# Convex Lattice Polygons

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Let $n \geq 3$ be an integer. A convex lattice $n$-gon is a polygon whose $n$ vertices are points on the integer lattice $\mathbb{Z}^2$ and whose interior angles are strictly less than $\pi$. Let $a_n$ denote the least possible area enclosed by a convex lattice $n$-gon, then [1, 2, 3]

$$\{a_n\}_{n=3}^{\infty} = \left\{\tfrac{1}{2}, 1, \tfrac{5}{2}, 3, \tfrac{13}{2}, 7, \tfrac{21}{2}, 14, x, 24, \tfrac{65}{2}, 40, y, 59, z, 87, w, 121, ...\right\}$$

where the unknown values $x$, $y$, $z$, and $w$ are known to satisfy

$$x \in \left\{\tfrac{39}{2}, \tfrac{41}{2}, \tfrac{43}{2}\right\}, \qquad y \in \left\{\tfrac{99}{2}, \tfrac{101}{2}, \tfrac{103}{2}\right\},$$

$$z \in \left\{\tfrac{147}{2}, \tfrac{149}{2}, \tfrac{151}{2}\right\}, \qquad w \in \left\{\tfrac{209}{2}, \tfrac{211}{2}, \tfrac{213}{2}\right\}.$$

On the one hand, Rabinowitz [4] and Colburn & Simpson [5] demonstrated that $a_n \leq Cn^3$ for some constant $C > 0$; Zunic [6] later proved that $C \leq 1/54$. On the other hand, Andrews [7] and Arnold [8] were the first to show that $a_n \geq cn^3$ for some $c > 0$; other proofs appear in [9, 10, 11, 12]. Bárány & Tokushige [13] succeeded in proving that $\lim_{n\to\infty} a_n/n^3$ actually exists and computed that

$$\lim_{n\to\infty} \frac{a_n}{n^3} = 0.0185067... < \frac{1}{54}$$

via a heuristic solution of $\approx 10^{10}$ constrained minimization problems. Further, the shape of the minimizing $n$-gon is approximated by that of the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

where $A = (0.003573...)n^2$ and $B = (1.656...)n$.

Much less can be said about the higher dimensional analog. A $d$-dimensional convex lattice polytope with $n$ vertices has volume $v_n$ satisfying [7, 9, 14, 15]

$$v_n \geq c_d n^{\frac{d+1}{d-1}}$$

but little else is known.

**0.1. Integer Convex Hulls.** Before discussing integer convex hulls, let us mention ordinary convex hulls. Given $n$ points chosen at random in the unit disk $D$, the convex hull $C_n$ is the intersection of all convex sets containing all $n$ points. The boundary of $C_n$ is a polygon; let $N_n$ denote the number of vertices of the polygon. It can be proved that [16, 17, 18]

$$\lim_{n\to\infty}\frac{\mathrm{E}(N_n)}{n^{1/3}}=2\pi\xi, \qquad \lim_{n\to\infty}\frac{\mathrm{Var}(N_n)}{n^{1/3}}=2\pi\eta$$

where

$$\xi=\left(\tfrac{3\pi}{2}\right)^{-\frac{1}{3}}\Gamma\left(\tfrac{5}{3}\right)=0.5384576135...,$$

$$\eta=\frac{16\pi^2\Gamma\left(\frac{2}{3}\right)^{-3}-57}{27}\xi=0.1316029298...=2(0.3350302716...)-\xi.$$

We point out that this is more complicated than the corresponding result when the unit disk is replaced by the unit square [16, 17, 19]:

$$\lim_{n\to\infty}\frac{\mathrm{E}(\tilde{N}_n)}{\ln(n)}=\frac{8}{3}, \qquad \lim_{n\to\infty}\frac{\mathrm{Var}(\tilde{N}_n)}{\ln(n)}=\frac{40}{27}.$$

In the integer case, we consider not $n$ random points in $D$, but rather *all* lattice points in $rD$, the disk of radius $r$, where $r$ is large. The convex hull $C_r$ of all these lattice points is clearly a convex lattice polygon, together with its interior. Motivation for studying this polygon comes from integer programming: When maximizing a linear function $\varphi$ on the lattice points in $rD$ (or any given convex set in $\mathbb{R}^2$), one looks for the maximum point of $\varphi$ on $C_r$. The size of the programming problem is hence proportional to $N_r$, the number of vertices of $C_r$, and thus we wish to have bounds on $N_r$.

Balog & Bárány [20, 21] proved that, for sufficiently large $r$,

$$0.33r^{2/3}\le N_r\le 5.54r^{2/3}$$

but confessed that it isn't clear whether $\lim_{r\to\infty}N_r r^{-2/3}$ exists. It is possible, however, to obtain asymptotics for the average value of $N_r$, defined in a special way:

$$\mathrm{E}_\theta(N_r)=\frac{1}{r^\theta}\int_r^{r+r^\theta}N_\rho\,d\rho$$

where the parameter $\theta$ satisfies $0<\theta<1$. (Actually, the only feature needed of $r^\theta$ is that it increases with $r$, but less rapidly than $r$ itself.) Balog & Deshouillers [22] proved that

$$\lim_{r\to\infty}\frac{\mathrm{E}_\theta(N_r)}{r^{2/3}}=\frac{6\cdot 2^{2/3}}{\pi}\chi=3.4536898915...$$

independently of $\theta$, where $\chi$ is defined later. The growth rate 2/3 is what we would expect on the basis of the probabilistic model (ordinary convex hull case), but the preceding constant 3.453... is slightly different from $2\pi\xi = 3.383....$ In this sense, lattice points do not behave in the same way as random points.

Another occurrence of the constant $\chi$ is as follows. For real $x$, let $||x||$ denote the distance from $x$ to the nearest integer. Then, for $0 \leq a < b \leq 1$, we have [22]

$$\lim_{\lambda \to 0^+} \frac{1}{(b-a)\lambda^{1/3}} \int_a^b \min_{t \neq 0} \left( ||\alpha t|| + \lambda t^2 \right) \, d\alpha = \frac{6}{\pi^2}\chi.$$

If $\lambda = 0$, the integral clearly is zero since, for any $\alpha$, the point $t = 1/\alpha$ gives the minimum. If $\lambda > 0$, this strategy no longer works because the penalty term $\lambda t^2 = \lambda/\alpha^2$ would be large.

Let $\Delta$ denote the triangular region bounded by the lines $y = x$, $y = 1 - x$ and $x = 1$. Partition $\Delta$ into four domains:

$$\Delta_1 = \left\{ (x,y) \in \Delta : 1 \leq xy(x+y) \right\},$$

$$\Delta_2 = \left\{ (x,y) \in \Delta : xy(x+y) \leq 1 \leq x(x+y)(x+2y) \right\},$$

$$\Delta_3 = \left\{ (x,y) \in \Delta : x(x+y)(x+2y) \leq 1 \leq x(x+y)(2x+y) \right\},$$

$$\Delta_4 = \left\{ (x,y) \in \Delta : x(x+y)(2x+y) \leq 1 \right\}.$$

Define $F : \Delta \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} 4 - x^3 - y^3 & \text{in } \Delta_1, \\ \dfrac{1}{xy(x+y)} + 2 - (x+y)(x-y)^2 & \text{in } \Delta_2, \\ \dfrac{1}{y(x+y)(x+2y)} + 6 - (x+y)(3x^2+2xy+y^2) & \text{in } \Delta_3, \\ \dfrac{1}{x(x+y)(2x+y)} + \dfrac{1}{y(x+y)(x+2y)} + 4 - (x+y)(x^2+xy+y^2) & \text{in } \Delta_4, \end{cases}$$

then $\chi$ is given by

$$\chi = \int_{1/2}^{1} \int_{1-x}^{x} F(x,y) \, dy \, dx.$$

Again, much less can be said about the higher dimensional analog. Let $B_d$ denote the $d$-dimensional unit ball. The number of vertices, $N_r$, of the integer convex hull of $rB_d$ satisfies [23]

$$c_d r^{\frac{d(d-1)}{d+1}} \leq N_r \leq C_d r^{\frac{d(d-1)}{d+1}}$$

but an asymptotic average value for $N_r$ is not known for any $d \geq 3$.

**0.2. Addendum.** The $d$-dimensional unit cube has $2^d$ vertices. Randomly select $n = n(d)$ vertices with replacement and form the ordinary convex hull of these points. If $V_d$ denotes its expected volume, then for any $\varepsilon > 0$, [24, 25]

$$\lim_{d\to\infty} V_d = \begin{cases} 0 & \text{if } n(d) \leq \left(2/\sqrt{e} - \varepsilon\right)^d, \\ 1 & \text{if } n(d) \geq \left(2/\sqrt{e} + \varepsilon\right)^d. \end{cases}$$

This is an interesting occurrence of the constant $2/\sqrt{e} = 1.2130613194...$, which is surprisingly small (relative to 2)! If instead the $n$ points are selected uniformly in the interior of the $d$-cube, then the same threshold phenomenon occurs, with constant $2/\sqrt{e}$ replaced by

$$\exp\left(\int_0^\infty \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)^2 dx\right) = 2.1396909474....$$

In fact, a closed-form expression is possible since

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)^2 dx = \ln(2\pi) - \gamma - \frac{1}{2} = 0.7606614015...$$

and the details underlying this formula appear in [26]. See [25] for relevant discussion of the $d$-dimensional unit ball.

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