Experiments and Discoveries in q-Trigonometry

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Abstract

We introduce a q-generalization of the sine and cosine functions, related to the ϑ functions, but (as revealed by computer experiments) possessing addition and multiplication formulas more analogous to those of ordinary sin and cos. These formulas then contribute identities to ϑ theory, and hint of a more natural formulation of ϑ functions as outgrowths of elementary functions. Nevertheless, this paper can be read without knowledge of ϑ functions—it was certainly written that way.

Preliminaries

In order to q-generalize the sin function, we start with the q-factorial,

$$z!_q := (1-q)^{-z} \prod_{n\ge 1} \frac{1-q^n}{1-q^{n+z}} \text{ “=” } \frac{1-q}{1-q}\frac{1-q^2}{1-q} \cdots \frac{1-q^z}{1-q}, \tag{q-Fac}$$

(the latter making sense when z is an integer), and try to q-generalize the reflection formula which connects ordinary factorial with ordinary sin:

$$(z-1)!\,(-z)! = \frac{\pi}{\sin \pi z}. \tag{Ref}$$

For q-newcomers, the connection between q-factorial and ordinary factorial $\big($or $\Gamma(z+1)\big)$ is

$$\lim_{q\to 1} z!_q = \lim_{q\to 1} \prod_{n\ge 1} \frac{1-q^n}{1-q^{n+z}} \left(\frac{1-q^{n+1}}{1-q^n}\right)^z = \prod_{n\ge 1} \frac{n}{n+z}\left(\frac{n+1}{n}\right)^z = \frac{z!}{1!^z} = z!$$

and, from (q-Fac),

$$\frac{z!_q}{(z-1)!_q} = (1-q)^{-1} \prod_{n\ge 1} \frac{1-q^{n+z-1}}{1-q^{n+z}} = \frac{1-q^z}{1-q}$$

for all z. (Most formulas will require $|q|<1$ for convergence.)

Fossickings and Findings

A good stab at q-generalizing (Ref) might be

$$(z-1)!_q\,(-z)!_q = \frac{f_q(z)}{\sin_q \pi z}, \tag{Stab}$$

with $f_q(z)$ chosen so that $\sin_q(z-\pi) = -\sin_q z$. But after a derivation completely parallel to that upcoming, we would discover that the resulting definition of $\sin_q$ would be more appropriate for $\sin_{\sqrt{q}}$. Thus warned, we rewrite (Stab):

$$(z-1)!_{q^2}\,(-z)!_{q^2} = \frac{f_q(z)}{\sin_q \pi z}. \tag{Hindsight}$$

If, for all q and z, we are to have

$$\sin_q \pi(z-1) = -\sin_q \pi z,$$

then we can divide (Hindsight) by itself with $z \leftarrow z-1$ to get

$$\frac{1-q^{2(z-1)}}{1-q^{2(1-z)}} = -\frac{f_q(z)}{f_q(z-1)}$$

or

$$\frac{f_q(z)}{f_q(z-1)} = q^{2(z-1)}.$$

Thus $f_q(z) = q^{z(z-1)}$ times some function of q which we shall call π_q, by analogy with (Ref). (The function $f_q(z)$ could also contain a period 1 factor, but this we absorb into $\sin_q \pi z$.) Solving (Hindsight) for $\sin_q$,

$$\sin_q \pi z := \frac{q^{z(z-1)}\pi_q}{(z-1)!_{q^2}\,(-z)!_{q^2}}. \tag{SinDef}$$

To determine π_q, put $z = \frac{1}{2}$ and require $\sin_q \frac{\pi}{2} = 1$:

$$\pi_q := q^{\frac{1}{4}}(-\tfrac{1}{2})!^2_{q^2} = (1-q^2)\,q^{\frac{1}{4}} \prod_{n\geq 1} \frac{(1-q^{2n})^2}{(1-q^{2n-1})^2}. \tag{πDef}$$

Incidentally, dividing both sides by $1+q$ gives a q-generalization of Wallis's product:

$$\frac{\pi_q}{1+q} = q^{\frac{1}{4}} \prod_{n\geq 1} \frac{1-q^{2n}}{1-q^{2n-1}}\,\frac{1-q^{2n}}{1-q^{2n+1}}.$$

Dividing (Sindef) by $q^{-z} - q^z$,

$$\frac{\sin_q \pi z}{q^{-z}-q^z} = \frac{q^{z^2}}{z!_{q^2}\,(-z)!_{q^2}}\,\frac{\pi_q}{1-q^2}.$$

Letting $z \to 0$,

$$-\frac{\pi \sin_q' 0}{2\ln q} = \lim_{z\to 0} \frac{\sin_q \pi z}{q^{-z}-q^z} = \frac{\pi_q}{1-q^2} =: \Pi_q, \tag{Small z}$$

corresponding to

$$\lim_{z\to 0} \frac{\sin \pi z}{z} = \pi.$$

In the formulas to follow, it will turn out that Π_q arises much more frequently than π_q, despite the former blowing up as $q \to 1$. Here is its connection with ϑ functions:

$$\begin{aligned}\Pi_q &= q^{\frac{1}{4}} \prod_{n\geq 1} \frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = \frac{\vartheta_2\vartheta_3}{2} = \frac{\vartheta_2^2(0, q^{\frac{1}{2}})}{4} \\ &= q^{\frac{1}{4}}(1+q+q^3+q^6+q^{10}+\cdots)^2. \end{aligned}\tag{ΠSer}$$

Rewriting (SinDef) with explicit products, we see its ϑ connection:

$$\sin_q \pi z = q^{(z-1/2)^2} \prod_{n\geq 1} \frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2} = iq^{z^2}\,\frac{\vartheta_1(iz\ln q)}{\vartheta_4} \tag{SinProd1}$$

$$= q^{z^2}\,\Pi_q(q^{-z}-q^z) \prod_{n\geq 1} \left(1 - (q^{-z}-q^z)^2 \frac{q^{2n}}{(1-q^{2n})^2}\right) \tag{SinProd2}$$

$$\begin{aligned}= q^{z^2}\,\Pi_q \Bigg(&(q^{-z}-q^z) - (q^{-z}-q^z)^3 \sum_{n\geq 1} \frac{q^{2n}}{(1-q^{2n})^2} \\ &+ (q^{-z}-q^z)^5 \frac{\left(\sum_{n\geq 1} \frac{q^{2n}}{(1-q^{2n})^2}\right)^2 - \sum_{n\geq 1} \frac{q^{4n}}{(1-q^{2n})^4}}{2} - \cdots \Bigg).\end{aligned} \tag{SinEx}$$

Expand (SinProd1) in powers of q:

$$\sin_q \pi z = q^{z^2+\frac{1}{4}} \left((q^{-z}-q^z) + (2q^{-z}-2q^z)q - (q^{-3z}-4q^{-z}+4q^z-q^{3z})q^2 - \cdots\right).$$

With the help of MACSYMA's `Taylor` (series) facility, we see the successive coefficients form the following (parabolic) tableau:

…	q^{-9z}	$-q^{-7z}$	q^{-5z}	$-q^{-3z}$	q^{-z}	$-q^{z}$	q^{3z}	$-q^{5z}$	q^{-7z}	$-q^{9z}$	…
					1	1					q^0
					2	2					q^1
				1	4	4	1				q^2
				2	8	8	2				q^3
				4	14	14	4				q^4
				8	24	24	8				q^5
			1	14	40	40	14	1			q^6
			2	24	64	64	24	2			q^7
			4	40	100	100	40	4			q^8
			8	64	154	154	64	8			q^9
			14	100	232	232	100	14			q^{10}
			24	154	344	344	154	24			q^{11}
		1	40	232	504	504	232	40	1		q^{12}
		2	64	344	728	728	344	64	2		q^{13}
		4	100	504	1040	1040	504	100	4		q^{14}
		8	154	728	1472	1472	728	154	8		q^{15}
		14	232	1040	2062	2062	1040	232	14		q^{16}
		24	344	1472	2864	2864	1472	344	24		q^{17}
		40	504	2062	3948	3948	2062	504	40		q^{18}
		64	728	2864	5400	5400	2864	728	64		q^{19}
	1	100	1040	3948	7336	7336	3948	1040	100	1	q^{20}
	2	154	1472	5400	9904	9904	5400	1472	154	2	q^{21}
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

On the plausible assumption that the coefficient sequence under $q^{\pm(2n+1)z}$ is just the displacement by $n(n+1)$ rows of that under q^z, we can write

$$\sin_q \pi z = g(q)\sum_{n\geq 0}(-1)^n q^{z^2+\frac{1}{4}+n(n+1)}(q^{-(2n+1)z} - q^{(2n+1)z}) \qquad \text{(SinSer)}$$

$$= g(q)\sum_{n>-\infty}(-1)^n q^{(n-z+\frac{1}{2})^2},$$

where

$$g(q) = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots .$$

Given a machine that will easily compute a couple of dozen terms of this series, we might think to reciprocate the above equation, "discovering"

$$\frac{1}{g(q)} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \cdots \qquad \text{(GaussKnewIt)}$$

$$= \sum_{n>-\infty}(-q)^{n^2}$$

$$= \vartheta_4.$$

Once before we have seen a quadratic progression of exponents—in the expansion of Π_q. Reciprocating (ΠSer),

$$\frac{q^{\frac{1}{4}}}{\Pi_q} = 1 - 2q + 3q^2 - 6q^3 + 11q^4 - 18q^5 + 28q^6 - 44q^7 + 69q^8 - \cdots .$$

This looks a little like $g(-q)$. Let's try dividing this last series into that of of $g(-q)$:

$$g(-q)\,\Pi_q = q^{\frac14}(1+q^2+q^6+q^{12}+q^{20}+\cdots).$$

But this is just $\sqrt{\Pi_{q^2}}$, so

$$g(-q) = \frac{\sqrt{\Pi_{q^2}}}{\Pi_q} = \prod_{n\ge1}\frac{1-q^{2n-1}}{1+q^{2n-1}}\frac{1+q^{2n}}{1-q^{2n}}.$$

Thus

$$\begin{aligned}g(q) &= \prod_{n\ge1}\frac{1+q^{2n-1}}{1-q^{2n-1}}\frac{1+q^{2n}}{1-q^{2n}} = \prod_{n\ge1}\frac{1+q^n}{1-q^n}\\ &= \prod_{n\ge1}\frac{1+q^n}{(1-q^{2n-1})(1-q^{2n})}\frac{1-q^n}{1-q^n} = \prod_{n\ge1}\frac{1}{(1-q^n)(1-q^{2n-1})}.\end{aligned}$$

(Combinatorially, g is the generating function for partitions into red integers and blue odd integers.) Another relation between $g(q)$ and Π_q arises from dividing both sides of (SinSer) by $q^{-z}-q^z$ and letting $z\to0$:

$$\Pi_q = g(q)\sum_{n\ge0}(-1)^n(2n+1)q^{(n+\frac12)^2}.$$

Using (GaussKnewIt), this becomes

$$\begin{aligned}q^{-\frac14}\frac{\Pi_q}{g(q)} &= \prod_{n\ge1}(1-q^{2n})^3\\ &= (1-2q+2q^4-2q^9+2q^{16}-\cdots)\left(1+q^{\frac{1\cdot2}{2}}+q^{\frac{2\cdot3}{2}}+q^{\frac{3\cdot4}{2}}+\cdots\right)^2\\ &= 1-3q^{1\cdot2}+5q^{2\cdot3}-7q^{3\cdot4}+\cdots.\end{aligned}$$

Double and Triple Angles

Before trying to guess how to q-generalize cosine, we can first test our $\sin_q$ by seeking analogues of

$$\sin 2z = 2\sin z\,\sqrt{1-(\sin z)^2}, \qquad \text{(Double)}$$

$$\sin 3z = 3\sin z - 4(\sin z)^3. \qquad \text{(Triple)}$$

To see why it is too much to expect that these identities should remain true with sin merely replaced by $\sin_q$, note that all of the terms of the latter contain a factor of q^{z^2}. Thus in (Triple), the $\sin_q 3z$ would have terms of the form $q^{9z^2+\frac14+jn\pm kz}$, which would be of a different order from those of the $3\sin_q z$ on the right. But suppose we try

$$\sin_q 3z = A_q\sin_{q^9} z - B_q(\sin_{q^3} z)^3, \qquad \text{(TriTry)}$$

with A_q and B_q to be determined. Then all three monomials have terms containing the same (9th) power of q^{z^2}. Solving for A_q,

$$A_q = \frac{\sin_q 3z}{\sin_{q^9} z} + B_q\frac{(\sin_{q^3} z)^3}{\sin_{q^9} z} = \frac{1-q^{6z}}{1-q^{18z}}\frac{1-q^{18z}}{\sin_{q^9} z}\left(\frac{\sin_q 3z}{1-q^{6z}} + B_q\frac{\sin_{q^3} z}{1-q^{6z}}(\sin_{q^3} z)^2\right).$$

Letting $z\to0$ $\big($using (Small z)$\big)$,

$$A_q = \frac{1}{3\,\Pi_{q^9}}\left(\Pi_q + B_q\,\Pi_{q^3}(\sin_{q^3} 0)^2\right) = \frac13\frac{\Pi_q}{\Pi_{q^9}}.$$

Now putting $z = \frac{\pi}{2}$ in (TriTry),

$$-1 = A_q - B_q,$$

so (TriTry) becomes

$$\sin_q 3z = \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}} \sin_{q^9} z - \left(1 + \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}}\right)(\sin_{q^3} z)^3. \tag{q-Triple}$$

But is it true? As of this writing, the author has not sought a formal proof, being more confident in MACSYMA's empirical confirmation to hundreds of terms than of a page or two of allegedly rigorous prose.

An alternative expression for B_q results from using the expansion (SinEx) and the value of A_q in (TriTry), dividing through by $(q^{-3z} - q^{3z})^3$, and finally letting $z \to 0$. We then have the equation

$$B_q = 1 + \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}} = \frac{\Pi_q}{\Pi_{q^3}^3}\left(\frac{1}{3} + \sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2} - 9\sum_{n\geq 1}\frac{q^{18n}}{(1-q^{18n})^2}\right).$$

Note that as $q \to 1$, (q-Triple) reverts to (Triple), and in general,

$$\lim_{q\to 1}\{\sin_q, \cos_q, \pi_q\} = \{\sin, \cos, \pi\},$$

$$\lim_{q\to 1}\frac{\Pi_{q^a}}{\Pi_{q^b}} = \frac{b}{a}.$$

What about (Double)? There is no way to choose a (nonzero) power of q^{z^2} to match that 1 under the $\sqrt{\quad}$, so let's get rid of the 1 by squaring and distributing:

$$(\sin 2z)^2 = 4(\sin z)^2 - 4(\sin z)^4.$$

Now we can match powers of q^{z^2}, using the form

$$(\sin_q 2z)^2 = A_q(\sin_{q^4} z)^2 - B_q(\sin_{q^2} z)^4. \tag{DubTry}$$

Substituting $z = 0$ and $z = \frac{\pi}{2}$ as we did in (TriTry), we find

$$A_q = \left(\frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}}\right)^2$$

and

$$0 = A_q - B_q.$$

Thus (DubTry) becomes

$$\sin_q 2z = \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}}\sqrt{(\sin_{q^4} z)^2 - (\sin_{q^2} z)^4}. \tag{q-Double}$$

As `Taylor` also confirms (q-Double) for hundreds of terms, we are encouraged to consider $\cos_q$. We can try to define $\cos_q$ by q-generalizing any or all of the following:

$$\cos z = \sqrt{1 - (\sin z)^2} \tag{One}$$

$$= \frac{\sin 2z}{2\sin z} \tag{Two}$$

$$= \sin(\tfrac{\pi}{2} - z). \tag{Phase}$$

(One) seems out because of the naked 1. (Two) holds hope, but it's not obvious which n to choose when introducing $\sin_{q^n}$. Also, (Two)'s q-generalization involves synthesizing a coefficient A_q which would then need to be determined by choosing an amplitude for $\cos_q$. Since we would obviously choose amplitude 1, and (Phase) already has that property, let's naively try

$$\cos_q z := \sin_q(\tfrac{\pi}{2} - z) \tag{CosDef}$$

and see if later we can find an interpretation of (Two). Combining (CosDef) with the (SinProd)s,

$$\begin{aligned}
\cos_q \pi z := \sin_q \pi(\tfrac{1}{2} - z) = q^{z^2} \prod_{n\geq 1} \frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2} &= q^{z^2}\,\frac{\vartheta_4(iz\ln q)}{\vartheta_4} \\
= q^{z^2} \prod_{n\geq 1}\left(1-(q^{-z}-q^{z})^2 \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)& \\
= q^{z^2}\left(1-(q^{-z}-q^{z})^2 \sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right.& \\
\left. + (q^{-z}-q^{z})^4 \frac{\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)^2 - \sum_{n\geq 1}\frac{q^{4n-2}}{(1-q^{2n-1})^4}}{2} - \cdots\right)&.
\end{aligned}$$

Expanding a few `Taylor` terms,

$$\cos_q \pi z = q^{z^2}\left(1-(q^{-2z}-2+q^{2z})q-(2q^{-2z}-4+2q^{2z})q^2-\cdots\right).$$

At greater length, the coefficient tableau is

$\cdots$	$-q^{-6z}$	q^{-4z}	$-q^{-2z}$	1	$-q^{2z}$	q^{4z}	$-q^{6z}$	$\cdots$
				1				q^0
			1	2	1			q^1
			2	4	2			q^2
			4	8	4			q^3
		1	8	14	8	1		q^4
		2	14	24	14	2		q^5
		4	24	40	24	4		q^6
		8	40	64	40	8		q^7
		14	64	100	64	14		q^8
	1	24	100	154	100	24	1	q^9
	2	40	154	232	154	40	2	q^{10}
	4	64	232	344	232	64	4	q^{11}
	8	100	344	504	344	100	8	q^{12}
	14	154	504	728	504	154	14	q^{13}
	24	232	728	1040	728	232	24	q^{14}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

and thus it appears that we can write

$$\begin{aligned}
\cos_q \pi z &= q^{z^2} g(q)\left(1+\sum_{n\geq 1}(-1)^n q^{n^2}(q^{-2nz}+q^{2nz})\right) \\
&= g(q) \sum_{n>-\infty} (-1)^n q^{(n-z)^2}.
\end{aligned}$$

Setting $z = 0$ corroborates (GaussKnewIt).

Now that we see that all the terms of $\cos_q z$, like those of $\sin_q z$, contain a factor of q^{z^2}, it becomes clear (within a constant) how to q-generalize (Two):

$$\cos_{q^2} z = A_q \frac{\sin_q 2z}{\sin_{q^2} z}. \tag{TwoTry}$$

Letting $z \to 0$ determines A_q, and solving for $\sin_q 2z$,

$$\sin_q 2z = \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2} z \cos_{q^2} z. \tag{q-Double$_2$}$$

We don't even need to `Taylor` this one, as it yields to straightforward rearrangements of the infinite products. Such manipulations confirm the more general results,

$$\begin{aligned}
&\sin_{q^k} \pi z \;\; \sin_{q^k} \pi(z+\tfrac{1}{k}) \;\; \sin_{q^k} \pi(z+\tfrac{2}{k}) \;\; \ldots \; \sin_{q^k} \pi(z+\tfrac{k-1}{k}) \\
&\qquad = q^{\frac{(k-1)(k+1)}{12}} \sin_q k\pi z \prod_{n\geq 1} \frac{(1-q^{2n-1})^2}{(1-q^{(2n-1)k})^{2k}} && \text{(Prod)} \\
&\qquad = \frac{\eta(q)^2}{\eta(q^2)^2} \frac{\eta(q^{2k})^{2k}}{\eta(q^k)^{2k}} \sin_q k\pi z && \text{(EtaProd)} \\
&\qquad = \frac{\Pi_{q^2}}{\Pi_q} \frac{\Pi_{q^4}^2}{\Pi_{q^2}^2} \frac{\Pi_{q^8}^4}{\Pi_{q^4}^4} \cdots \frac{\Pi_{q^k}^{\frac{k}{2}}}{\Pi_{q^{\frac{k}{2}}}^{\frac{k}{2}}} \sin_q k\pi z && \text{(Pow2)} \\
&\qquad = \frac{\Pi_{q^k}^k}{\Pi_q} \frac{\sin_q k\pi z}{q^{\frac{(k-1)(k+1)}{6}} \prod_{n\geq 1} \frac{\Pi_{q^{2^n}}^{2^{n-1}}}{\Pi_{q^{2^n k}}^{2^{n-1}k}} \, q^{2^{2n-3}(k-1)(k+1)}}, && \text{(UnTel)}
\end{aligned}$$

where $\eta(q)$ is the Dedekind eta function:

$$\eta(q) := q^{\frac{1}{24}} \prod_{n\geq 1} 1 - q^n, \qquad \Pi_q = \frac{\eta(q^2)^4}{\eta(q)^2}, \qquad \eta(q)^6 = \frac{\Pi_q^5}{\Pi_{q^2}^2} - 16\, \Pi_q \, \Pi_{q^2}^2 \, .$$

(This last relation is equivalent to Jacobi's "*æquatio identica satis abstrusa*".)

The relation (Pow2), which is restricted to when k is a power of 2, follows from (q-Double$_2$). (UnTel) regeneralizes (Pow2) to arbitrary integer k by "untelescoping" (Pow2) after factoring out sufficient powers of q for convergence. It retelescopes to (EtaProd) when reexpressed in η functions.

Having now found $\cos_q z := \sin_q(\frac{\pi}{2} - z)$ compatible with both (Two) and (Phase), we can use (q-Double$_2$) to determine that

$$\cos_{q^2} \tfrac{\pi}{4} = \sin_{q^2} \tfrac{\pi}{4} = \sqrt{\frac{\Pi_{q^2}}{\Pi_q}}. \tag{Root2}$$

Further emboldened, we use the same method as in (q-Triple) to determine

$$\cos_q 2z = (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2, \tag{q-Double$_3$}$$

and again find empirical confirmation. Writing $z \leftarrow \frac{\pi}{2} - z$ in (q-Double),

$$\sin_q 2z = \frac{1}{2} \frac{\Pi_q}{\Pi_{q^4}} \sqrt{(\cos_{q^4} z)^2 - (\cos_{q^2} z)^4}. \tag{q-Double$_4$}$$

Eliminating $\sin_q$ (as opposed to $\sin_{q^2}$) and $\cos_q$ from (q-Double)^2, (q-Double$_3$), and (q-Double$_4$)^2, and then writing q for q^2, we find

$$\cos_q 2z = (\cos_q z)^4 - (\sin_q z)^4. \qquad (q\text{-Double}_5)$$

Since this result involves the same power of q throughout, it is not surprising to learn that it restates a familiar formula of ϑ theory.

We can now attack

$$\sin 3z = 3(\cos z)^2 \sin z - (\sin z)^3,$$

which will come out different from (q-Triple) because there is no direct equivalent of $\cos^2 + \sin^2 = 1$. In fact,

$$\sin_q 3z = \frac{\Pi_q}{\Pi_{q^3}} (\cos_{q^3} z)^2 \sin_{q^3} z - (\sin_{q^3} z)^3. \qquad (q\text{-Triple}_2)$$

Plugging (Root2) into (q-Double) and squaring, we find an unexpected relationship among the Π_{q^n}:

$$4 = \frac{\Pi_q^2}{\Pi_{q^4}\,\Pi_{q^2}} - \frac{\Pi_{q^2}^2}{\Pi_{q^4}^2}. \qquad (\mathrm{Pi}_{1,2,4})$$

Digression on Computing π

My colleague Eugene Salamin points out that (Pi$_{1,2,4}$) provides a quadratically convergent method for computing π (or (other) logarithms). Solve (Pi$_{1,2,4}$) for Π_q and write $q^{2^{-n}}$ for q:

$$\Pi_{q^{2^{-n}}} = \Pi_{q^{2^{1-n}}} \sqrt{\frac{\Pi_{q^{2^{1-n}}}}{\Pi_{q^{2^{2-n}}}} + 4\frac{\Pi_{q^{2^{2-n}}}}{\Pi_{q^{2^{1-n}}}}}$$

or

$$a_n = a_{n-1} \sqrt{\frac{1}{2}\left(\frac{a_{n-1}}{a_{n-2}} + \frac{a_{n-2}}{a_{n-1}}\right)},$$

if

$$a_n := 2^{-n}\,\Pi_{q^{2^{-n}}} = \frac{\pi_{q^{2^{-n}}}}{2^n(1 - q^{2^{1-n}})}.$$

Then

$$\lim_{n\to\infty} a_n = -\frac{\pi}{2\ln q}.$$

If q is chosen to be 2^{-8}, say, then a_0 and a_1 can be immediately written down in binary:

$$a_0 = \tfrac{1}{4}(1.\overbrace{00\ldots01}^{8}\overbrace{00\ldots01}^{16}\overbrace{00\ldots01}^{24}\ldots)_2^2,$$

$$a_1 = \tfrac{1}{4}(1.\overbrace{0001}^{4}\overbrace{00\ldots01}^{8}\overbrace{00\ldots01}^{12}\ldots)_2^2.$$

Empirically, one then finds

$$2(\ln 256)a_n = \pi + 10^{-2^{n+.59\ldots}}.$$

End of π Digression

Relations even stranger than (Pi$_{1,2,4}$) arise from combining (Root2) and the (q-Triple)s. In fact, it seems that for any three (or more) distinct integers n, $\{\Pi_{q^n}\}$ satisfy homogeneous polynomials. One can narrow the search for such relationships by noting that the factor of $q^{\frac{n}{4}}$ in Π_{q^n} will require that in each term of the form $\Pi_{q^a}\,\Pi_{q^b}\,\Pi_{q^c}\ldots$, the sum $\pmod 4$ of $a+b+c+\cdots$ must be the same. One then proposes a polynomial with undetermined coefficients, Taylor expands it, and attempts to solve the linear system resulting from equating powers of q. (To reject unsolvable systems, you can first take an approximate Wronskian of the proposed set of polynomial terms to see if they seem linearly interdependent. In MACSYMA, one may form a matrix of the first few derivatives of the `Taylor` expansions, and then simply see if the `Determinant` $= 0 + \ldots$. Alternatively, use numerical approximations with various values of q.) Here are some typical identities so discovered (but not proved!):

$$\Pi_{q^2}^2 + 2\,\Pi_{q^2}\,\Pi_{q^6} = \Pi_q\,\Pi_{q^3} + 3\,\Pi_{q^6}^2$$

$$\frac{\Pi_{q^2}\,\Pi_{q^3}^2}{\Pi_{q^6}\,\Pi_q^2} = \frac{\Pi_{q^2} - \Pi_{q^6}}{\Pi_{q^2} + 3\,\Pi_{q^6}}$$

$$\Pi_{q^3}^2 + 3\,\Pi_q\,\Pi_{q^9} = \sqrt{\Pi_q\,\Pi_{q^9}}\,\left(\Pi_q + 3\,\Pi_{q^9}\right)$$

$$\sqrt{\Pi_{q^2}\,\Pi_{q^6}}\,\left(\Pi_q^2 - 3\,\Pi_{q^3}^2\right) = \sqrt{\Pi_q\,\Pi_{q^3}}\,\left(\Pi_{q^2}^2 + 3\,\Pi_{q^6}^2\right)$$

$$\Pi_{q^2}\,\Pi_{q^3}^4 = \Pi_{q^6}\left(\Pi_{q^2} - \Pi_{q^6}\right)^3\left(\Pi_{q^2} + 3\,\Pi_{q^6}\right)$$

$$\Pi_{q^6}\,\Pi_q^4 = \Pi_{q^2}\left(\Pi_{q^2} - \Pi_{q^6}\right)\left(\Pi_{q^2} + 3\,\Pi_{q^6}\right)^3$$

$$\Pi_q\,\Pi_{q^3}\left(\Pi_q^2 \pm 4\,\Pi_{q^2}^2\right)^2 = \Pi_{q^2}^2\left(\Pi_q \mp \Pi_{q^3}\right)\left(\Pi_q \pm 3\,\Pi_{q^3}\right)^3$$

$$\Pi_q\,\Pi_{q^3}\left(\Pi_{q^3}^2 \pm 4\,\Pi_{q^6}^2\right)^2 = \Pi_{q^6}^2\left(\Pi_q \mp \Pi_{q^3}\right)^3\left(\Pi_q \pm 3\,\Pi_{q^3}\right)$$

$$\Pi_{q^2}\,\Pi_{q^5}^4\left(16\,\Pi_{q^{10}}^4 - \Pi_{q^5}^4\right) = \Pi_{q^{10}}^3\left(5\,\Pi_{q^{10}} - \Pi_{q^2}\right)\left(\Pi_{q^2} - \Pi_{q^{10}}\right)^5$$

$$\Pi_{q^{10}}\,\Pi_q^4\left(16\,\Pi_{q^2}^4 - \Pi_q^4\right) = \Pi_{q^2}^3\left(5\,\Pi_{q^{10}} - \Pi_{q^2}\right)^5\left(\Pi_{q^2} - \Pi_{q^{10}}\right)$$

$$\Pi_q\,\Pi_{q^5}\left(16\,\Pi_{q^2}^4 - \Pi_q^4\right)^2 = \Pi_{q^2}^4\left(5\,\Pi_{q^5} - \Pi_q\right)^5\left(\Pi_{q^5} - \Pi_q\right)$$

$$\Pi_q\,\Pi_{q^5}\left(16\,\Pi_{q^{10}}^4 - \Pi_{q^5}^4\right)^2 = \Pi_{q^{10}}^4\left(5\,\Pi_{q^5} - \Pi_q\right)\left(\Pi_{q^5} - \Pi_q\right)^5$$

$$\Pi_{q^2}\,\Pi_{q^{10}}\left(\Pi_{q^5} - \Pi_q\right)\left(5\,\Pi_{q^5} - \Pi_q\right) = \left(\Pi_q\,\Pi_{q^{10}} - \Pi_{q^2}\,\Pi_{q^5}\right)^2.$$

The $\pm$ signs indicate that there are two ways to write the given equation (e.g. $(x \pm y)(x \mp y) = z$), rather than that there are two equations. Thus one cannot use the above results (nor any others known to this writer) to establish an algebraic relation involving only two of the Π_{q^n}.

What about $\sin_q \frac{\pi}{6}$? Plug $z = \frac{\pi}{3}$ into (q-Triple$_2$) and (q-Double$_5$), the latter becoming

$$\begin{aligned}\cos_q \tfrac{\pi}{3} &= (\cos_q \tfrac{\pi}{6})^4 - (\sin_q \tfrac{\pi}{6})^4\\ &= (\sin_q \tfrac{\pi}{3})^4 - (\cos_q \tfrac{\pi}{3})^4.\end{aligned}$$

Writing q^3 for q and eliminating the $\sin_{q^3} \frac{\pi}{3}$, we find

$$(\cos_{q^3} \tfrac{\pi}{3})^3 = (\sin_{q^3} \tfrac{\pi}{6})^3 = \frac{1}{\left(\dfrac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1} = q\prod_{n\ge 1}\frac{(1+q^{3n})^9}{(1+q^n)^3},$$

the latter following from $z = \frac{1}{6}$ in (SinProd1). Eliminating instead the $\cos_{q^3} \frac{\pi}{3}$ would have lead to

$$(\sin_{q^3} \tfrac{\pi}{3})^3 = (\cos_{q^3} \tfrac{\pi}{6})^3 = \frac{\left(\dfrac{\Pi_q}{\Pi_{q^3}}\right)^{\frac{3}{2}}}{\left(\dfrac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1} = q^{\frac{1}{4}}\prod_{n\ge 1}\frac{(1-q^{2n})^3(1+q^{3n})^3}{(1-q^{3n})^3},$$

the latter following from $z = \frac{1}{3}$ in (SinProd1).

Addition Formulas

Might it be possible to generalize (q-Double$_2$) into something like

$$\sin_q(x+y) = A_q(x,y)\sin_{q^2} x \cos_{q^2} y + B_q(x,y)\cos_{q^2} x \sin_{q^2} y?$$

Because of the q^{z^2} factor in $\sin_q z$, A_q and B_q must contain a factor of $q^{-(x-y)^2}$. This suggests

$$\sin_q(x+y) = A_q(x-y)\sin_{q^2} x \cos_{q^2} y + B_q(x-y)\cos_{q^2} x \sin_{q^2} y.$$

Substituting $x \leftarrow x + \frac{\pi}{2}$, $y \leftarrow y + \frac{\pi}{2}$ reveals that $A_q(x-y) = B_q(x-y)$. Letting $y = x - t$ renders A_q independent of x, which is empirically confirmed by solving for $A_q(t)$ and noting the absence of x from the terms of the Taylor series. By (q-Double$_2$), we know that

$$A_q(0) = \frac{\Pi_q}{2\,\Pi_{q^2}}. \tag{Clue}$$

Choosing instead $y = x - \frac{\pi}{2}$ and comparing with (q-Double$_3$) reveals

$$A_q(\tfrac{\pi}{2}) = 1,$$

and similar arguments show that

$$A_q(t+\pi) = A_q(t).$$

The evanescent appearance of Π_q in (Clue) should recollect (Small z), and we quickly discover that

$$A_q(t) = \frac{\sin_q t}{\sin_{q^2} t}$$

meets all requirements. Extensive Taylor expansion then reassures us that indeed

$$\sin_q(x+y) = \frac{\sin_q(x-y)}{\sin_{q^2}(x-y)}\left(\sin_{q^2} x \cos_{q^2} y + \cos_{q^2} x \sin_{q^2} y\right). \tag{Add}$$

Shifting $y \leftarrow y + \frac{\pi}{2}$ gives the corresponding cosine formula. It is interesting that no obvious applications of (Add) yield either the (Triple) angle formulas, or the quintuple formula in the summary tables at the end of this paper.

Mysteries

What is the $\sin_q(x+y+z)$ addition formula generalizing (q-Triple$_2$)?

Are there $\sinh_q$ and $\cosh_q$, perhaps corresponding to ϑ_2 and ϑ_3?

Is there a q-generalization of $x - \frac{x^3}{3!} + \cdots$ equal to $\sin_q x$?

Is there a q-exponential and a q-deMoivre's theorem that would simplify everything?

Bibliographic Notes

In "A Basic-sine and cosine with symbolical solutions of certain differential equations", *Proceedings of the Edinburgh Mathematical Society* **XXII** (1904), 28-39, The Rev. Dr. F. H. Jackson q-generalizes $\sin x := x - \frac{x^3}{3!} + \cdots$ to get a function not obviously related to our $\sin_q$.

In "Pseudo-periodic Functions Analogous to the Circular Functions", *The Messenger of Mathematics* **34** (1905), 32-39, Jackson defines

$$\left.\begin{aligned} S_p(x) &:= p^{-\frac{1}{2}(\frac{x}{\omega}-\frac{1}{2})^2} \sin_{p^{-\frac{1}{2}}} \pi\frac{x}{\omega}, \\ \omega &:= p^{\frac{7}{8}}\pi_{p^{-\frac{1}{2}}} \end{aligned}\right. \qquad |p| \geq 1,$$

when converted to our notation.

Then, in "The Basic Gamma-Function and the Elliptic Functions", *Proceedings of the Royal Society of London* Series **A 76** (1905), 127-144 he defines

$$S_p(x,\omega) := p^{-\frac{\omega}{2}(\frac{x}{\omega}-\frac{1}{2})^2} \sin_{p^{\frac{\omega}{2}}} \pi\frac{x}{\omega}.$$

But for neither of these S_p does he offer so much as a double angle formula.

Whittaker and Watson's *Modern Analysis* (1902-1978), 462-490 is an informative introduction to ϑ functions. It provides powerful methods for formula verification, if not discovery.

Collected Definitions and Identities

$$z!_q := (1-q)^{-z}\prod_{n\ge1}\frac{1-q^n}{1-q^{n+z}} \text{ ``='' } \frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^z}{1-q} = q\text{-factorial}(z)$$

$$\sin_q \pi z := \frac{q^{z(z-1)}\pi_q}{(z-1)!_{q^2}\,(-z)!_{q^2}} = q^{(z-1/2)^2}\prod_{n\ge1}\frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2} = iq^{z^2}\,\frac{\vartheta_1(iz\ln q)}{\vartheta_4}$$

$$= q^{z^2}\,\Pi_q(q^{-z}-q^z)\prod_{n\ge1}\left(1-(q^{-z}-q^z)^2\frac{q^{2n}}{(1-q^{2n})^2}\right) = -\sin_q \pi(z+1)$$

$$= q^{z^2}\,\Pi_q\left((q^{-z}-q^z)-(q^{-z}-q^z)^3\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}\right.$$

$$\left.+(q^{-z}-q^z)^5\frac{\left(\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}\right)^2-\sum_{n\ge1}\frac{q^{4n}}{(1-q^{2n})^4}}{2}-\cdots\right)$$

$$= \frac{\sum_{n>-\infty}(-1)^n q^{(n-z+\frac12)^2}}{\sum_{n>-\infty}(-q)^{n^2}} = q^{(z-1/2)^2}\prod_{|n|<\infty}q^{z-1/2}\frac{1-q^{2n-2z}}{1-q^{2n-1}} = -\frac{\sin_q \pi\left(z+\frac{i\pi}{\ln q}\right)}{e^{2\pi iz-\pi^2/\ln q}},$$

$$\cos_q \pi z := \sin_q \pi(\tfrac12 - z) = q^{z^2}\prod_{n\ge1}\frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2} = q^{z^2}\,\frac{\vartheta_4(iz\ln q)}{\vartheta_4}$$

$$= q^{z^2}\prod_{n\ge1}\left(1-(q^{-z}-q^z)^2\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)$$

$$= q^{z^2}\left(1-(q^{-z}-q^z)^2\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right.$$

$$\left.+(q^{-z}-q^z)^4\frac{\left(\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)^2-\sum_{n\ge1}\frac{q^{4n-2}}{(1-q^{2n-1})^4}}{2}-\cdots\right)$$

$$= \frac{\sum_{n>-\infty}(-1)^n q^{(n-z)^2}}{\sum_{n>-\infty}(-q)^{n^2}} = q^{z^2}\prod_{|n|<\infty}q^{z}\frac{1-q^{2n-2z-1}}{1-q^{2n-1}} = \frac{\cos_q \pi\left(z+\frac{i\pi}{\ln q}\right)}{e^{2\pi iz-\pi^2/\ln q}},$$

$$\Pi_q := \frac{\pi_q}{1-q^2} := \lim_{z\to0}\frac{\sin_q \pi z}{q^{-z}-q^z} = q^{\frac14}\frac{(-\frac12)!^2_{q^2}}{1-q^2} = q^{\frac14}\prod_{n\ge1}\frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = \frac{q\text{-Wallis product}}{1-q}$$

$$= q^{\frac14}(1+q+q^3+q^6+q^{10}+\cdots)^2 = \frac{\vartheta_2\vartheta_3}{2} = \frac{\vartheta_2^2(0,q^{\frac12})}{4} = \frac{\sum_{n>-\infty}(4n+1)q^{(2n+\frac12)^2}}{\sum_{n>-\infty}(-q)^{n^2}}$$

Then

$$
\begin{aligned}
\cos_q 2z &= (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2 \\
&= (\cos_q z)^4 - (\sin_q z)^4 = \cos 2z \prod_{n\geq 0} \left(\sin^2_{q^{2^{-n}}} z + \cos^2_{q^{2^{-n}}} z\right) \\
\sin_q 2z &= \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2} z \cos_{q^2} z \\
&= \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}} \sqrt{(\sin_{q^4} z)^2 - (\sin_{q^2} z)^4} = \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}} \sqrt{(\cos_{q^4} z)^2 - (\cos_{q^2} z)^4} \\
\sin_q 3z &= \frac{\Pi_q}{\Pi_{q^3}} (\cos_{q^3} z)^2 \sin_{q^3} z - (\sin_{q^3} z)^3 \\
&= \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}} \sin_{q^9} z - \left(1 + \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}}\right) (\sin_{q^3} z)^3 \\
\sin_q 5z &= \frac{\Pi_q}{\Pi_{q^5}} (\cos_{q^5} z)^4 \sin_{q^5} z - \sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3} - 2\frac{\Pi_q^2}{\Pi_{q^5}^2} + 5\frac{\Pi_q}{\Pi_{q^5}}}\, (\cos_{q^5} z)^2 (\sin_{q^5} z)^3 + (\sin_{q^5} z)^5 \\
\operatorname{ccs}_q \pi z &:= \frac{\cos_{q^2} \pi z}{\cos_q \pi z} = \sin^2_{q^2} \tfrac{\pi}{2} z + \cos^2_{q^2} \tfrac{\pi}{2} z = q^{z^2} \prod_n q^z \frac{1 + q^{2n-2z-1}}{1 + q^{2n-1}}, \\
\operatorname{ssn}_q \pi z &:= \operatorname{ccs}_q \pm \left(\pi(z \pm \tfrac{1}{2})\right) = \frac{\sin_{q^2} \pi z}{\sin_q \pi z} = q^{(z-1/2)^2} \prod_n q^{z-1/2} \frac{1 + q^{2n-2z}}{1 + q^{2n-1}}, \\
\operatorname{ssn}_1 z &= \operatorname{ccs}_1 z = 1, \quad (2z \not\equiv 0 \pmod{\pi}), \\
\sin_q(x+y) &= \frac{\sin_{q^2} x \cos_{q^2} y + \cos_{q^2} x \sin_{q^2} y}{\operatorname{ssn}_q (x-y)}, \\
\cos_q(x+y) &= \frac{\cos_{q^2} x \cos_{q^2} y - \sin_{q^2} x \sin_{q^2} y}{\operatorname{ccs}_q (x-y)}.
\end{aligned}
$$

$$
\sin_q(x+y)\sin_q(x-y) = \sin^2_q x \cos^2_q y - \cos^2_q x \sin^2_q y
$$

$$
\begin{aligned}
\sin_q(a+b+x+y) \operatorname{ccs}_q(x-y) \sin_q(a-b) = {}& \sin_q(a+x)\sin_q(a+y)\operatorname{ccs}_q(b+x)\operatorname{ccs}_q(b+y) \\
&- \sin_q(b+x)\sin_q(b+y)\operatorname{ccs}_q(a+x)\operatorname{ccs}_q(a+y),
\end{aligned}
$$

$$
\begin{aligned}
\sin_q\left(\sum_{1\leq j\leq n} c_j\right) &= \prod_{1\leq i\leq n} \frac{\operatorname{ssn}_q a_i}{\operatorname{ssn}_q(a_i - c_i)} \sum_{1\leq i\leq n} \frac{\sin_q c_i}{\operatorname{ssn}_q a_i} \operatorname{ssn}_q\left(a_i - \sum_{1\leq j\leq n} c_j\right) \prod_{1\leq j\leq n}^{j\neq i} \frac{\sin_q(c_j + a_i - a_j)}{\sin_q(a_i - a_j)}, \\
&= \prod_{1\leq i\leq n} \frac{\sin_q a_i}{\sin_q(a_i - c_i)} \sum_{1\leq i\leq n} \frac{\sin_q c_i}{\sin_q a_i} \sin_q\left(a_i - \sum_{1\leq j\leq n} c_j\right) \prod_{1\leq j\leq n}^{j\neq i} \frac{\sin_q(c_j + a_i - a_j)}{\sin_q(a_i - a_j)}.
\end{aligned}
$$

Then $c_j := a_j - b_j,\ a_1 + \cdots + a_n = b_1 + \cdots + b_n$ in the latter gives

$$\sum_{1\le i\le n} \sin_q(a_i - b_i) \prod_{1\le j\le n}^{j\ne i} \frac{\sin_q(a_i - b_j)}{\sin_q(a_i - a_j)} = 0.$$

E.g., when $n := 4, c_j := (-)^j \pi/2, a_1 := w, a_2 := x, a_3 := y, a_4 := z$,

$$\begin{aligned}
&\cos_q(z-x)\cos_q(x-w)\cos_q(x-y)\sin_q(y-w)\sin_q(y-z)\sin_q(z-w)\\
&+\sin_q(z-x)\sin_q(x-w)\cos_q(x-y)\cos_q(y-w)\cos_q(y-z)\sin_q(z-w)\\
&+\cos_q(z-x)\sin_q(x-w)\sin_q(x-y)\sin_q(y-w)\cos_q(y-z)\cos_q(z-w)\\
&+\sin_q(z-x)\cos_q(x-w)\sin_q(x-y)\cos_q(y-w)\sin_q(y-z)\cos_q(z-w) = 0.
\end{aligned}$$

While if $q \to 1$ in the ssn-ccs version,

$$\sin\left(\sum_{1\le i\le n} c_i\right) = \sum_{1\le i\le n} \sin c_i \prod_{1\le j\le n}^{j\ne i} \frac{\sin(c_j + a_i - a_j)}{\sin(a_i - a_j)}.$$

$$\begin{aligned}
\frac{\sin_{q^4} x}{\sin_{q^2} x}(\cos_q y)^2 + \frac{\cos_{q^4} x}{\cos_{q^2} x}(\sin_q y)^2 &= \frac{\sin_{q^4} y}{\sin_{q^2} y}(\cos_q x)^2 + \frac{\cos_{q^4} y}{\cos_{q^2} y}(\sin_q x)^2\\
\frac{\sin_{q^3} x}{\sin_q x}(\cos_q y)^2 + \frac{\cos_{q^3} x}{\cos_q x}(\sin_q y)^2 &= \frac{\sin_{q^3} y}{\sin_q y}(\cos_q x)^2 + \frac{\cos_{q^3} y}{\cos_q y}(\sin_q x)^2\\
\sin_{q^3} x \sin_q(2x-y) - \sin_{q^3} y \sin_q(2y-x) &= \sin_{q^3}(x-y)\sin_q(x+y)\\
\sin_{q^3} x \sin_q(2y-x) - \sin_{q^3} y \sin_q(2x-y) &= \cos_{q^3} y \cos_q(2x-y) - \cos_{q^3} x \cos_q(2y-x)
\end{aligned}$$

$$\begin{aligned}
\sin_q' 0 &= -\frac{2\ln q}{\pi}\,\Pi_q\\
\text{Fixthis!}\cos_q'' 0 = (\sin_q' 0)^2 + \frac{\cos''_{\sqrt{q}} 0}{2} &= \frac{2\ln q}{\pi^2}\left(1 - 4\ln q \sum_{n\ge1} \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)
\end{aligned}$$

$$\begin{aligned}
(\sin_q' z)^2 &= \sin_q z \sin_q'' z + (\sin_q' 0)^2(\cos_q z)^2 - \cos_q'' 0\,(\sin_q z)^2\\
\sin_q' z &= \frac{\sin_{q^2}' z + \sin_{q^2}' 0 \cos_{q^2} z}{2\,\mathrm{ssn}_q z}
\end{aligned}$$

$$\sum_{n\ge1} \frac{q^n}{(1-q^n)^2} = \sum_{n\ge1} \sigma_1(n) q^n = \sum_{n\ge1} \frac{nq^n}{1-q^n}$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-2\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}=\frac{1}{24}\left(\frac{\Pi_q^4}{\Pi_{q^2}^2}-1\right)+\frac{2}{3}\Pi_{q^2}^2$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-3\sum_{n\ge1}\frac{q^{3n}}{(1-q^{3n})^2}=\frac{(\Pi_q^2+3\Pi_{q^3}^2)^2}{12\,\Pi_q\,\Pi_{q^3}}-\frac{1}{12}$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-4\sum_{n\ge1}\frac{q^{4n}}{(1-q^{4n})^2}=\frac{1}{8}\left(\frac{\Pi_q^4}{\Pi_{q^2}^2}-1\right)$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\ge1}\frac{q^{5n}}{(1-q^{5n})^2}=\frac{1}{6}\left((\Pi_q^2+2\Pi_q\Pi_{q^5}+5\Pi_{q^5}^2)\sqrt{\frac{\Pi_q}{\Pi_{q^5}}-2+5\frac{\Pi_{q^5}}{\Pi_q}}-1\right)$$

$$\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}-9\sum_{n\ge1}\frac{q^{18n}}{(1-q^{18n})^2}=\frac{\Pi_{q^3}^3}{\Pi_q}+\frac{1}{3}\left(\frac{\Pi_{q^3}^3}{\Pi_{q^9}}-1\right)$$

$$\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-2\sum_{n\ge1}\frac{q^{4n-2}}{(1-q^{4n-2})^2}=\Pi_{q^2}^2=\sum_{n\ge1}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}$$

$$\frac{\displaystyle\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-3\sum_{n\ge1}\frac{q^{6n-3}}{(1-q^{6n-3})^2}}{\Pi_{q^3}^2}=\frac{\Pi_q}{\Pi_{q^3}}$$

$$\frac{\displaystyle\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\ge1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}}{\Pi_{q^5}^2}=\frac{\dfrac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\dfrac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\dfrac{\Pi_q}{\Pi_{q^5}}-4-\dfrac{\Pi_{q^5}}{\Pi_q}}=\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}$$

$$\frac{\displaystyle\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-9\sum_{n\ge1}\frac{q^{18n-9}}{(1-q^{18n-9})^2}}{\Pi_{q^9}^2}=\left(\frac{\Pi_q}{\Pi_{q^9}}+3\right)\sqrt{\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{\frac{3}{2}}-3\frac{\Pi_q}{\Pi_{q^9}}+3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{\frac{1}{2}}}$$

$$\Pi_q^4=6\sum_{n\ge1}\frac{q^{4n-2}}{(1-q^{2n-1})^4}+\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}=\sum_{n\ge1}\frac{n^3q^n}{1-q^{2n}}\ ^*$$

$$\sum_{n\ge1}\frac{nq^{2n}}{1+q^{2n}}\sum_{n\ge1}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}=\sum_{n\ge1}\frac{B_3(n)}{3}\frac{q^{2n-1}}{1-q^{4n-2}},$$

$$\tfrac{B_3(n)}{3}:=\tfrac{(n-1)(n-\frac12)n}{3}=1^2+2^2+\cdots+(n-1)^2$$

Implying

$$\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-4\sum_{n\ge1}\frac{q^{8n-4}}{(1-q^{8n-4})^2}=\frac{1}{8}\left(\frac{\Pi_q^4}{\Pi_{q^2}^2}-\frac{\Pi_{q^2}^4}{\Pi_{q^4}^2}\right)=\Pi_{q^2}^2+2\Pi_{q^4}^2$$

$$\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-6\sum_{n\ge1}\frac{q^{12n-6}}{(1-q^{12n-6})^2}=\Pi_{q^2}^2+2\Pi_{q^2}\Pi_{q^6}=\Pi_q\Pi_{q^3}+3\Pi_{q^6}^2$$

*$I.e.$, the number of ways to express $n-1$ as the sum of 8 triangular numbers equals n^3 times the sum of the cubes of the reciprocals of the odd divisors of n.

$$\cos_{q^2}\tfrac{\pi}{4} = \sin_{q^2}\tfrac{\pi}{4} = \sqrt{\frac{\Pi_{q^2}}{\Pi_q}}$$

$$(\cos_{q^3}\tfrac{\pi}{3})^3 = (\sin_{q^3}\tfrac{\pi}{6})^3 = \frac{1}{\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1} = q\prod_{n\geq 1}\frac{(1+q^{3n})^9}{(1+q^n)^3}$$

$$(\sin_{q^3}\tfrac{\pi}{3})^3 = (\cos_{q^3}\tfrac{\pi}{6})^3 = \frac{\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^{\frac{3}{2}}}{\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1} = q^{\frac{1}{4}}\prod_{n\geq 1}\frac{(1-q^{2n})^3(1+q^{3n})^3}{(1-q^{3n})^3}$$

$$(\cos_{q^5}\tfrac{\pi}{5})^5 = \Pi_{q^5}\,\frac{\Pi_q^2 - \Pi_q\,\Pi_{q^5} + 2\,\Pi_{q^5}^2 + \Pi_q\sqrt{\Pi_q^2 - 2\,\Pi_q\,\Pi_{q^5} + 5\,\Pi_{q^5}^2}}{2(\Pi_q - \Pi_{q^5})^3}$$

$$4 = \frac{\Pi_q^2}{\Pi_{q^4}\,\Pi_{q^2}} - \frac{\Pi_{q^2}^2}{\Pi_{q^4}^2}$$

$$\Pi_{q^3}^2 + 3\,\Pi_q\,\Pi_{q^9} = \sqrt{\Pi_q\,\Pi_{q^9}}\,\left(\Pi_q + 3\,\Pi_{q^9}\right)$$

$$\sqrt{\Pi_{q^2}\,\Pi_{q^6}}\,\left(\Pi_q^2 - 3\,\Pi_{q^3}^2\right) = \sqrt{\Pi_q\,\Pi_{q^3}}\,\left(\Pi_{q^2}^2 + 3\,\Pi_{q^6}^2\right)$$

$$\Pi_{q^2}\,\Pi_{q^3}^4 = \Pi_{q^6}\left(\Pi_{q^2} - \Pi_{q^6}\right)^3\left(\Pi_{q^2} + 3\,\Pi_{q^6}\right)$$

$$\Pi_{q^6}\,\Pi_q^4 = \Pi_{q^2}\left(\Pi_{q^2} - \Pi_{q^6}\right)\left(\Pi_{q^2} + 3\,\Pi_{q^6}\right)^3$$

$$\Pi_q\,\Pi_{q^3}\left(\Pi_q^2 \pm 4\,\Pi_{q^2}^2\right)^2 = \Pi_{q^2}^2\left(\Pi_q \mp \Pi_{q^3}\right)\left(\Pi_q \pm 3\,\Pi_{q^3}\right)^3$$

$$\Pi_q\,\Pi_{q^3}\left(\Pi_{q^3}^2 \pm 4\,\Pi_{q^6}^2\right)^2 = \Pi_{q^6}^2\left(\Pi_q \mp \Pi_{q^3}\right)^3\left(\Pi_q \pm 3\,\Pi_{q^3}\right)$$

$$\Pi_{q^2}\,\Pi_{q^5}^4\left(16\,\Pi_{q^{10}}^4 - \Pi_{q^5}^4\right) = \Pi_{q^{10}}^3\left(5\,\Pi_{q^{10}} - \Pi_{q^2}\right)\left(\Pi_{q^2} - \Pi_{q^{10}}\right)^5$$

$$\Pi_{q^{10}}\,\Pi_q^4\left(16\,\Pi_{q^2}^4 - \Pi_q^4\right) = \Pi_{q^2}^3\left(5\,\Pi_{q^{10}} - \Pi_{q^2}\right)^5\left(\Pi_{q^2} - \Pi_{q^{10}}\right)$$

$$\Pi_q\,\Pi_{q^5}\left(16\,\Pi_{q^2}^4 - \Pi_q^4\right)^2 = \Pi_{q^2}^4\left(5\,\Pi_{q^5} - \Pi_q\right)^5\left(\Pi_{q^5} - \Pi_q\right)$$

$$\Pi_q\,\Pi_{q^5}\left(16\,\Pi_{q^{10}}^4 - \Pi_{q^5}^4\right)^2 = \Pi_{q^{10}}^4\left(5\,\Pi_{q^5} - \Pi_q\right)\left(\Pi_{q^5} - \Pi_q\right)^5$$

$$\Pi_{q^2}\,\Pi_{q^{10}}\left(\Pi_{q^5} - \Pi_q\right)\left(5\,\Pi_{q^5} - \Pi_q\right) = \left(\Pi_q\,\Pi_{q^{10}} - \Pi_{q^2}\,\Pi_{q^5}\right)^2$$

$$\frac{\Pi_{q^2}\,\Pi_{q^3}^2}{\Pi_{q^6}\,\Pi_q^2} = \frac{\Pi_{q^2} - \Pi_{q^6}}{\Pi_{q^2} + 3\,\Pi_{q^6}}$$

The $1, 3, 9$ case can be rewritten

$$\sqrt{\frac{\Pi_q}{\Pi_{q^9}}} = 1 + \sqrt[3]{\left(\frac{\Pi_{q^3}}{\Pi_{q^9}}\right)^2 - 1}$$

$$= \frac{1 + \sqrt[3]{9\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1}}{3}.$$

Also

$$6\left(\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-5\sum_{n\geq 1}\frac{q^{5n}}{(1-q^{5n})^2}\right)+1=\left(\frac{\Pi_q}{\Pi_{q^5}}+2+5\frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\geq 1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$

$$3\left(\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}-9\sum_{n\geq 1}\frac{q^{9n}}{(1-q^{9n})^2}\right)+1=\left(\sqrt{\frac{\Pi_q}{\Pi_{q^9}}}+3\sqrt{\frac{\Pi_{q^9}}{\Pi_q}}\right)\left(\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-9\sum_{n\geq 1}\frac{q^{18n-9}}{(1-q^{18n-9})^2}\right)$$

$$=(\Pi_q+3\,\Pi_{q^9})\frac{3(\Pi_q\,\Pi_{q^9}+\Pi_{q^3}^2)^2+\Pi_q\,\Pi_{q^9}\,(\Pi_q-3\,\Pi_{q^9})^2}{4\,\Pi_q\,\Pi_{q^3}\,\Pi_{q^9}}$$