Characterization of Numbers n with 4 Regions of Width 1 with 2 Regions meeting at the Center of the Dyck Path in the Symmetric Representation of sigma(n)

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Definitions of notations can be found in the paper referenced in the LINK section of A241561.

THEOREM:

For every number $n \in \mathbb{N}$:
$c_n = 4$ & $w_n = 1$ & two regions meet at the center of the Dyck path
$\Leftrightarrow$ $n = 2^m \times p \times (2^{m+1} \times p + 1)$, where $m \geq 0$, $2^{m+1} < p$ and p as well as $2^{m+1} \times p + 1$ are prime.

In this case the first region has $2^{m+1} - 1$ legs and the second starts with leg p and ends with leg r_n.
Furthermore, their respective areas are:
$\frac{1}{2} \times (2^{m+1} - 1) \times (2^{m+1} \times p^2 + p + 1)$ and $\frac{1}{2} \times (2^{m+1} - 1) \times (2^{m+1} \times p + p + 1)$.
Therefore $v_n = \left(2^{m+1} - 1\right) \times (p + 1) \times (2^{m+1} \times p + 2) = \sigma(n)$.

PROOF:

"$\Leftarrow$": Let $n = 2^m \times p \times (2^{m+1} \times p + 1)$, where $m \geq 0$, $2^{m+1} < p$ and p as well as $2^{m+1} \times p + 1$ are prime. Similar to the calculations in the proof of Theorem in the link of A262259 we get $r_n = 2^{m+1} \times p$ and 1's in positions 1, 2^{m+1}, p and $2^{m+1} \times p$ of the n-th row of irregular triangle A237048. Therefore, there are two regions of width 1 along the first half of the Dyck path for n with the second region ending at the center.

"$\Rightarrow$": Let $n = 2^m \times s$ where s is odd and $m \geq 0$. Since $c_n = 4$ there are four 1's in the n-th row of irregular triangle A237048. Since $w_n = 1$ and two regions meet at the center the 1's must be in positions $1 < 2^{m+1} < p < 2^{m+1} \times p = r_n$ where p|s, p > 2 is the smallest prime divisor of n and divisor $q = \frac{s}{p} > r_n$ is represented by the 1 in position r_n. With $n = 2^m \times p \times q$ we get $2^{m+1} \times q + 1 \leq \frac{s}{q} = p < 2^{m+1} \times q + 3 + \frac{1}{2^m \times q}$ similar to the computations in Theorem in the link of A262259. In other words, $q = 2^{m+1} \times p + 1$ or $q = 2^{m+1} \times p + 3$. Again, since n must be a triangular number, $q = 2^{m+1} \times p + 1$.
■

Corollary:

The set of numbers $n = p \times (2 \times p + 1)$ with $p > 2$ and $2 \times p + 1$ primes from the Theorem form sequence A156592, except for A156592(1) = 10 since $c_{10} = 2$. In this case v_n simplifies to $v_n = 2 \times (p + 1)^2$.