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CONFERENCE PAPER · JANUARY 2004

DOI: 10.1007/978-3-540-24664-0_19 · Source: DBLP

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Models and Symmetry Breaking for 'Peaceable Armies of Queens'

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Abstract. We discuss a difficult optimization problem on a chess-board, requiring equal numbers of black and white queens to be placed on the board so that the white queens cannot attack the black queens. We show how the symmetry of the problem can be straightforwardly eliminated using SBDS, allowing a set of non-isomorphic optimal solutions to be found. We present three different ways of modelling the problem in constraint programming, starting from a basic model. An improvement on this model reduces the number of constraints in the problem by introducing ancillary variables representing the lines on the board. The third model is based on the insight that only the white queens need be placed, so long as there are sufficient unattacked squares to accommodate the black queens. We also discuss variable ordering heuristics: we present a heuristic which finds optimal solutions very quickly but is poor at proving optimality, and the opposite heuristic for which the reverse is true. We suggest that in designing heuristics for optimization problems, the different requirements of the two tasks (finding an optimal solution and proving optimality) should be taken into account.

1 Introduction

Robert Bosch introduced the "Peaceably Coexisting Armies of Queens" problem in his column in Optima in 1999 [1]. It is a variant of a class of problems requiring pieces to be placed on a chessboard, with requirements on the number of squares that they attack: Martin Gardner [3] discusses more examples of this class. In the "Armies of Queens" problem, we are required to place two equal-sized armies of black and white queens on a chessboard so that the white queens do not attack the black queens (and necessarily v.v.) and to find the maximum size of two such armies. Bosch asked for an integer programming formulation of the problem and how many optimal solutions there would be for a standard 8×8 chessboard.

Here we discuss a range of possible models of the problem as a CSP, and show how Symmetry-Breaking During Search (SBDS) [4] can be used to eliminate the symmetry in each model, and hence find all non-isomorphic optimal solutions.

We have implemented some of the models in both ECL^{i}PS^e and ILOG Solver. Constraint programming tools can differ, for instance in the way that apparently equivalent constraints propagate: by comparing two different implementations of our models, we can draw conclusions that are independent of a particular constraint programming tool.

2 Basic Model

In a later issue of Optima, Bosch gives an IP formulation due to Frank Plastria. This has two binary variables for each square of the board:

$$\begin{aligned} b_{ij} &= 1 \text{ if there is a black queen on square } (i,j) \\ &= 0 \text{ otherwise} \\ w_{ij} &= 1 \text{ if there is a white queen on square } (i,j) \\ &= 0 \text{ otherwise} \end{aligned}$$

For the general case of an $n \times n$ board:

$$\begin{aligned} \text{maximize } & \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} \\ \text{subject to } & \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \\ & b_{i_1 j_1} + w_{i_2 j_2} \leq 1 \text{ for all } ((i_1,j_1),(i_2,j_2)) \in M \\ & b_{ij}, w_{ij} \in \{0,1\} \text{ for all } 1 \leq i,j \leq n \end{aligned}$$

where M is the the set of ordered pairs of squares that share a line (row, column or diagonal) of the board. Bosch reported that finding an optimal solution for an 8×8 board (with value 9) took just over 4 hours using CPLEX on a 200 MHz Pentium PC.

A straightforward model of the problem as a CSP is similar to this IP formulation. There is no difficulty in having variables with more than 2 values, so the number of variables can be reduced to n^2:

$$\begin{aligned} s_{ij} &= 2 \text{ if there is a white queen on square } (i,j) \\ &= 1 \text{ if there is a black queen on square } (i,j) \\ &= 0 \text{ otherwise} \end{aligned}$$

We can express the 'non-attacking' constraints as:

$$\begin{aligned} & s_{i_1 j_1} = 1 \Rightarrow s_{i_2 j_2} \neq 2 \\ \text{and } & s_{i_1 j_1} = 2 \Rightarrow s_{i_2 j_2} \neq 1 \text{ for all } ((i_1,j_1),(i_2,j_2)) \in M \end{aligned}$$

or more compactly as:

$$s_{i_1 j_1} + s_{i_2 j_2} \neq 3 \text{ for all } ((i_1, j_1), (i_2, j_2)) \in M$$

In both ECL^{i}PS^e and Solver, the single constraint gives the same number of backtracks as the two implication constraints, but is faster.

Constrained variables w, b count the number of white and black queens respectively (using the counting constraints provided in constraint programming tools such as ECL^{i}PS^e and Solver). We then have the constraint $w = b$, and the objective is to maximize w. This is achieved by adding a lower bound on w whenever a solution is found, so that future solutions must have a larger value of w; when there are no more solutions, the last one found has been proved optimal.

The model has n^2 search variables and approximately $4n^3$ binary constraints, as well as the counting constraints which have arity n^2.

3 Symmetry

The problem has the symmetry of the chessboard, as in the familiar n-queens problem; in addition, in any solution we can swap all the white queens for all the black queens, and we can combine these two kinds of symmetry. Hence the problem has 16 symmetries. It is well-known that symmetry in CSPs can result in redundant search, since subtrees may be explored which are symmetric to subtrees already explored. If only one solution is required, these difficulties do not always arise in practice. However, if a complete traversal of the search tree is required, either because there is no solution, or because all solutions are wanted, symmetry must lead to wasted search unless dealt with. This means that symmetry will cause difficulties in optimization problems, where proving optimality entails a complete search to prove that there is no better solution.

Symmetry Breaking During Search [4] is ideal for problems such as this since it only requires a simple function to describe the effect of each symmetry (other than identity) on the assignment of a value to a variable. Hence, in this case, just 15 such functions are required. Briefly, on backtracking to a choice point in the search, represented by the two constraints $var = val$ and $var \neq val$, SBDS adds a constraint to the second branch for any symmetry which has not yet been broken along the path from the root of the search tree to this node. The constraint is the symmetric equivalent of $var \neq val$ and prevents exploration of partial solutions equivalent under this symmetry to those which have already been explored following the choice $var = val$. If the effect of each individual symmetry is described, SBDS will eliminate all symmetry: all solutions produced are non-isomorphic to each other, and the search never explores any part of the search tree which is symmetric to a subtree already explored.

The seven board symmetries for which symmetry functions are required can be labelled `x`, `y`, `d1`, `d2`, `r90`, `r180` and `r270` (reflection in the horizontal, vertical and both diagonal axes, and rotations through 90°, 180° and 270°, respectively). An assignment $s_{ij} = v$ is passed to each function as a constraint, and the equivalent constraint under the relevant symmetry is returned. For instance, if the rows

and columns of the board are numbered $1, .., n$, $\texttt{r90}(s_{ij} = v)$ is the constraint $s_{j,n+1-i} = v$. The symmetry which interchanges the black and white queens, `bw`, returns $s_{ij} = v'$, where $v' = 0$ if v =0, and otherwise $v' = 3 - v$. We also need to describe the 7 symmetries which combine a board symmetry with interchanging black and white: for instance, the symmetry $\texttt{bw} \circ \texttt{r90}$ returns $s_{j,n+1-i} = v'$.

4 Basic Model: Results

The square variables are assigned in a predefined (static) order: top row, left to right, 2nd row, left to right, and so on. To ensure that good solutions are found early, values are assigned in descending order; otherwise, the first solution found has 0 assigned to every variable, corresponding to no queens of either colour, which is valid but far from optimal. The running times given relate to a 1.6GHz Pentium 4 PC for ECL^{i}PS^e and a 1.7GHz Celeron PC with 128MB RAM for Solver. The implementation of SBDS in ECL^{i}PS^e used in these experiments is due to Warwick Harvey.

Tables 1 and 2 show that using SBDS makes a huge difference to the time required to prove optimality, although not to the time to find the optimal solution. There is a more than 10-fold reduction in the number of fails, except for the smallest values of n, though the reduction in running time is less. It would be possible to achieve some of the speed-up without SBDS, by adding constraints to the model, for instance that the top half of the board contains more white queens than the bottom, but simple constraints of this kind cannot remove all the symmetry. Table 3 compares finding all solutions with and without symmetry breaking using SBDS. It proved impracticable to find all solutions for the 8×8 board without any symmetry breaking: there are evidently hundreds of possible solutions, although only 71 are distinct.

Table 1. Search effort and running time to find an optimal solution to the armies of queens problem, with no symmetry breaking. Value = optimal number of queens of each colour; F = number of fails (backtracks) to find the optimal solution; P = number of fails to prove optimality; sec. = running time in seconds

n	Value	ECL^{i}PS^e			ILOG Solver		
		F	P	sec.	F	P	sec.
2	0	7	7	<0.01	7	14	< 0.01
3	1	6	18	0.01	6	24	< 0.01
4	2	0	134	0.01	0	148	< 0.01
5	4	25	978	0.13	30	1,031	0.05
6	5	10	21,469	3.2	9	24,210	1.48
7	7	64	393,806	78	51	435,598	38.1
8	9	4339	10,846,300	3,500	5,270	12,002,608	1,660

Table 2. Search effort and running time to find an optimal solution to the armies of queens problem, with SBDS.

n	Value	ECL^{i}PS^e			ILOG Solver		
		F	P	sec.	F	P	sec.
2	0	1	1	< 0.01	1	2	< 0.01
3	1	4	6	0.01	4	11	0.01
4	2	0	15	0.04	0	16	0.01
5	4	24	96	0.16	29	131	0.03
6	5	10	1,609	1.7	9	1,854	0.35
7	7	64	29,255	27	51	33,529	7.96
8	9	4,339	806,056	640	5,270	924,505	331

Table 3. Search effort and running time to find all optimal solutions to the armies of queens problem, with ECL^{i}PS^e.

	No symmetry breaking			With SBDS		
n	Solutions	backtracks	sec.	Solutions	backtracks	sec.
2	1	0	< 0.01	1	0	< 0.01
3	16	15	0.02	1	2	0.01
4	112	219	0.05	10	18	0.06
5	18	1856	0.02	3	169	0.24
6	560	44400	5.8	35	3306	3.0
7	304	822108	130	19	59876	48
8	*not completed*			71	1604456	1130

5 Combining Squares and Lines

The basic model has a constraint between two variables if they represent squares which are on the same line (row, column or diagonal) of the board. We could consider an alternative model in which the lines are also represented by variables in the CSP. Any line must have either only white queens on it, or only black queens, or be empty, so we could create the line variables with three values corresponding to these possibilities. More compactly, we can have two possible values for each line variable: 0 means that there is no white queen on the line, and 1 means that there is no black queen on the line (unoccupied lines can have either value).

The advantage of adding the line variables is that we can reduce the number of constraints. Whenever a queen is placed on a square the values of the corresponding line variables are set accordingly. Thereafter, a queen of opposite colour cannot be placed on any of these lines, and we no longer need the constraints between square variables to enforce this.

Taking the rows as an example, we have n variables $r_1, ..., r_n$, and constraints:

$$s_{ij} = 1 \Rightarrow r_i = 0$$
$$\text{and } s_{ij} = 2 \Rightarrow r_i = 1 \text{ for all } 1 \leq i, j \leq n$$

As before, we can reduce the pair of constraints to a single constraint:

$$s_{ij} + r_i \neq 2 \text{ for all } 1 \leq i, j \leq n$$

The combined model has more variables than the basic model (another $6n$, approximately), but we can still use just the n^2 square variables as the search variables. There are approximately $4n^2$ constraints, each between a line variable and a square variable, rather than $4n^3$ constraints between pairs of square variables as before.

Adding the line variables to the model makes no difference to the number of fails in ECL^{i}PS^e (there are some differences in Solver), but reduces the running time to solve the problems optimally by about one-third in Solver and about one-sixth in ECL^{i}PS^e, for $n = 8$.

Note that since the search variables, and hence the branching decisions made during search, are unchanged, SBDS is unaffected by the change in the model.

6 Combined Model: Discussion

Figure 1 compares the constraints required in the two models: it shows the constraints required to express that row i cannot have queens of different colours, in the case $n = 5$. The solid lines show the clique of constraints required between the variables corresponding to the squares on row i in the basic model. The dotted lines are the constraints that replace them in the combined model: we have replaced an n-clique of constraints by just n.

In addition to the constraints expressing that we cannot have queens of different colours on any line, we also need constraints on the number of queens of each colour. These would be difficult if not impossible to express solely in terms of the line variables. Hence, adding the line variables to the basic model is not exactly analogous to the idea of redundant modelling [2]: in redundant modelling, two

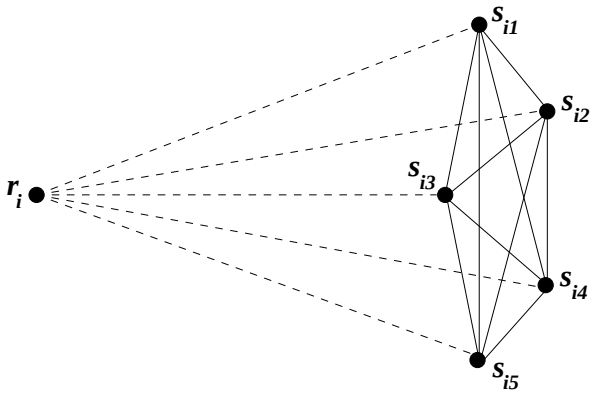

Fig. 1. Constraints in the two encodings of the 'armies of queens' problem

models are combined, either of which could be used independently. Moreover, the claimed advantage of redundant modelling is that constraint propagation within either model can feed through to the other, via the channelling constraints which link them. Here, there are no constraints between the line variables, and in the combined model, the only constraints between the square variables are those counting the number of queens. We are replacing all other constraints of the basic model by the channelling constraints linking the line and square variables. This is somewhat similar to combining models of permutation problems, where the benefit comes from propagation of the channelling constraints, which in that case can replace $\neq$ constraints between the variables of the original model [5].

Although we cannot have a model with line variables alone, we could in theory have a model with both line and square variables in which we search on the line variables and not the square variables. This is an attractive idea, since there would only be $6n$ search variables, approximately, rather than n^2. However, in practice it leads to a number of difficulties. It would introduce new symmetries, since the values 0 and 1 are interchangeable if a line is unoccupied. We could avoid this by having three values rather than two for the line variables, but even then, a complete assignment to the line variables does not always uniquely determine the values of the square variables, so that not all non-isomorphic optimal solutions would be found.

7 Counting Unattacked Squares

In trying to solve the armies of queens problem by hand, it becomes apparent that we need only place the queens of one colour, say white, provided that we check as each queen is placed that the number of squares not so far attacked is at least equal to the number of white queens on the board. A black queen can be placed on any square which is not attacked by any white queen; hence if there are k white queens on the board and at least k unattacked squares, we can extend the current assignment to a complete solution with value k.

This leads to a new model of the problem. As in the basic model, there is a variable s_{ij} for each square on the board, but now with possible values 0 and 1, where 0 signifies that the square is either empty or occupied by a black queen and 1 that it contains a white queen. For each square, we also construct the set of squares, A_{ij}, that a queen placed on this square would attack. A set variable U represents the unattacked squares on the board. If $s_{ij} = 0$ but $(i, j) \notin U$, square (i, j) must be empty.

The constraints are:

$$s_{ij} = 1 \Leftrightarrow A_{ij} \cap U = \emptyset \quad 1 \leq i, j \leq n$$

The minimum of the number of white queens and the number of unattacked squares gives the size of the two equal-sized armies; this minimum is the objective to be maximised, as before.

The unattacked squares model does not affect the optimal value, but it does lead to a different way of counting distinct optimal solutions. The earlier models produce solutions with equal numbers of white and black queens, but now solutions with different numbers of the two colours can be produced. For example, it is possible to place 9 white queens and 10 black queens on an 8×8 board: such a solution leads to 10 distinct solutions with exactly 9 of each colour, all obtained by omitting one black queen. In the new model we count such a configuration as just one solution, but we note that an unequal number of queens appear. That is to say, we report only solutions in which no queen of either colour can be added to any square, even if this unbalances the numbers. All previous solutions can be obtained from the smaller number of solutions we now find. Unfortunately, the number of solutions in the previous model cannot be calculated trivially from the number in the new model, a point we will discuss further below.

Optimal solutions with different numbers of black and white queens led to other subtle problems that we had to address in implementation. We added constraints so that a square being unattacked by a white queen is equivalent to having a black queen on it, and v.v. Without these constraints, the model produces spurious solutions in which an unattacked square does not have a black queen on it. This is incorrect as we seek only solutions in which no queens of either colour can be added to the solution. The constraints are implemented by introducing a matrix of Boolean variables b_{ij} to indicate if a black queen is on square (i, j), and a set variable W to represent the squares occupied by white queens. We then add the constraints:

$$s_{ij} = 1 \Leftrightarrow (i, j) \in W \quad 1 \leq i, j \leq n$$

$$b_{ij} = 1 \Leftrightarrow (i, j) \in U \quad 1 \leq i, j \leq n$$

$$b_{ij} = 1 \Leftrightarrow A_{ij} \cap W = \emptyset \quad 1 \leq i, j \leq n$$

The first two constraints simply give the intended meanings to the sets W and U, while the last (together with the primary constraint given above) equates squares occupied by black queens with squares not attacked by white queens.

These additional constraints are expensive to reason with and rarely have an effect during search. To save runtime, we only add them when an assignment satisfying all the other constraints has been found, backtracking if this then causes a failure. For example, when $n = 8$, these constraints eliminated just one solution out of 46 candidates; thus, they are not important except to get an exact count of genuinely non-isomorphic solutions.

The model has $4n^2+1$ binary constraints, as opposed to $O(n^3)$ for the squares model and about $4n^2$ for the combined model with line and square variables. However, as just described, only $n^2 + 1$ of these constraints are active during most of the search. The search variables are the n^2 s_{ij} variables. This is the same number as in the previous models, but now each variable has only two possible values rather than three.

It is easy to implement SBDS in the unattacked squares model. The functions for the seven board symmetries are exactly as in the previous models. For symmetries that swap the black and white queens, we could obviously make $s_{ij} = 1$ equivalent to $b_{ij} = 1$. However, this would necessitate the inclusion of the inefficient constraints throughout search, so instead we make $s_{ij} = 1$ map to $(i, j) \in U$. That is, since black queens are on all unattacked squares, the symmetric version of a white queen existing on a square is that the same square is unattacked by any white queen. This combines with the symmetries of the board to give eight more functions.

Table 4 shows the results for this model using ILOG Solver. In reporting the number of solutions, we give the total number of distinct solutions found, but also note how many of these have unbalanced numbers of queens of each colour. As discussed above, in these cases the number of solutions in this table is different to that reported in Table 3. We can obtain a set of solutions with equal numbers of each by dropping any extra queens in all possible ways, but we cannot guarantee these are all symmetrically distinct without further checking. For example, for $n = 3$ there are two distinct ways to have two queens of one colour and one of the other, but when the extra queen is removed in both possible ways, the four resulting solutions are all symmetrically equivalent to the unique solution with one of each colour. In short, there is no trivial way to match the numbers of solutions in Table 3 and Table 4, although it would be possible to generate all solutions with equal numbers of queens from the set of solutions in the unattacked squares model and discard symmetric duplicates.

In comparison with the previous models, the number of fails is almost quartered for $n = 8$, and the running time also greatly improved compared to the model combining square and line variables (171 sec. to 60 sec.). The difference is still larger when $n = 9$: the combined model takes over 31 million fails and 5500 sec. to find and prove the optimal solution.

Table 4. Search effort and running time to solve the armies of queens problem and find all non-isomorphic optimal solutions, with the unattacked squares model, SBDS and lexicographic ordering, using ILOG Solver. The number of solutions in brackets is the number found that have more queens of one colour.

		Finding & proving optimal solution			Finding all solutions		
n	Value	F	P	sec.	Solutions	fails	sec.
2	0	0	1	< 0.01	1 (1)	1	< 0.01
3	1	0	4	0.01	2 (2)	3	< 0.01
4	2	0	12	< 0.01	7 (2)	8	0.01
5	4	26	112	0.02	3	101	0.03
6	5	8	1,062	0.27	21 (3)	1,462	0.42
7	7	39	12,318	3.21	19	17,587	4.85
8	9	1,150	185,504	60.4	45 (3)	260,237	81.3
9	12	591,486	3,101,026	1,040	18	3,283,004	1,320

7.1 A Puzzle for the Standard Chessboard

In the solution set for the standard chessboard, we noticed how remarkably close to placing 10 queens of each colour we can get. Of the three solutions with 10 of one colour and 9 of the other, two differ in just one place, as illustrated in Figure 2 (left).

From this layout we can derive that shown on the right of Figure 2, containing the king and all nine possible queens of each colour, i.e. the original queen and eight promoted pawns. Thus, the figure solves the following puzzle, and in fact gives the unique solution up to symmetry: *Put the king and the 9 queens of each colour on a chessboard in such a way that no queen is on the same row, column or diagonal as any piece of the opposite colour.*

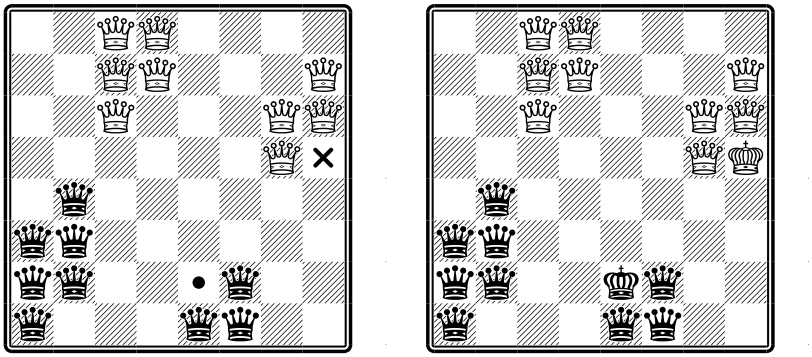

Fig. 2. (Left) the optimal solution for an 8×8 board that is closest to having 10 queens of each colour. A white queen could be placed on the square marked $\times$ *or* a black queen on the square marked $\bullet$, but not both, since they would then attack each other. (Right) a possible chess position, with the king and all nine possible queens of each colour.

8 Variable Ordering

The unattacked squares model was derived from trying to solve the problem by hand. This also led to an algorithm for constructing a solution and from that a variable ordering heuristic. The algorithm places a white queen on the square attacking fewest squares that are not already attacked; hence, it tries to keep the number of unattacked squares as large as possible. The algorithm terminates when no more white queens can be placed without reducing the number of unattacked squares below the number of white queens. Often, the solution found is optimal or near optimal. The first white queen placed is in a corner square, and the lexicographic ordering used so far assigns the variable representing the top left corner first. However, after assigning the first variable, the lexicographic ordering diverges from the algorithm. We have therefore experimented with a

dynamic variable ordering heuristic that chooses next the variable representing a square which is already attacked itself and where a white queen would attack fewest unattacked squares.

The fewest-unattacked-squares heuristic finds optimal solutions very quickly, but is worse than lexicographic ordering at proving optimality. For instance, when n=8, it finds an optimal solution immediately, with no backtracking, but then takes more than 320,000 fails and 78 sec. to prove optimality. It is also poor at enumerating solutions: for $n = 8$ and 9 it takes 118 sec. and 2,020 sec. respectively, again both worse than lexicographic ordering.

Since this heuristic is so poor at proving optimality, it seemed worthwhile to try exactly the opposite heuristic, i.e. choose the square where a white queen will attack *most* unattacked squares. Not surprisingly, this takes much longer to find the optimal solution (though not as long as lexicographic ordering for the larger values of n), but it is overall much faster than either the fewest-unattacked-squares heuristic or lexicographic ordering. The results are shown in Table 5. For 8×8 and 9×9, this heuristic runs more than 10 times faster than lexicographic ordering.

Table 5. Search effort and running time to solve the armies of queens problem and find all non-isomorphic optimal solutions, with the unattacked squares model, SBDS and the most-unattacked-squares variable ordering heuristic, using ILOG Solver.

		Finding & proving optimal solution			Finding all solutions		
n	Value	F	P	sec.	Solutions	fails	sec.
2	0	0	1	< 0.01	1 (1)	1	< 0.01
3	1	3	10	0.01	2 (2)	6	< 0.01
4	2	2	18	< 0.01	7 (2)	11	0.02
5	4	4	46	0.03	3	64	0.03
6	5	47	598	0.13	21 (3)	681	0.22
7	7	504	3,387	1.06	19	5,401	1.56
8	9	2,890	40,751	15.1	45 (3)	55,723	24.8
9	12	40,195	320,589	141	18	482,537	217
10	14	14,752	4,581,194	3,030	149 (17)	7,498,801	5,180
11	17	208,573	61,076,593	43,400	168 (25)	95,112,446	67,600
12	21	131,988,279	717,853,580	681,000	*insufficient time to run*		

The 10×10, 11×11 and 12×12 problems can now be solved, although the latter takes more than a week of cpu time. In the first two cases, we have found all optimal solutions; there are solutions with an extra queen of one colour, and a solution with two extra queens in the 11×11 case. The fewest-unattacked-squares heuristic again finds optimal solutions very quickly for these two problems.

Since the most-unattacked-squares heuristic performs so much better than either of the other variable orderings considered, and yet is not especially good at finding optimal solutions, it is worth trying to explain why it does well. Figure

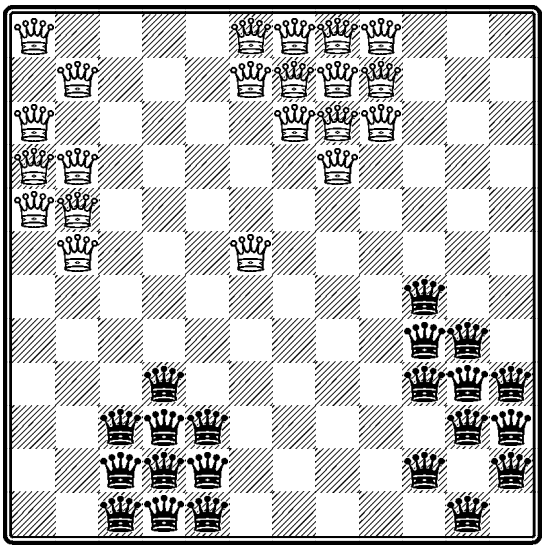

Fig. 3. An optimal solution for a 12 × 12 board, with 21 queens of each colour.

4 shows a configuration with 6 white queens attacking all but 6 squares on a 11 × 11 board, and an optimal solution with 17 queens of each colour.

The most-unattacked-squares heuristic is biased towards producing configurations with a few white queens attacking all but a few squares, as on the left of Figure 4. However, once an optimal solution has been found, such configurations become infeasible. In fact, it would no longer be possible to place 6 white queens as shown: a branch of the search tree leading to this configuration would be pruned as soon as fewer than 17 unattacked squares are left. Hence, we conjecture that the heuristic is successful because it can prune branches of the search tree when only a few variables have been assigned, i.e. it tends to find small nogoods. On the other hand, a heuristic which tries to place as many white queens as possible before leaving fewer than the optimal number of unattacked squares (as the fewest-unattacked-squares heuristic does) will tend to prune the search much lower down the tree.

Both heuristics could be used with the earlier models, but would be expensive to implement, since the information on unattacked squares is not readily available. Here, we compute $|A_{ij} \cap U|$ for each unassigned variable s_{ij}, and choose the variable for which this is smallest or largest, depending on the heuristic.

9 Discussion

The peaceable armies of queens problem is a difficult optimization problem that was hard to solve using an integer programming model. The constraint programming models considered here have all done reasonably well in solving the 8 ×

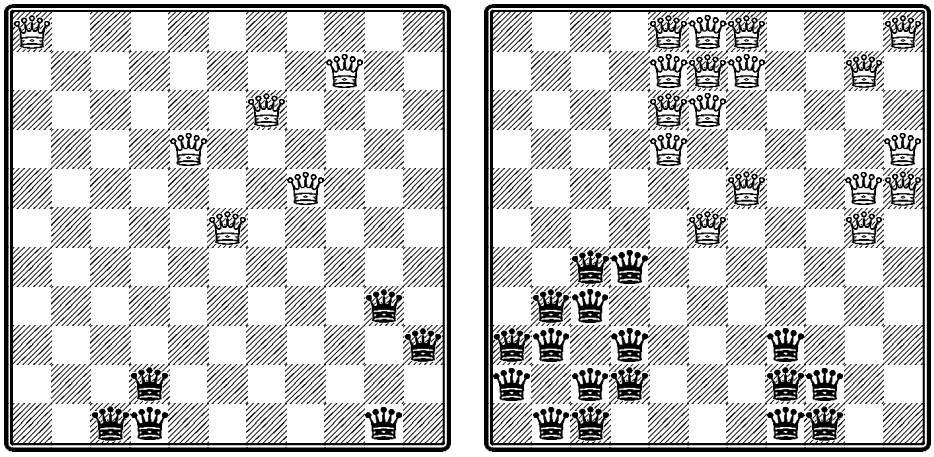

Fig. 4. Equal sized armies of queens on a 11 × 11 board. Left, 6 white queens attack all but 6 squares on the board. Right, an optimal solution with 17 queens of each colour.

8 problem; even so, problems larger than 10 × 10 are taking a very long time to solve, even for the best model we have found. Related problems have been investigated by Velucchi [6]. His results give optimal solutions for the armies of queens problem up to 10 × 10 and for 12×12, but it is unclear if optimality was proved, and certainly the number of optimal solutions is not reported. This suggests that constraint programming is competitive with other methods that have been tried for this problem, and indeed is capable of obtaining new results in the field. However, the problem has no practical importance and it is the experience of trying to solve it that is useful, rather than the solutions themselves.

Starting from a basic constraint programming model with no symmetry breaking, we have shown that the time to solve the 8 × 8 problem can be reduced from 1,660 sec. to 15 sec. (using ILOG Solver), a more than 100-fold improvement. The results for 8×8 and 9×9 are summarized in Table 6. Note that even our initial technique is similar in speed to Plastria's IP solution, allowing for differences in machine, so our final technique is literally a hundred times better. With increasing size the improvement ratio gets better: for 9 × 9 we saw a 400-fold runtime improvement from the first model (65,300sec.) to the last (141sec.), using the same environment on the same machine.

A major part of the improvement is due to eliminating the symmetry using SBDS. Given an implementation of SBDS, it requires no ingenuity on the part of the user to write the 15 functions to describe the effects of the individual symmetries of the problem. For the 8 × 8 problem, eliminating the symmetry reduces the time to solve the problem optimally from 1,660 sec. to 331 sec.; it also allows a set of non-isomorphic solutions to be found, whereas without symmetry breaking, it took too long to find all the possible solutions, which would in any case have been uninformative.

Table 6. Performance of different models in solving the 8×8 and 9×9 armies of queens problems.

8×8	Model	F	P	sec.
	Basic model, no SBDS	5,270	12,002,608	1,660
	Basic model, with SBDS	5,270	924,505	331
	Combined model, with SBDS	5,270	924,505	171
	Unattacked squares model, with SBDS	1,150	185,504	60
	Unattacked squares model, with SBDS & most-unattacked-squares heuristic	2,890	40,751	15
9×9	Model	F	P	sec.
	Basic model, no SBDS	8,049,706	273,820,671	65,300
	Basic model, with SBDS	7,502,848	31,407,249	12,500
	Combined model, with SBDS	7,502,848	31,407,249	6,620
	Unattacked squares model, with SBDS	591,486	3,101,026	1,040
	Unattacked squares model, with SBDS & most-unattacked-squares heuristic	40,195	320,589	141

Further reductions in running time are due to remodelling the problem. We have described three different ways of modelling it, starting from a basic model not very different from an integer programming formulation. The combined model introduces ancillary variables (one for each row, column or diagonal) in order to reduce the number of constraints, from $4n^3$ to $4n^2$, approximately. This significantly reduces running time, although the search effort is largely unaffected.

The unattacked squares model has the same number of search variables as the other models, but with fewer possible values, so that the number of possible assignments is reduced. The model also has fewer constraints than the previous models, which probably contributes to the reduction in running time. However, the binary constraints are between an integer variable and a set variable, so that constraint propagation may be more expensive than with binary constraints involving two integer variables.

Devising new models does require ingenuity. The different models we have presented can be seen as viewing the problems at different levels. The basic model expresses that a single white queen and a single black queen are inconsistent if they are on the same row, column or diagonal. The combined model takes the perspective of a line (row, column or diagonal) of the board: any number of queens can be placed on a line provided that they are all the same colour. The unattacked squares model expresses that any number of white queens can be placed anywhere on the board, as long as there are at least as many unattacked squares as white queens. Hence, each model takes a broader view of the problem than the previous model. Moreover, whereas the first two models are only concerned with whether the white and black queens attack each other, the final model also has something of the optimization criterion built into it: not only must the white and black queens not attack each other, but there must be enough of each of them. Trying to view the problem from several different

angles is likely to be a fruitful source of ideas for remodelling; we found that constructing solutions by hand facilitated this and gave useful insights into key features of the problem.

The final improvement in modelling the problem came from a variable ordering heuristic. We have presented two: one finds optimal or near-optimal solutions very quickly, but is poor at proving optimality. The other is its exact opposite and takes much longer to find an optimal solution, but then is much better at proving optimality. Although it is intuitively clear that finding optimal solutions and proving optimality are different in nature, it is surprising to see it demonstrated in such a clear-cut way. Again, the first heuristic was inspired by trying to construct solutions by hand. There may be other problems where a good heuristic for proving optimality is the exact opposite of a good heuristic for finding an optimal solution, and this will be investigated further. Variable ordering heuristics have hitherto mainly been investigated in the context of constraint satisfaction rather than optimization: our experience with this problem suggests that variable ordering heuristics for satisfaction problems and for optimization problems may need to be designed separately. For some optimization problems, it may be better to use two different heuristics, the first to find a good solution and the second to improve that solution if possible and to prove optimality.

Acknowledgments

We are extremely grateful to Warwick Harvey for his advice and help in using ECL^{i}PS^e, and for his implementation of SBDS. The authors are members of the APES and CP Pod research groups (`http://www.dcs.st-and.ac.uk/~apes` and `http://www.dcs.st-and.ac.uk/~cppod`) and would like to thank the other members, as well as Graeme Bell. This work is supported by EPSRC grants GR/R29666 and GR/R29673, as well as a Royal Society of Edinburgh SEELLD Support Fellowship to the third author.

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