-2Q)x^2+Q^2x^4}$$
(3)

$$= \frac{x + Qx^3}{1 - (R - 2Q)x^2 + Q^2x^4} + \frac{x^2}{1 - (R - 2Q)x^2 + Q^2x^4}.$$
 (4)

as a sum of odd and even functions.

The Lehmer sequence $L_n(R,Q)$ may also be found by concatenating the first rows of the 2 x 2 matrices $\frac{1}{-Q}M^n$, n=1,2,..., where

$$M = \left[\begin{array}{cc} -Q & -Q \\ R & R - Q \end{array} \right].$$

We also note the product formula

$$L_n(R,Q) = \prod_{k=1}^{(n-1)/2} \left(R - 4Q\cos^2\left(\frac{k\pi}{n}\right) \right).$$

2-periodic continued fractions

Let P_1, P_2 and Q be nonzero integers and consider the 2-periodic generalized continued fraction

$$\frac{Q}{P_{1+}} \frac{Q}{P_{2+}} \frac{Q}{P_{1+}} \frac{Q}{P_{2+}} \dots$$
 (5)

The sequence of convergents to the continued fraction begins

$$\frac{0}{1}, \frac{Q}{P_1}, \frac{QP_2}{P_1P_2 + Q}, \frac{Q(P_1P_2 + Q)}{P_1(P_1P_2 + 2Q)}, \frac{QP_2(P_1P_2 + 2Q)}{P_1^2P_2^2 + 3QP_1P_2 + Q^2}, \frac{Q(P_1^2P_2^2 + 3QP_1P_2 + Q^2)}{P_1(P_1P_2 + Q)(P_1P_2 + 3Q)}, \cdots$$

Let $D_n = D_n(P_1, P_2, Q)$ (resp. $N_n = N_n(P_1, P_2, Q)$) denote the denominator (resp. the numerator) of the *n*th convergent of the continued fraction (5). It can be shown inductively that the denominators satisfy the recurrence

$$D_{n+1}(P_1, P_2, Q) = P_1 D_n(P_2, P_1, Q) + Q D_{n-1}(P_1, P_2, Q)$$

By considering the (n+1)th convergent of (5) we see that

$$\frac{N_{n+1}(P_1, P_2, Q)}{D_{n+1}(P_1, P_2, Q)} = \frac{Q}{P_1 + \frac{N_n(P_2, P_1, Q)}{D_n(P_2, P_1, Q)}},$$

from which, by another induction argument, we find

$$N_{n+1}(P_1, P_2, Q) = QD_n(P_2, P_1, Q).$$
(7)

We will show that the sequence of denominators $(D_n)_{n\geq 1}$ is essentially a Lehmer sequence. It will follow from this that the denominator sequence $(D_n)_{n\geq 1}$ is a linear divisibility sequence of the fourth order; furthermore, if both P_1 and P_2 are relatively prime to Q then we will show that the sequence $(D_n)_{n\geq 1}$ is a strong divisibility sequence, that is, $\gcd(D_n,D_m)=D_{\gcd(n,m)}$ for all natural numbers n and m. It will then follow from (7) that the same divisibility results also hold for the sequence of numerators $(N_{n+1})_{n\geq 1}$.

Recurrence equations and generating function for (D_n)

From the general theory of continued fractions the denominator sequence (D_n) satisfies the pair of second-order linear recurrence equations $D_0 = 0$, $D_1 = 1$ and, for $n \ge 1$,

$$D_{2n} = P_1 D_{2n-1} + Q D_{2n-2}$$

$$D_{2n+1} = P_2 D_{2n} + Q D_{2n-1}.$$
(8)

We can use these recurrence equations to extend the sequence D_n to negative suffices. The result is

 $D_{-n} = (-1)^{n-1} \frac{D_n}{O^n}.$ (9)

The pair of recurrences (8) may be combined into the single fourth-order linear recurrence equation

$$D_n = (P_1 P_2 + 2Q) D_{n-2} - Q^2 D_{n-4}, \quad n \ge 4.$$
(10)

Using (10), we can easily show the generating function for the sequence of denominators is

$$\sum_{n\geq 1} D_n x^n = \frac{x(1+P_1x-Qx^2)}{1-(P_1P_2+2Q)x^2+Q^2x^4}$$

$$= \frac{x-Qx^3}{1-(P_1P_2+2Q)x^2+Q^2x^4} + \frac{P_1x^2}{1-(P_1P_2+2Q)x^2+Q^2x^4}$$
(11)

$$= \frac{x - Qx^3}{1 - (P_1P_2 + 2Q)x^2 + Q^2x^4} + \frac{P_1x^2}{1 - (P_1P_2 + 2Q)x^2 + Q^2x^4}$$
(12)

as a sum of odd and even functions. Comparing (12) with (4), we see that the sequence of denominators (D_n) of the convergents to the continued fraction (5) is essentially the Lehmer sequence $L_n(P_1P_2, -Q)$ with integer parameters P_1P_2 and -Q, except that the even-indexed denominators D_{2n} have an extra factor of P_1 :

$$D_n(P_1, P_2, Q) = \begin{cases} L_n(P_1 P_2, -Q) & \text{for } n \text{ odd} \\ P_1 L_n(P_1 P_2, -Q) & \text{for } n \text{ even.} \end{cases}$$
 (13)

By (7), there is a corresponding result for the sequence of numerators:

$$N_{n+1}(P_1, P_2, Q) = \begin{cases} QL_n(P_1P_2, -Q) & \text{for } n \text{ odd} \\ QP_2L_n(P_1P_2, -Q) & \text{for } n \text{ even.} \end{cases}$$
(14)

In particular, if $P_1 = 1$ we have

$$D_n(1, P_2, Q) = L_n(P_2, -Q),$$

while if $P_2 = 1$ we have

$$N_{n+1}(P_1, 1, Q) = QL_n(P_1, -Q).$$

Divisibility properties of D_n

In [1] Lehmer states two theorems concerning the divisibility properties of Lehmer sequences. We sketch the proofs in the Appendix.

Theorem 1. The Lehmer sequence $L_n = L_n(R,Q)$ is a divisibility sequence, that is, L_n divides L_m whenever n divides m (provided $L_n \neq 0$). \square

Corollary 1. Let P_1, P_2 and Q be nonzero integers. The sequence of denominators $(D_n(P_1, P_2, Q))_{n \geq 1}$ and the sequence of numerators $(N_{n+1}(P_1, P_2, Q))_{n \geq 1}$ of the convergents of the periodic continued fraction

$$\frac{Q}{P_1+} \frac{Q}{P_2+} \frac{Q}{P_1+} \frac{Q}{P_2+} \dots$$

are divisibility sequences.

Proof. Immediate from Theorem 1 applied to (13) and (14). \square

Theorem 2. If R and Q are relatively prime integers then the Lehmer sequence $L_n = L_n(R, Q)$ is a strong divisibility sequence; that is, for all positive integers n, m we have

$$\gcd(L_n, L_m) = L_{\gcd(n,m)}. \square$$

Corollary 2. Let P_1, P_2 and Q be nonzero integers. Suppose further that both P_1 and P_2 are relatively prime to Q. Then the sequence of denominators $(D_n(P_1, P_2, Q))_{n\geq 1}$ and the sequence of numerators $(N_{n+1}(P_1, P_2, Q))_{n\geq 1}$ of the convergents of the periodic continued fraction

$$\frac{Q}{P_1+}\,\frac{Q}{P_2+}\,\frac{Q}{P_1+}\,\frac{Q}{P_2+}...$$

are strong divisibility sequences.

Proof.

We prove the result for the sequence of denominators; the proof for the sequence of numerators will then follow from (7). The proof is on a case-by-case basis depending on the parity of n and m. For example, let us show that $\gcd(D_n, D_m) = D_{\gcd(n,m)}$ in the case when n is odd and m is even; the remaining cases are dealt with in a similar manner and are left for the reader.

First we prove the following property of the Lehmer sequence $L_n = L_n(P_1P_2, Q)$: P_1 is coprime to L_n when n odd, that is to say,

$$gcd(L_{2n-1}, P_1) = 1$$
 for all positive integers n . (15)

The proof is by induction on n. We make the inductive hypothesis

$$gcd(L_{2n-1}, P_1) = 1$$
 for some n .

This is clearly true when n = 1 since $L_1 = 1$. Using the recurrence equation for Lehmer sequences (2) we find

$$\gcd(L_{2n+1},P_1) = \gcd(P_1P_2L_{2n} - QL_{2n-1},P_1)$$

$$= \gcd(-QL_{2n-1},P_1)$$

$$= \gcd(L_{2n-1},P_1) \text{ since by assumption } P_1 \text{is relatively prime to } Q$$

and the induction goes through.

Now by assumption, n is odd and m is even, and so we have from (13)

$$\gcd(D_n, D_m) = \gcd(L_n, P_1 L_m)$$

$$= \gcd(L_n, L_m) \text{ using (15)}$$

$$= L_{\gcd(n,m)} \text{ by Theorem 2}$$

$$= D_{\gcd(n,m)}$$

using (13) again, since gcd(n, m) is odd. \square

Appendix

In this section we sketch proofs of Lehmer's Theorem 1 and Theorem 2. We follow the treatment of the corresponding results for Lucas sequences given by Norfleet [2, Theorem 3]. It will be convenient to define the Lehmer sequence L_n by means of the recurrence equations (2), which we write in a condensed form as

$$L_n = f_n L_{n-1} - Q L_{n-2}, \quad [L_0 = 0, L_1 = 1],$$
 (16)

where we define $f_i = 1$ if j is even and $f_i = R$ if j is odd.

Theorem 1. The Lehmer sequence $L_n = L_n(R, Q)$, with integer parameters R and Q, is a divisibility sequence.

Sketch proof. It is not difficult to establish inductively the following generalization of the defining recurrences (16):

for $n \ge 1$ and $k \ge 1$ there holds

$$L_{n+k} = \begin{cases} f_k L_{k+1} L_n - Q L_k L_{n-1} & \text{for } n \text{ even} \\ L_{k+1} L_n - f_{k+1} Q L_k L_{n-1} & \text{for } n \text{ odd.} \end{cases}$$
 (17)

Assuming this and choosing k to be a multiple of n in (17), say k = mn, gives an expression for $L_{(m+1)n}$ as a linear combination of L_n and L_{mn} . Therefore, if L_n divides L_{mn} then L_n divides $L_{(m+1)n}$. It follows by an induction argument that L_n is a divisibility sequence. \square

Theorem 2. If R and Q are relatively prime integers then the Lehmer sequence $L_n = L_n(R,Q)$ is a strong divisibility sequence; that is, for all positive integers n, m we have

$$\gcd(L_n, L_m) = L_{\gcd(n,m)}.$$

Sketch proof. Firstl, one uses induction arguments to prove that for $n \geq 1$

$$\gcd(L_n, Q) = 1 \tag{18}$$

and

$$\gcd(L_{n+1}, L_n) = 1. \tag{19}$$

We need to prove the strong divisibility property

$$\gcd(L_n, L_m) = L_{\gcd(n,m)} \tag{20}$$

holds for all natural numbers n, m. We can assume without loss of generality that $n \ge m$. Let k = n - m. We begin by establishing the result

$$\gcd(L_n, L_m) = \gcd(L_{n-m}, L_m). \tag{21}$$

Suppose first that m is even. Then

$$\gcd(L_{n}, L_{m}) = \gcd(L_{m+k}, L_{m})$$

$$= \gcd(f_{k}L_{k+1}L_{m} - QL_{k}L_{m-1}, L_{m}) \text{ by (17)}$$

$$= \gcd(QL_{k}L_{m-1}, L_{m})$$

$$= \gcd(L_{k}L_{m-1}, L_{m}) \text{ using (18)}$$

$$= \gcd(L_{k}, L_{m}) \text{ using (19)}$$

$$= \gcd(L_{(n-m)}, L_{m}).$$

The proof of (21) when m is odd is exactly similar.

We are now ready to prove the strong divisibility property (20) by means of a strong induction argument on n+m. Clearly, (20) is true for the base case n=m=1. We make the inductive hypothesis that (20) is true for all n,m with $n+m \leq N$. Then if n+m=N+1

$$\gcd(L_n, L_m) = \gcd(L_{n-m}, L_m)$$
 by (21)
= $L_{\gcd(n-m,m)}$ by the inductive hypothesis
= $L_{\gcd(n,m)}$

and hence the induction goes through. \square

If we examine the above proofs of Theorem 1 and Theorem 2 we see that they only use the fact that the ring of integers $\mathbb Z$ is an integral domain with a greatest common divisor function. Thus these two theorems may be generalized to Lehmer sequences defined by the recurrence equations (2) where now R and Q are taken to be elements of an arbitrary GCD domain [3]. For example, the sequence of bivariate polynomials $P_n(x,y)$ defined by the linear recurrence equations $P_0=0, P_1=1$ and for $n\geq 1$

$$P_{2n} = P_{2n-1} - xyP_{2n-2}$$

$$P_{2n+1} = (x+y)^2 P_{2n} - xyP_{2n-1}$$
(22)

will be a strong divisibility sequence in the polynomial ring $\mathbb{Z}[x,y].$ An explicit formula is

$$P_{n}(x,y) = \begin{cases} (x^{n} - y^{n}) / (x - y) & \text{for } n \text{ odd} \\ (x^{n} - y^{n}) / (x^{2} - y^{2}) & \text{for } n \text{ even.} \end{cases}$$
 (23)

LINKS

- [1] D. H. Lehmer, An extended theory of Lucas' Functions, Annals of Mathematics Second Series, Vol. 31, No. 3 (July 1930), 419-448.
- [2] M. Norfleet, Characterization of second-order strong divisibility sequences of polynomials, The Fibonacci Quarterly, Vol. 43, No. 2 (May 2005), 166-169.
- [3] Wikipedia, GCD domain
- [4] Wikipedia, Lehmer number
- [5] Wikipedia, Lucas sequence