Modular forms and two new integer sequences at level 7

A Thesis presented in partial fulfilment of the requirements for the degree of

Master of Science
in
Mathematics

at Massey University, Albany, New Zealand

Lynette Anne O'Brien

2016

> No mathematician can be a complete mathematician unless he is also something of a poet. - K.Weierstrass [55]

In the magical world of modular forms
Double series of theta functions transforms
To Fourier expansions of functions of q.
Some gaps in the theory left plenty to do.

So in level seven I fossicked around
Two infinite integer sequences found
In functions with coefficients polynomial
But, I could not find a form binomial.

Recurrence relations were produced
From this the theorems were deduced.
The proofs I have will be revealed
By reading this thesis in which they're concealed.

Lynette O'Brien (6/7/16)

ABSTRACT

Integer sequences resulting from recurrence relations with polynomial coefficients are rare. Two new integer sequences have been discovered and are the main result in this thesis. They consist of a three-term quadratic recurrence

$$(n+1)^2 c_7(n+1) = (26n^2+13n+2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1)$$

with initial conditions $c_7(-1) = 0$ and $c_7(0) = 1$, and a five-term quartic recurrence

$$(n+1)^4 u_7(n+1) = -Pu_7(n) - Qu_7(n-1) - Ru_7(n-2) - Su_7(n-3)$$

where

$$\begin{aligned} P &= 26n^4 + 52n^3 + 58n^2 + 32n + 7, \\ Q &= 267n^4 + 268n^2 + 18, \\ R &= 1274n^4 - 2548n^3 + 2842n^2 - 1568n + 343, \\ S &= 2401(n-1)^4 \end{aligned}$$

with initial conditions $u_7(0) = 1$ and $u_7(-1) = u_7(-2) = u_7(-3) = 0$. The experimental procedure used in their discovery utilizes the mathematical software Maple. Proofs are given that rely on the theory of modular forms for level 7, Ramanujan's Eisenstein series, theta functions and Euler products. Differential equations associated with theta functions are solved to reveal these recurrence relations. Interesting properties are investigated including an analogue of Clausen's identity, asymptotic behaviour of the sequences and finally two conjectures for congruence properties are given.

Acknowledgements

It has been a privilege to have Associate Professor Shaun Cooper as my supervisor. His excellent teaching, patience and attention to detail has guided me through this thesis. I also want to thank my husband Graeme O'Brien who has encouraged me and for his proof-reading of my thesis.

PREFACE

This thesis is the original work of the author Lynette A. O'Brien. It consists of ten chapters. The introductory chapter gives a brief outline of the main results and the motivation. In Chapter 2 we give a brief historical overview of modular forms and some background theory. In Chapters 3–5 we give definitions, derivatives, differential equations and proofs. Our main results are revealed in Chapter 6. Then in Chapters 7–9 we look at consequences of our findings. First we discuss an analogue to Clausen's identity for the square of the hypergeometric series, then look at asymptotics and congruences. We finish with conclusions and suggestions for further work.

TABLE OF CONTENTS

Chapter 1

Introduction

Sequences of integers have been known for centuries. A famous sequence studied by Fibonacci in the middle ages is generated by a recurrence relation where the next term depends on the previous two terms:

$$F_{n+1} = F_n + F_{n-1}.$$

To get the sequence started the first two terms need to be specified. So we begin with $F_{-1} = 0$ and $F_0 = 1$. Then the integer sequence begins

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}.$$

This is a prototype example. Another sequence which is similar in that each term depends on the previous terms is the Apéry sequence, however, the coefficients are cubic polynomials. The recurrence relation is given by

$$(n+1)^3 a_{n+1} = (2n+1)(17n^2 + 17n + 5)a_n - n^3 a_{n-1}$$

with initial conditions $a_{-1} = 0$ and $a_0 = 1$. The terms in the Apéry sequence may be given by a binomial sum

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{1.1}$$

These terms, called the Apéry numbers, are also integers although this is not obvious from the recurrence relation. The first few Apéry numbers are as follows:

$$\{1, 5, 73, 1445, 33001, \ldots\}.$$

The Apéry numbers were used to prove $\zeta(3)$ is irrational where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

is the Riemann zeta function. This is well explained in the entertaining article by van der Poorten in [58].

Zagier [66] was motivated by Apéry's example to do a computer search for sequences of the form

$$(n+1)^2 s_{n+1} = (an^2 + an + b)s_n - cn^2 s_{n-1}, \quad s_{-1} = 0,\ s_0 = 1.$$

There are only six known tuples (a, b, c), together with the initial conditions $a_{-1} = 0,\ a_0 = 1$, that give non trivial integral solutions. That is they are neither terminating nor polynomial. They correspond to

$$(a, b, c) = (11, 3, -1), (-17, 6, 72), (10, 3, 9), (7, 2, -8), (12, 4, 32), (-9, -3, 27).$$

Zagier also conjectured that that was all there were.

The major result of my research reported in this thesis is the discovery of two sequences similar to those found by Zagier. They have polynomial coefficients that result in integer sequences. Like the Apéry numbers the first sequence is a three-term recurrence relation but this time the polynomials are quadratic and of a different form to Zagier's six examples. The first sequence is defined by the recurrence relation

$$(n+1)^2 c_7(n+1) = (26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1) \qquad (1.2)$$

with initial conditions $c_7(-1) = 0$ and $c_7(0) = 1$. The sequence occurs in the power series expansion of

$$z_7 = \sum_{n=0}^{\infty} c_7(n) X^n$$

where z is defined as a theta series

$$z_7 = z_7(q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}, \qquad |q| < 1 \qquad (1.3)$$

and X is given in terms of eta quotients by

$$X_7 \;=\; \frac{w_7}{1 + 13w_7 + 49w_7^2} \qquad (1.4)$$

where

$$w_7 = q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4}. \qquad (1.5)$$

The second new sequence is defined by a five-term quartic recurrence relation

$$(n+1)^4 u_7(n+1) = -Pu_7(n) - Qu_7(n-1) - Ru_7(n-2) - Su_7(n-3)$$

where

$$\begin{aligned} P &= 26n^4 + 52n^3 + 58n^2 + 32n + 7, \\ Q &= 267n^4 + 268n^2 + 18, \\ R &= 1274n^4 - 2548n^3 + 2842n^2 - 1568n + 343, \\ S &= 2401(n-1)^4 \end{aligned}$$

with initial conditions $u_7(0) = 1$ and $u_7(-1) = u_7(-2) = u_7(-3) = 0$. This sequence occurs in the power series expansion of

$$y_7 = \sum_{n=0}^{\infty} u_7(n) w_7^n$$

where

$$y_7 = \prod_{j=1}^{\infty} \frac{(1-q^j)^7}{(1-q^{7j})}$$

and

$$w_7 = q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4}.$$

The properties of these sequences are the subject of this thesis.

This thesis was motivated by questions that arose from recent work on a sequence $\{t_7(n)\}$ that was discovered by my supervisor, S. Cooper in 2012 [24]. He performed a computer search much like Zagier's. One of three sequences he found at that time was:

$$(n+1)^3 t_7(n+1) = (2n+1)(13n^2 + 13n + 4)t_7(n) + 3n(9n^2 - 1)t_7(n-1) \tag{1.6}$$

with one initial condition $t_7(0) = 1$ which is sufficient in this case. This is a three term cubic recurrence relation where the coefficients are all integers. The sequence $\{t_7(n)\}$ occurs in the expansion of

$$z_7^2 = \sum_{n=0}^{\infty} t_7(n) X_7^n$$

where z_7 and X_7 are given by (1.3) and (1.4).

A binomial sum was later found by W. Zudilin [24, p. 171] for this sequence as follows:

$$t_7(n) \quad = \quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} \binom{n+k}{k}. \tag{1.7}$$

We are now able to automate the process of finding a recurrence relation if we know the binomial sum. In the Maple computer software we use the sumtools package and the command sumrecursion which produces a recurrence relation from sums of binomial coefficients. Using the example of Equation (1.7) as follows:

```
> with(sumtools):
> sumrecursion(binomial(n,k)^2*binomial(2*k,n)
        *binomial(k+n,k),k,t(n));
```

the output is the recurrence relation

$$-3\,(n-1)\,(3\,n-4)\,(3\,n-2)\,t_{n-2} - (2\,n-1)\left(13\,n^2 - 13\,n + 4\right)t_{n-1} + t_n n^3$$

that is equal to zero. Unfortunately there is no mathematical technique, hence no algorithm yet to produce a binomial sum given a recurrence relation.

Another aspect of the project was to see if there was an analogue to Clausen's identity. Clausen was working nearly 200 years ago on special functions using hypergeometric series; see (6.1) for a definition. He found a very useful identity where he had a series which was the square of another series, that is

$$\left\{ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} ; x \right) \right\}^2 = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix} ; x \right).$$

We wanted to see if there is an analogue of Clausen's identity. We took the two power series formed from the recurrence relation for $\{c_7(n)\}$ Equation (1.2) and $\{t_7(n)\}$ Equation (1.6). We found if we square one we get the other

$$\left(\sum_{n=0}^{\infty} c_7(n) X^n \right)^2 \quad = \quad \sum_{n=0}^{\infty} t_7(n) X^n.$$

If we take any sequence defined by a recurrence relation and try to square it, in general we are not going to get anything nice. So this is a striking result for our new sequence and hence shows we do have a Clausen type analogue.

We look at some properties of the sequences. We look at asymptotics, that is we are looking for an approximation for a term in the sequence for a large n. The asymptotic formula for the sequence in Equation (1.2) is given by

$$c_n \sim Cn^{-\frac{3}{2}}27^n \qquad \text{as } n \to \infty$$

where we estimate the constant C to be

$$C \approx 0.09552230526812671465130791078 7029\ldots.$$

We also investigate congruences in this thesis. About a hundred and fifty years ago Edouard Lucas [48] found a congruence satisfied by the binomial coefficients. Suppose p is a prime, n and k are non negative integers with base p representations given by

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r$$

and

$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r.$$

Then

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r} \pmod{p}.$$

Gessel in [33] has shown that for any prime p and any integer n the Apéry sequence in Equation (1.1) satisfies the property

$$a_n \equiv a_{n_0} a_{n_1} \ldots a_{n_r} \pmod{p}.$$

We thought it would be interesting to see if sequence $\{c_7(n)\}$ in Equation (1.2) had any such congruent property. A check was done using Maple and we were excited to find that in half of the first two hundred primes a Lucas-type congruence holds. We were able to make the conjecture that

$$c_n \equiv c_{n_0} c_{n_1} \ldots c_{n_r} \qquad \pmod{p}$$

holds if and only if p is a prime congruent to $0, 1, 2$ or 4, modulo 7. The main reason we haven't been able to prove the conjecture yet is that there is no known binomial sum for $\{c_7(n)\}$.

Chapter 2

Background

2.1 Modular forms

The functions we are studying in this thesis are from a class of functions called modular forms which are interesting analytic functions defined on the upper half of the complex plane. They have been described by Zagier [67] as beautiful and magical and have a lot of symmetry and nice analytical properties. Modular forms also have a Fourier expansion where the coefficients are often interesting sequences. A modular form constructed in one way can be equal to a modular form constructed in another way which may lead to interesting identities. A series for which the exponents of q, a complex variable, are quadratic expressions are called theta functions. Theta functions are examples of modular forms and are introduced in the next section. In this chapter we give a brief historical overview of modular forms, look at the relevant theory of modular groups, modular forms and their transformation properties.

2.2 Historical overview of modular forms

Theta functions were first used by Jakob Bernoulli and are recorded in his book, *Ars Conjectandi*, published posthumously in 1713; see [12, p. 55]. The theta functions were used as answers to questions in probability that arose when the Dutch physicist, Christiaan Huygens around 1657, posed a number of problems. A generalization of one of these problems is analysed in Hald's book [34, p. 185] as follows:

> *Problem 1*
> *Players A and B play with a die on the condition that he who first throws an ace wins. Player A throws once, then player B throws once; thereafter A throws two times in succession and then B throws two times; then A throws three times and B also three times,and so on. What is the ratio of their chances?*
> Bernoulli gave the probability, P, of winning as the infinite series
>
> $$P = 1 + q^2 + q^6 + q^{12} + q^{20} + \cdots - q - q^4 - q^9 - q^{16} - \cdots . \quad (2.1)$$

We notice the exponents $\{2, 6, 12, 20, \ldots\}$ are numbers of the form $n(n + 1)$ and the exponents $\{1, 4, 9, 16, \ldots\}$ are square numbers.

We can draw a tree diagram and with each roll of the die either we get a 1 with probability $\frac{1}{6}$ and the game ends, or not a 1 with probability $\frac{5}{6}$ and the game remains alive. Now player A can only win on the first roll of the die or the third or fourth roll or seventh, eighth or ninth rolls and so on. To find who wins we need to add up the probabilities for each player. Written mathematically

$$P = \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^6 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^8 \left(\frac{1}{6}\right) + \cdots .$$

Now generalizing, let $q = \frac{5}{6}$ so that $1 - q = \frac{1}{6}$. Substituting these in the above equation we get

$$P = (1-q) + q^2(1-q) + q^3(1-q) + q^6(1-q) + q^7(1-q) + q^8(1-q) + \cdots .$$

If we now take a factor out and group the remaining terms we can begin to see a pattern

$$P = (1-q)(1 + (q^2 + q^3) + (q^6 + q^7 + q^8) + (q^{12} + q^{13} + q^{14} + q^{15}) + \cdots).$$

Each of these groups can be regarded as a geometric series

$$P = (1-q)\left(\frac{1-q}{1-q} + q^2\frac{(1-q^2)}{1-q} + q^6\frac{(1-q^3)}{1-q} + q^{12}\frac{(1-q^4)}{1-q} + \cdots\right).$$

We can cancel the factors $1 - q$ to get

$$P = q^0(1-q) + q^2(1-q^2) + q^6(1-q^3) + q^{12}(1-q^4) + \cdots ,$$

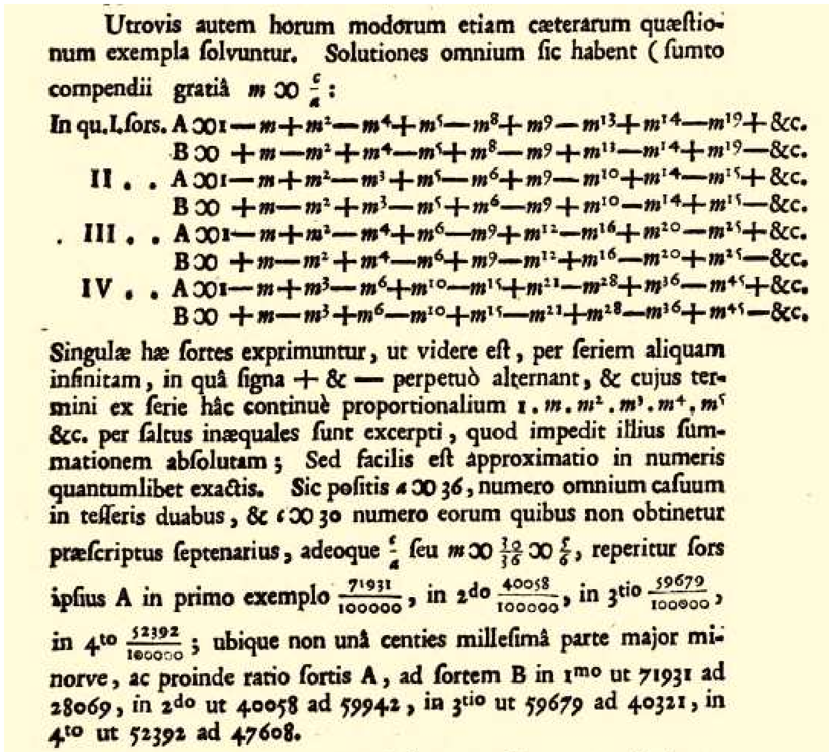
that is

$$P = \sum_{n=0}^{\infty} q^{n(n+1)}(1 - q^{n+1}).$$

Expanding the series and rearranging gives us

$$\sum_{n=0}^{\infty} q^{n(n+1)} - \sum_{n=0}^{\infty} q^{(n+1)^2} = 1 + q^2 + q^6 + q^{12} \cdots - q - q^4 - q^9 - q^{16} \cdots$$

as set out in Hald's work. Bernoulli did not use the modern q notation but rather powers of m as found in his original work [12] in the quotation below:



Utrovis autem horum modorum etiam cæterarum quæſtionum exempla ſolvuntur. Solutiones omnium ſic habent (ſumto compendii gratiâ m ꝏ $\frac{c}{a}$:

In qu. I. ſors. A ꝏ $1 - m + m^2 - m^4 + m^5 - m^8 + m^9 - m^{13} + m^{14} - m^{19} +$ &c.
B ꝏ $+ m - m^2 + m^4 - m^5 + m^8 - m^9 + m^{13} - m^{14} + m^{19} -$ &c.
II . . A ꝏ $1 - m + m^2 - m^3 + m^5 - m^6 + m^9 - m^{10} + m^{14} - m^{15} +$ &c.
B ꝏ $+ m - m^2 + m^3 - m^5 + m^6 - m^9 + m^{10} - m^{14} + m^{15} -$ &c.
III . . A ꝏ $1 - m + m^2 - m^4 + m^6 - m^9 + m^{12} - m^{16} + m^{20} - m^{25} +$ &c.
B ꝏ $+ m - m^2 + m^4 - m^6 + m^9 - m^{12} + m^{16} - m^{20} + m^{25} -$ &c.
IV . . A ꝏ $1 - m + m^3 - m^6 + m^{10} - m^{15} + m^{21} - m^{28} + m^{36} - m^{45} +$ &c.
B ꝏ $+ m - m^3 + m^6 - m^{10} + m^{15} - m^{21} + m^{28} - m^{36} + m^{45} -$ &c.

Singulæ hæ ſortes exprimuntur, ut videre eſt, per ſeriem aliquam infinitam, in quâ ſigna + & — perpetuò alternant, & cujus termini ex ſerie hâc continuè proportionalium $1 . m . m^2 . m^3 . m^4 , m^5$ &c. per ſaltus inæquales ſunt excerpti, quod impedit illius ſummationem abſolutam; Sed facilis eſt approximatio in numeris quantumlibet exactis. Sic poſitis a ꝏ 36, numero omnium caſuum in teſſeris duabus, & c ꝏ 30 numero eorum quibus non obtinetur præſcriptus ſeptenarius, adeoque $\frac{c}{a}$ ſeu m ꝏ $\frac{30}{36}$ ꝏ $\frac{5}{6}$, reperitur ſors ipſius A in primo exemplo $\frac{71931}{100000}$, in 2do $\frac{40058}{100000}$, in 3tio $\frac{59679}{100000}$, in 4to $\frac{52392}{100000}$; ubique non unâ centies milleſimâ parte major minorve, ac proinde ratio ſortis A, ad ſortem B in 1mo ut 71931 ad 28069, in 2do ut 40058 ad 59942, in 3tio ut 59679 ad 40321, in 4to ut 52392 ad 47608.

Theta functions with two variables q and z were studied by Carl Jacobi in the 19^{th} century; see [41]. He gave us a very useful tool known as Jacobi's triple product identity which takes sums to products. If $|q| < 1$ and $z \neq 0$, then

$$\sum_{j=-\infty}^{\infty} q^{j^2} z^j = \prod_{j=1}^{\infty} (1 + zq^{2j-1})(1 + z^{-1}q^{2j-1})(1 - q^{2j}). \tag{2.2}$$

Proofs can be found in [1, p. 497], [5, p. 319] and [35, p. 282].
Another useful tool to manipulate infinite products is known as Euler's product identity. If $|q| < 1$, then

$$\prod_{j=1}^{\infty} (1 + q^j) = \prod_{j=1}^{\infty} \frac{1}{(1 - q^{2j-1})}. \tag{2.3}$$

This is proved by multiplying numerator and denominator of the left hand side by $(1 - q^j)$ to give

$$\prod_{j=1}^{\infty} (1 + q^j) = \prod_{j=1}^{\infty} (1 + q^j) \frac{(1 - q^j)}{(1 - q^j)}.$$

Expanding the numerator and taking the difference of squares we obtain

$$\prod_{j=1}^{\infty}(1+q^j) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})}{(1-q^j)}.$$

Then separating the denominator into even and odd terms

$$\prod_{j=1}^{\infty}(1+q^j) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})}{(1-q^{2j})(1-q^{2j-1})}.$$

Cancelling completes the proof.

Another useful identity is

$$\prod_{j=1}^{\infty}(1-(-q)^j) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^3}{(1-q^j)(1-q^{4j})}. \tag{2.4}$$

To prove this we can separate the left hand side of (2.4) into even and odd terms and multiply the numerator and denominator by $(1+q^{2j})$ to obtain

$$\begin{aligned}\prod_{j=1}^{\infty}(1-(-q)^j) &= \prod_{j=1}^{\infty}(1-q^{2j})(1+q^{2j-1})\frac{(1+q^{2j})}{(1+q^{2j})} \\ &= \prod_{j=1}^{\infty}(1-q^{2j})(1+q^j)\frac{1}{(1+q^{2j})}.\end{aligned}$$

Now multiply the numerator and denominator by $(1-q^j)(1-q^{2j})$ to give

$$\prod_{j=1}^{\infty}(1-(-q)^j) = \prod_{j=1}^{\infty}(1-q^{2j})\frac{(1+q^j)}{(1+q^{2j})}\frac{(1-q^j)}{(1-q^j)}\frac{(1-q^{2j})}{(1-q^{2j})}.$$

Hence by difference of squares and simplifying we complete the proof.

If we define the function E by

$$E(q) = \prod_{j=1}^{\infty}(1-q^j), \tag{2.5}$$

then (2.4) can be written as

$$E(-q) = \frac{E^3(q^2)}{E(q)E(q^4)}. \tag{2.6}$$

Early in the 20^{th} century Srinivasa Ramanujan [6, p. 36] developed several special functions of complex variables. They are known as Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ and are defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} \cdots$$

and

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = 1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots .$$

To obtain an infinite product for $\varphi(q)$ we substitute $z = 1$ into (2.2) to obtain

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \prod_{j=1}^{\infty}(1 + q^{2j-1})^2(1 - q^{2j}). \tag{2.7}$$

To derive an infinite product formula for $\psi(q)$ we start by replacing q with $q^{\frac{1}{2}}$ and z with $q^{\frac{1}{2}}$ in (2.2) thus

$$\begin{aligned} \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} &= \prod_{j=1}^{\infty}(1 + q^j)(1 + q^{j-1})(1 - q^j) \\ &= 2\prod_{j=1}^{\infty}(1 + q^j)^2(1 - q^j). \end{aligned} \tag{2.8}$$

The series side is summed over all integers. To obtain a sum that is over the non-negative integers, write

$$\sum_{j=-\infty}^{\infty} q^{j(j+1)/2} = \sum_{j=0}^{\infty} q^{j(j+1)/2} + \sum_{j=-1}^{-\infty} q^{j(j+1)/2}.$$

In the second sum on the right hand side we replace j with $-1 - k$. Thus

$$\sum_{j=-\infty}^{\infty} q^{j(j+1)/2} = \sum_{j=0}^{\infty} q^{j(j+1)/2} + \sum_{k=0}^{\infty} q^{k(k+1)/2} = 2\sum_{k=0}^{\infty} q^{k(k+1)/2}.$$

On the product side of (2.8) multiply numerator and denominator by $(1 - q^j)$ to obtain

$$2\prod_{j=1}^{\infty} \frac{(1 + q^j)^2(1 - q^j)^2}{1 - q^j} \quad = \quad 2\prod_{j=1}^{\infty} \frac{(1 - q^{2j})^2}{1 - q^j}.$$

So we have a series and infinite product representation for ψ as

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2}{(1-q^j)}. \tag{2.9}$$

It is interesting to note that the solution to the Bernoulli problem in Equation (2.1) can be expressed as

$$P = \psi(q^2) - \frac{1}{2}\varphi(q) + \frac{1}{2}.$$

Another identity we require is obtained by manipulating Jacobi's triple product so

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2} = \prod_{j=1}^{\infty} (1-q^j). \tag{2.10}$$

To prove this identity we replace q with $q^{\frac{3}{2}}$ then take $z = -q^{\frac{1}{2}}$ on series side of Equation (2.2). We can replace j with $-j$ since we are summing over all integers so

$$\begin{aligned}\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2} &= \prod_{j=1}^{\infty} (1-q^{3j-1})(1-q^{3j-2})(1-q^{3j}) \\ &= \prod_{j=1}^{\infty} (1-q^j).\end{aligned}$$

Another form of Equation (2.10) is obtained by replacing j with $-j$ on the series side of Equation (2.10) and multiplying both sides by $q^{\frac{1}{24}}$ to complete the square thus

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2/24} = q^{\frac{1}{24}} \prod_{j=1}^{\infty} (1-q^j). \tag{2.11}$$

The series side of (2.11) gives a Fourier expansion, that is a sum of complex exponentials where τ is a complex number with $\text{Im}(\tau) > 0$ and $q = e^{2\pi i \tau}$. The product on the right hand side is called Dedekind's eta function, defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{j=1}^{\infty} (1-q^j). \tag{2.12}$$

More generally for any positive integer m let $\eta_m = \eta(m\tau)$ be defined by

$$\eta_m = q^{\frac{m}{24}} \prod_{j=1}^{\infty} (1-q^{mj}). \tag{2.13}$$

The Dedekind eta function was first reported in an 1887 journal article by Richard Dedekind [29, p. 285]. Srinivasa Ramanujan used Dedekind's eta function to develop a series called Ramanujan's Eisenstein series. He used logarithmic differentiation to obtain

$$\begin{aligned} P &= q\frac{d}{dq}\log(q\prod_{n=1}^{\infty}(1-q^n)^{24}) \\ &= 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \end{aligned} \tag{2.14}$$

This function P is one of three functions given by Ramanujan. The other two, Q and R, are defined as

$$Q = 1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \tag{2.15}$$

and

$$R = 1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{2.16}$$

They satisfy a system of three differential equations as described by Ramanujan in [53]. The following identity shows the relationship between Ramanujan's Eisenstein series and Dedekind's eta function with

$$Q^3-R^2 = 1728q\prod_{j=1}^{\infty}(1-q^j)^{24}.$$

Ramanujan's φ function can be expressed in terms of Dedekind's eta function by

$$\varphi(q) = \frac{\eta_2^5}{\eta_1^2\eta_4^2}.$$

This is obtained by replacing q with $-q$ in Equation (2.7). The result is

$$\varphi(-q) = \prod_{j=1}^{\infty}(1-q^{2j-1})^2(1-q^{2j}). \tag{2.17}$$

Multiply numerator and denominator of (2.17) by $(1-q^{2j})$ to get

$$\varphi(-q) = \prod_{j=1}^{\infty}(1-q^{2j-1})^2(1-q^{2j})^2\frac{1}{(1-q^{2j})}$$

and simplify using (2.5) to obtain

$$\varphi(-q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-q^{2j})} = \frac{(E(q))^2}{E(q^2)}.$$

Replacing q with $-q$ again, we find

$$\varphi(q) = \frac{(E(-q))^2}{E(q^2)}$$

and using (2.6) we obtain

$$\varphi(q) = \left(\frac{E^3(q^2)}{E(q)E(q^4)}\right)^2 \frac{1}{E(q^2)} = \frac{E^5(q^2)}{E^2(q)E^2(q^4)} = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^5}{(1-q^j)^2(1-q^{4j})^2}.$$

Finally, using (2.13) gives us Ramanujan's $\varphi(q)$ in terms of an infinite series and also as a product of eta functions

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^5}{(1-q^j)^2(1-q^{4j})^2} = \frac{\eta_2^5}{\eta_1^2\eta_4^2}.$$

We can also write $\psi(q)$ found in (2.9) in terms of eta functions, however, we need a $q^{\frac{1}{8}}$ to balance the equation so

$$q^{\frac{1}{8}}\psi(q) = \sum_{j=0}^{\infty} q^{(2j+1)^2/8} = q^{\frac{1}{8}} \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2}{(1-q^j)} = \frac{\eta_2^2}{\eta_1}.$$

Gotthold Eisenstein (1823-1852) developed an infinite series known as Eisenstein series, an example of modular forms, that we define as

$$G_k = -\frac{B_k}{2k} + \sum_{j=1}^{\infty} \frac{j^{k-1}q^j}{1-q^j}$$

where B_k are the Bernoulli numbers defined by

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Ramanujan's functions P, Q and R are multiples of Eisenstein series G_2, G_4 and G_6, respectively. That is $P = -24G_2$, $Q = 240G_4$ and $R = -504G_6$. A good explanation of Eisenstein series can be found in [4, p. 12].

Ramanujan built on Jacobi's work on theta functions and alluded to three corresponding theories. He gave some results in these theories but without proofs in the unorganized pages in his second notebook [54, p. 257–262]. They are now referred to as Ramanujan's theories of elliptic functions to alternative bases. He mentioned the theories in a letter he sent to Hardy [7] while in his paper "Modular forms and approximations to π" [52] Ramanujan gave these three theories but did not develop them. Late in the 20^{th} century these theories were discovered and developed by J. M. Borwein and P. B. Borwein; see [13]. The cubic theory is based on $a(q)$, $b(q)$ and $c(q)$ defined below. The Borweins' work was extended by B. C. Berndt, S. Bhargava and F. G. Garvan a few years latter; see [8] and [10]. Then M. D. Hirschhorn and F. G. Garvan developed further identities in [40].

We let $\omega = \exp\left(\frac{2\pi i}{3}\right)$ then the cubic theta functions a, b and c are defined by the Borweins in [13] as

$$a(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2}, \tag{2.18}$$

$$b(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \omega^{j-k} q^{j^2+jk+k^2} \tag{2.19}$$

and

$$c(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{(j+\frac{1}{3})^2+(j+\frac{1}{3})(k+\frac{1}{3})+(k+\frac{1}{3})^2}. \tag{2.20}$$

The Borweins' cubic theta functions $a(q)$, $b(q)$ and $c(q)$ are analogues of Ramanujan's theta functions

$$\varphi^2(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+k^2},$$

$$\varphi^2(-q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^{j+k} q^{j^2+k^2}$$

and

$$4q^{\frac{1}{2}}\psi^2(q^2) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{(j+\frac{1}{2})^2+(k+\frac{1}{2})^2}$$

respectively.

The function $a(q)$ has a Lambert series given by

$$a(q) = 1 + 6\sum_{j=1}^{\infty}\left(\frac{j}{3}\right)\frac{q^j}{1-q^j} \tag{2.21}$$

where

$$\left(\frac{j}{3}\right) = \begin{cases} +1 & \text{if} \quad j = 1 \quad \bmod 3 \\ -1 & \text{if} \quad j = -1 \quad \bmod 3 \\ 0 & \text{if} \quad j = 0 \quad \bmod 3. \end{cases}$$

The functions $b(q)$ and $c(q)$ have product formulae in terms of infinite products and eta functions. Thus

$$b(q) = \prod_{j=1}^{\infty}\frac{(1-q^j)^3}{(1-q^{3j})} = \frac{\eta_1^3}{\eta_3} \tag{2.22}$$

and

$$c(q) = 3q^{\frac{1}{3}}\prod_{j=1}^{\infty}\frac{(1-q^{3j})^3}{(1-q^j)} = 3\frac{\eta_3^3}{\eta_1}. \tag{2.23}$$

Proofs of these functions can be found in [13], [15] and [22].

2.3 Weight and level of modular forms

Now we look at the objects that help our classification of modular forms. The modular group can be defined as a group of 2×2 matrices where the entries are integers and the determinant is 1. It is called the special linear group and is denoted by $\mathrm{SL}_2(\mathbb{Z})$ or Γ, that is

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\right\}.$$

The upper half plane $\mathcal{H}$ is the set of all complex numbers $z = x + iy$ where y is positive that is

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The set Γ is a group and the group operation is matrix multiplication. The group can be generated by the matrices S and T where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Further details can be found in [4, Ch. 2].

We give a definition of modular functions as follows:

Definition 2.1. *Let $f : \mathcal{H} \to \mathbb{C}$. Then f is said to be a modular function for Γ if the following conditions are satisfied:*

(1) *$f(\tau)$ is meromorphic in $\mathcal{H}$.*

(2) *$f(\tau)$ acts on the upper half plane $\mathcal{H}$ by way of Möbius transformations*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \text{ for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(3) *$f(\tau)$ is meromorphic at the cusps. At the cusp $i\infty$, this means*

$$f(\tau) = \sum_{n=-m}^{\infty} a_n e^{2\pi i n\tau} = \sum_{n=-m}^{\infty} a_n q^n$$

where $q = e^{2\pi i\tau}$ and m is a non negative integer. Finitely many negative powers of q are allowed in the expansion.

For details see [4, p. 34] and [44, p. 108]. We now tighten condition (1) of modular functions to give a definition for modular forms.

2.3.1 Weight

We now give a definition for modular form.

Definition 2.2. *Let $f : \mathcal{H} \to \mathbb{C}$. Then f is said to be a modular form of weight k for Γ if the following conditions are satisfied:*

(1) *$f(\tau)$ is holomorphic in $\mathcal{H}$.*

(2) *$f(\tau)$ acts on the upper half plane $\mathcal{H}$ by way of Möbius transformations*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \quad = \quad (c\tau+d)^k f(\tau) \ \textit{ for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

(3) *$f(\tau)$ is holomorphic at the cusps. At the cusp $i\infty$, this means $f(\tau)$ has a Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau} = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i\tau}$.

For "holomorphic at the cusps" see [30, p. 16]. Also see [44, Chapter 3] for more information on modular forms.

For example, condition (2) for matrix S gives the transformation

$$f\left(\frac{-1}{\tau}\right) \quad = \quad f(\tau)$$

while for matrix T we see the function is periodic with period 1

$$f(\tau+1) \quad = \quad f(\tau).$$

We provide an informal discussion without rigorous checks that each example meets the above conditions. The prototype example is Dedekind's eta function. We take the 24^{th} power using (2.12) so that all the powers in the q expansion are integral

$$\eta^{24}(\tau) = q\prod_{j=1}^{\infty}(1-q^j)^{24} = e^{2\pi i\tau}\prod_{j=1}^{\infty}(1-e^{2\pi i\tau j})^{24}.$$

Since

$$e^{2\pi i(\tau+1)} = e^{2\pi i\tau}$$

it follows that

$$\eta^{24}(\tau+1) = \eta^{24}(\tau)$$

hence, η^{24} is periodic, that is, it satisfies property (2) for the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T$$

for any integer k.
Another important property of Dedekind's eta functions is the transformation

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)$$

found in [4, p. 48], [22] and [51]. When we take 24^{th} power of eta we get

$$\eta^{24}\left(\frac{-1}{\tau}\right) = \tau^{12}\eta^{24}(\tau).$$

Hence η^{24} satisfies property (2) for the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$$

for $k = 12$. Since T and S generate the whole group then property (2) holds for all matrices in the modular group Γ so η^{24} is a modular form of weight 12; see [4, Ch. 3].

As another example, we state without proof the transformation formulae for Ramanujan's Eisenstein series P, Q and R

$$\begin{aligned} P(r) &= (c\tau+d)^2 P(q) - \frac{6ic(c\tau+d)}{\pi}, \\ Q(r) &= (c\tau+d)^4 Q(q) \end{aligned} \tag{2.24}$$

and

$$R(r) = (c\tau+d)^6 R(q)$$

where

$$r = \exp\left(2\pi i \frac{a\tau+b}{c\tau+d}\right) \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The functions Q and R are modular forms of weight 4 and 6 respectively. They satisfy the three conditions above. The function P is not a modular form because of the presence of the extra term $6ic(c\tau+d)/\pi$ in the transformation formula. However, $P(r)$, termed a quasi modular form, may be used to construct modular forms of weight 2 which we will discuss shortly. Proofs may be found in [25, Ch. 2] and [65, p. 19].

The Fourier expansions for $P(q)$, $Q(q)$ and $R(q)$, that is condition (3), are given in terms of divisor sums. By expanding (2.14)–(2.16) into geometric series, we observe that

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = 1 - 24\sum_{N=1}^{\infty}\sigma_1(N)q^N,$$
$$Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240\sum_{N=1}^{\infty}\sigma_3(N)q^N$$

and

$$R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504\sum_{N=1}^{\infty}\sigma_5(N)q^N$$

where

$$\sigma_k(N) = \sum_{d|N} d^k$$

and the sum is over divisors d of N; see [4, p. 20].

2.3.2 Level

Next we introduce the classification of modular forms by level.

Definition 2.3. *Let ℓ be a positive integer. The group $\Gamma_0(\ell)$ is defined by*

$$\Gamma_0(\ell) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad - bc = 1, c \equiv 0 \pmod{\ell}\right\}$$

A function f is a modular form of weight k with respect to $\Gamma_0(\ell)$ if the following conditions apply.

(1) *$f(\tau)$ is holomorphic in $\mathcal{H}$.*

(2) *$f(\tau)$ acts on the upper half plane $\mathcal{H}$ by way of Möbius transformations*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell).$$

(3) *$f(\tau)$ is holomorphic at the cusps. At $i\infty$ $f(\tau)$ has a Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau} = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i\tau}$.

Informally, we will refer to modular form of weight k with respect to $\Gamma_0(\ell)$ simply as a modular form of weight k and level ℓ.

Suppose $f(\tau)$ is a modular form of weight k, ℓ is a positive integer and $g(\tau) = f(\ell\tau)$. We show that g is a modular form of weight k and level ℓ. We need to show the three conditions above are satisfied. Condition (1) is satisfied since if f is holomorphic, then g is holomorphic by composition of holomorphic functions. Condition (3) is satisfied since the Fourier expansion of g is the Fourier expansion of f with q^ℓ in place of q. So we need only check the transformation property, condition (2). In the group $\Gamma_0(\ell)$

$$c \equiv 0 \mod \ell \quad \text{so} \quad c = e\ell \quad \text{where } e \in \mathbb{Z}.$$

Let

$$g(\tau) = f(\ell\tau).$$

Since

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

it follows that

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = f\left(\ell\frac{a\tau + b}{c\tau + d}\right) = f\left(\frac{a\ell\tau + b\ell}{c\tau + d}\right) = f\left(\frac{a(\ell\tau) + b\ell}{e(\ell\tau) + d}\right).$$

Let $t = \ell\tau$ to get

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = f\left(\frac{at + b\ell}{et + d}\right). \tag{2.25}$$

Since

$$\det\begin{pmatrix} a & b\ell \\ e & d \end{pmatrix} = ad - b\ell e = ad - bc = 1$$

by property (2), (2.25) becomes

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = f\left(\frac{at + b\ell}{et + d}\right) = (et + d)^k f(t).$$

But $g(\tau) = f(\ell\tau)$, $t = \ell\tau$ and $c = e\ell$, so

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = f\left(\frac{a\ell\tau + b\ell}{c\tau + d}\right) = (c\tau + d)^k f(\ell\tau) = (c\tau + d)^k g(\tau).$$

Therefore $g(\tau)$ is a modular form of level ℓ and weight k.

Here is another example, where the quasi modular form P is used to construct a modular form of weight 2 and level ℓ. Let

$$Z_\ell = \frac{\ell P(q^\ell) - P(q)}{\ell - 1}. \tag{2.26}$$

We want to show Z_ℓ is a modular form. We need to show (2.26) fulfils the criteria in Definition 2.3. The Equation Z_ℓ is holomorphic since P is holomorphic. Z_ℓ has a Fourier expansion so we need only show that Z_ℓ transforms by way of Möbius transformations, that is

$$Z_\ell\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 Z_\ell(\tau) \quad \text{for every} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell). \tag{2.27}$$

We will work in terms of τ. Define

$$\tilde{P}(\tau) = P(e^{2\pi i \tau}),$$

then by definition

$$Z_\ell\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\ell \tilde{P}\left(\ell \frac{a\tau + b}{c\tau + d}\right) - \tilde{P}\left(\frac{a\tau + b}{c\tau + d}\right)}{\ell - 1}.$$

By Equation (2.24) we have

$$\tilde{P}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \tilde{P}(\tau) - \frac{6ic(c\tau + d)}{\pi}. \tag{2.28}$$

Since $c \equiv 0 \mod \ell$ we can write $c = e\ell$ where $e \in \mathbb{Z}$. Then let $t = \ell\tau$. So

$$\begin{aligned}
\tilde{P}\left(\ell \frac{a\tau + b}{c\tau + d}\right) &= \tilde{P}\left(\frac{a\ell\tau + b\ell}{e\ell\tau + d}\right) \\
&= \tilde{P}\left(\frac{at + b\ell}{et + d}\right) \\
&= (et + d)^2 \tilde{P}(t) - \frac{6ie(et + d)}{\pi},
\end{aligned}$$

where the last line comes from (2.24) with the matrix $\begin{pmatrix} a & b\ell \\ e & d \end{pmatrix} \in \Gamma_0(\ell)$.
Now converting back to c and τ we have

$$\tilde{P}\left(\ell\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\tilde{P}(\ell\tau) - \frac{6ie(c\tau+d)}{\pi}. \tag{2.29}$$

Substituting (2.28) and (2.29) into (2.26) gives

$$\begin{aligned} Z_\ell\left(\frac{a\tau+b}{c\tau+d}\right) &= \frac{(c\tau+d)^2\ell\tilde{P}(\ell\tau) - \frac{6ie\ell(c\tau+d)}{\pi} - \left((c\tau+d)^2\tilde{P}(\tau) - \frac{6ic(c\tau+d)}{\pi}\right)}{\ell-1} \\ &= \frac{(c\tau+d)^2\ell\tilde{P}(\ell\tau) - \frac{6ic(c\tau+d)}{\pi} - \left((c\tau+d)^2\tilde{P}(\tau) - \frac{6ic(c\tau+d)}{\pi}\right)}{\ell-1} \\ &= (c\tau+d)^2\frac{\left(\ell\tilde{P}(\ell\tau) - \tilde{P}(\tau)\right)}{\ell-1}. \end{aligned}$$

This proves that Z_ℓ given by (2.26) is a modular form of weight 2 and level ℓ.

A heuristic way of determining the level is to look at the discriminant of the quadratic power of q in the theta function. For example

$$z = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}, \qquad |q| < 1.$$

If

$$am^2 + bmn + cn^2 \tag{2.30}$$

is the quadratic polynomial, then the discriminant is

$$b^2 - 4ac.$$

So the discriminant of $m^2 + mn + 2n^2$ is -7 where 7 is the level.

For a comprehensive guide and proofs of modular groups and modular forms the reader is refereed to any of the following books and articles [4], [14], [22], [43], [56] or [65].

Chapter 3

Functions of modular form

Functions of modular forms satisfy linear differential equations. In preparation for developing differential equations we give relevant definitions and identities for level 3 (cubic) theta functions, level 7 (septic) theta functions and their Eisenstein series as there are many similarities between the two theories.

3.1 Level 3: cubic theta functions

3.1.1 Definitions

We begin this section with four definitions. Let q be a complex number with $|q| < 1$, then the definition of z_3 is given in terms of the Borweins' cubic theta function (2.18) and the double sum where the discriminant of the polynomial exponent of q indicates the level:

$$z_3 = a(q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}. \tag{3.1}$$

Now we define Z_3 in terms of Ramanujan's Eisenstein series (2.14) as

$$Z_3 = \frac{1}{2}\left(3P(q^3) - P(q)\right). \tag{3.2}$$

The definition of x_3 is given in terms of the Borweins' cubic theta functions (2.18) and (2.20) as

$$x_3 = \left(\frac{c(q)}{a(q)}\right)^3. \tag{3.3}$$

So the definition of X_3 given in terms of eta functions and Z_3 (3.2) is

$$X_3 = \left(\frac{\eta_1^2\eta_3^2}{Z_3}\right)^3 \tag{3.4}$$

where η_1 and η_3 are defined by (2.13).

3.1.2 Eisenstein series

We met $P(q)$ and $Q(q)$ in Equations (2.14) and (2.15) in our historical overview and now that we have defined z_3 and x_3 we can express Ramanujan's Eisenstein Series $P(q)$ and $P(q^3)$ in terms of these.

Theorem 3.1. *Let z_3 and x_3 be defined by* (3.1) *and* (3.3)*. Then*

$$\begin{aligned} P(q) &= \frac{12q}{z_3}\frac{dz_3}{dq} + (1-4x_3)z_3^2, \\ 3P(q^3) &= \frac{12q}{z_3}\frac{dz_3}{dq} + (3-4x_3)z_3^2 \end{aligned}$$

and

$$\begin{aligned} Q(q) &= z_3^4(1+8x_3), \\ Q(q^3) &= z_3^4(1-\frac{8}{9}x_3). \end{aligned}$$

Proof. Proofs can be found in [10] and [22]. □

3.1.3 Properties

In this section we look at properties of the cubic theta functions and accordingly seek identities involving the square, third and fourth powers of z_3 and their equivalent powers of $a(q)$. These properties were known to Ramanujan; see his second notebook [54, Chapter, 21 Entry 5] and developed and proved by B. Berndt [6, p. 467]. We begin with z_3^2 and z_3^4 in terms of the Borweins' cubic theta functions and the Eisenstein series.

Theorem 3.2. *Let z_3 and $a(q)$ be defined by* (3.1)*. Then*

$$z_3^2 = a^2(q) = Z_3 = \frac{1}{2}\left(3P(q^3) - P(q)\right) \tag{3.5}$$

and

$$z_3^4 = a^4(q) = \frac{1}{10}(9Q(q^3) + Q(q)). \tag{3.6}$$

Proof. The result follows from Theorem 3.1. □

The next identity in this section is the Borweins' cubic identity given by

Theorem 3.3. *Let $a(q)$, $b(q)$, $c(q)$ and z_3 be defined by* (2.21), (2.22), (2.23) *and* (3.1), *respectively. Then*

$$z_3^3 = a^3(q) = b^3(q) + c^3(q). \tag{3.7}$$

Proof. Proofs are given in [10], [13] and [15]. □

We now give a consequence of Theorem 3.3

Corollary 3.4. *Let the Borweins' cubic theta functions $a(q)$, $b(q)$, $c(q)$ and x_3 be defined by* (2.21), (2.22), (2.23) *and* (3.3), *respectively. Then*

$$1 - x_3 = 1 - \left(\frac{c(q)}{a(q)}\right)^3 = \left(\frac{b(q)}{a(q)}\right)^3. \tag{3.8}$$

Proof. The proof follows from Theorem 3.3. □

We can relate X_3 to x_3 by the following theorem.

Theorem 3.5. *Let x_3 and X_3 be defined by* (3.3) *and* (3.4)*. Then*

$$X_3 = \frac{x_3(1 - x_3)}{27}. \tag{3.9}$$

Proof. Using the Borweins' cubic theta functions (2.21), (2.22), (2.23) and (3.4) we have

$$x_3(1 - x_3) = \frac{b(q)^3 c(q)^3}{a(q)^6} = \frac{27\eta_1^6\eta_3^6}{z_3^6} = 27X_3.$$

Rearranging completes the proof. □

We end this section by solving (3.9) for x_3. Rearranging gives

$$x_3 = \frac{1}{2} \pm \frac{\sqrt{1 - 108X_3}}{2}.$$

From (3.3) and (3.4) we see $q = 0$ when $x_3 = X_3 = 0$ so the only root that satisfies this condition is

$$x_3 = \frac{1}{2} - \frac{\sqrt{1 - 108X_3}}{2}. \tag{3.10}$$

3.2 Level 7: septic theta functions

3.2.1 Definitions

In this section we give five definitions. The first two are definitions of the functions y_7 and w_7 in terms of eta functions (2.13).

$$y_7 = \frac{\eta_1^7}{\eta_7} = \prod_{j=1}^{\infty} \frac{(1-q^j)^7}{(1-q^{7j})} \tag{3.11}$$

and

$$w_7 = \frac{\eta_7^4}{\eta_1^4} = q\prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4}. \tag{3.12}$$

Next we define z_7 in terms of a double sum theta function where the discriminant of the exponent indicates the level as shown in (2.30)

$$z_7 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}. \tag{3.13}$$

The definition of Z_7 is given in terms of Ramanujan's Eisenstein series (2.14)

$$Z_7 = \frac{7P(q^7)-P(q)}{6}. \tag{3.14}$$

Now we give a definition of X_7 in terms of eta functions (2.13) and Eisenstein series (3.14)

$$X_7 = \frac{\eta_1^3\eta_7^3}{Z_7^{\frac{3}{2}}}. \tag{3.15}$$

3.2.2 Eisenstein series

Ramanujan's Eisenstein Series $P(q)$ and $P(q^7)$ can be expressed in terms of w_7 and z_7 as follows:

Theorem 3.6. *Let w_7 and z_7 be defined by* (3.12) *and* (3.13). *Then*

$$P(q) = \left(\frac{1-39w_7-343w_7^2}{1+13w_7+49w_7^2}\right)z_7^2 + 12w_7z_7\frac{dz_7}{dw_7},$$
$$P(q^7) = \left(\frac{1+\frac{39}{7}w_7-7w_7^2}{1+13w_7+49w_7^2}\right)z_7^2 + \frac{12}{7}w_7z_7\frac{dz_7}{dw_7},$$
$$Q(q) = \left(\frac{1+245w_7+2401w_7^2}{1+13w_7+49w_7^2}\right)z_7^4$$

and

$$Q(q^7) = \left(\frac{1+5w_7+w_7^2}{1+13w_7+49w_7^2}\right) z_7^4.$$

Proof. Proofs are found in [17] and [27]. □

3.2.3 Properties

Now we look at properties of the level 7 theta functions. As with level 3, we need identities involving second, third and fourth powers. We note Theorems 3.7, 3.8 and 3.9 are analogues for the level 3 Equations (3.5), (3.7) and (3.6), respectively. The following identity relates z_7 to Z_7.

Theorem 3.7. *Let z_7 and Z_7 be defined by* (3.13) *and* (3.14)*. Then*

$$z_7^2 = Z_7 \tag{3.16}$$

that is

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^2 = \frac{7P(q^7)-P(q)}{6}.$$

Proof. Ramanujan knew of this expression; see his second notebook [54, Chapter 21, Entry 5(i)]. It was proved by Andrews and Berndt in [2, p. 404]. Further proofs are found in [21] and [27]. □

The following useful identity gives a representation of z_7^3 in terms of y_7 and w_7.

Theorem 3.8. *Let y_7, w_7 and z_7 be defined by* (3.11), (3.12) *and* (3.13)*, respectively. Then*

$$z_7^3 = (1+13w_7+49w_7^2)y_7.$$

Proof. This was known to Ramanujan; see his second notebook [54, Chapter 21, Entry 5]. It was developed and proved by B. Berndt in [6, p. 467]. Further proofs can be found in [21], [27] and [47]. □

Next an identity for z_7^4 is given in terms of w_7 and $Q(q)$.

Theorem 3.9. *Let w_7, z_7 and $Q(q)$ be defined by* (3.12) *and* (3.13) *and Theorem 3.6, respectively. Then*

$$z_7^4 = \frac{1+13w_7+49w_7^2}{1+245w_7+2401w_7^2} Q(q).$$

Proof. A proof is found in [27]. □

The following identity is a representation of X_7 in terms of w_7.

Theorem 3.10. *Let w_7 and X_7 be defined by* (3.12) *and* (3.15). *Then*

$$X_7 \;=\; \frac{w_7}{1+13w_7+49w_7^2}.$$

Proof. Substituting (3.11) into Theorem 3.8 gives us

$$z_7^3 \;=\; Z_7^{\frac{3}{2}} = (1+13w_7+49w_7^2)\frac{\eta_1^7}{\eta_7}. \tag{3.17}$$

By rearranging (3.15) and equating with (3.17) we obtain

$$X_7 = \left(\frac{1}{1+13w_7+49w_7^2}\right)\frac{\eta_7^4}{\eta_1^4}.$$

Using (3.12) completes the proof. □

We end this section with the following two identities.

Corollary 3.11. *Let w_7 and X_7 be defined as* (3.12) *and* (3.15). *Then*

$$1+X_7 = \frac{(1+7w_7)^2}{1+13w_7+49w_7^2} \tag{3.18}$$

and

$$1-27X_7 = \frac{(1-7w_7)^2}{1+13w_7+49w_7^2}. \tag{3.19}$$

Proof. Using Theorem 3.10

$$\begin{aligned} 1+X_7 &= 1+\frac{w_7}{1+13w_7+49w_7^2} && (3.20)\\ &= \frac{1+14w_7+49{w_7}^2}{1+13w_7+49w_7^2} \end{aligned}$$

and

$$\begin{aligned} 1-27X_7 &= 1-27\frac{w_7}{1+13w_7+49w_7^2} && (3.21)\\ &= \frac{1-14w_7+49{w_7}^2}{1+13w_7+49w_7^2}. \end{aligned}$$

$$\tag{3.22}$$

Factoring the numerators completes the proof. □

Chapter 4

Derivatives

In order to derive differential equations in the next chapter we must compute derivatives of the functions defined in Chapter 3. Hence, we give derivatives with respect to the parameter q for x_3, X_3 and z_3 for level 3 and $\log w_7$, $\log X_7$, $\log y_7$ and $\log z_7$ for level 7. We will use the operator $q\frac{d}{dq}$ where

$$q\frac{d}{dq} = q\frac{d}{d\tau}\frac{d\tau}{dq} = q\frac{d}{d\tau}\Big/\left(\frac{dq}{d\tau}\right) = q\frac{d}{d\tau}\Big/(2\pi i q) = \frac{1}{2\pi i}\frac{d}{d\tau}$$

for $q = e^{2\pi i\tau}$ and $\mathrm{Im}\tau > 0$.

We start with Ramanujan's derivative for $P(q)$.

Theorem 4.1. *Let $P(q)$ and $Q(q)$ be defined by* (2.14) *and* (2.15). *Then*

$$q\frac{dP(q)}{dq} = \frac{P^2(q) - Q(q)}{12}.$$

Proof. Ramanujan's proof can be found in his paper; see [53] □

One of the techniques we will be using is logarithmic differentiation; for a good description see [5, p. 322].

4.1 Level 3 derivatives

The derivative for x_3 is as follows:

Theorem 4.2. *Let z_3 and x_3 be defined by* (3.1) *and* (3.3). *Then*

$$q\frac{dx_3}{dq} = z_3^2 x_3(1 - x_3).$$

Proof. Using (2.13), (2.22), (2.23), (3.3) and (3.8) we obtain

$$\frac{x_3}{1-x_3} = \left(\frac{c(q)}{b(q)}\right)^3 = \left(3\frac{\eta_3^4}{\eta_1^4}\right)^3 = 27q\prod_{j=1}^{\infty}\left(\frac{1-q^{3j}}{1-q^j}\right)^{12}.$$

Using logarithmic differentiation gives us

$$\left(\frac{1}{x_3} + \frac{1}{1-x_3}\right) q\frac{dx_3}{dq} = \frac{3P(q^3) - P(q)}{2}.$$

Then rearranging the left hand side and using (3.5) we obtain

$$\frac{1}{x_3(1-x_3)} q\frac{dx_3}{dq} = z_3^2.$$

Rearranging completes the proof. □

We now want to establish the derivative for X_3.

Theorem 4.3. *Let* X_3 *and* Z_3 *be defined by* (3.4) *and* (3.5)*. Then*

$$q\frac{dX_3}{dq} = Z_3 X_3 \sqrt{1 - 108X_3}.$$

Proof. As established in Theorem 3.5 we can write X_3 as

$$X_3 = \frac{x_3(1-x_3)}{27}.$$

We apply the operator $q\frac{d}{dq}$ to get

$$q\frac{dX_3}{dq} = \frac{1}{27}(1-2x_3) q\frac{dx_3}{dq}.$$

Using Theorem 4.2 and Equations (3.5) and (3.9) we obtain

$$q\frac{dX_3}{dq} = Z_3 X_3 (1-2x_3)$$

then use Equation (3.10) to complete the proof. □

In order to compute derivatives for z_3 we require the following lemma:

Lemma 4.4. *Let* z_3 *and* x_3 *be defined as* (3.1) *and* (3.3)*. Then*

$$z_3^{12} x_3 (1-x_3)^3 = 27q \prod_{j=1}^{\infty} (1-q^j)^{24}$$

and

$$z_3^{12} x_3^3 (1-x_3) = 27^3 q^3 \prod_{j=1}^{\infty} (1-q^{3j})^{24}.$$

Proof. Using (2.21), (2.22), (2.23), (3.1), (3.3) and (3.8) we have

$$z_3^{12}x_3(1-x_3)^3 = a^{12}(q)\times\frac{c^3(q)}{a^3(q)}\times\left(\frac{b^3(q)}{a^3(q)}\right)^3 = b^9(q)c^3(q) = 27q\prod_{j=1}^{\infty}(1-q^j)^{24}$$

and

$$z_3^{12}x_3^3(1-x_3) = a^{12}(q)\times\left(\frac{c^3(q)}{a^3(q)}\right)^3\times\frac{b^3(q)}{a^3(q)} = b^3(q)c^9(q) = 27^3q^3\prod_{j=1}^{\infty}(1-q^{3j})^{24}.$$

□

Accordingly, we compute two different expressions for the derivative of z_3.

Theorem 4.5. *Let* P, z_3 *and* x_3 *be defined as* (2.14), (3.1) *and* (3.3), *respectively. Then*

$$q\frac{dz_3}{dq} = \frac{z_3(P(q)-(1-4x_3)z_3^2)}{12} \tag{4.1}$$

and

$$q\frac{dz_3}{dq} = \frac{z_3(3P(q^3)-(3-4x_3)z_3^2)}{12}. \tag{4.2}$$

Proof. Take logarithms of the identities of Lemma 4.4 and apply the operator $q\frac{d}{dq}$. Then using Theorem 4.2 and Equation (2.14) we obtain

$$P(q) = \frac{12q}{z_3}\frac{dz_3}{dq} + (1-4x_3)z_3^2$$

and

$$3P(q^3) = \frac{12}{z_3}q\frac{dz_3}{dq} + (3-4x_3)z_3^2.$$

Rearranging again completes the proofs. □

4.2 Level 7 derivatives

Next we compute the derivative for w_7.

Theorem 4.6. *Let* w_7, z_7 *and* Z_7 *be defined by* (3.12), (3.13) *and* (3.14), *respectively. Then*

$$q\frac{d}{dq}\log w_7 = Z_7 = z_7^2.$$

Proof. Using (3.12)

$$w_7 = q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^4}{(1-q^j)^4}$$

and by logarithmic differentiation we obtain

$$q\frac{d}{dq}\log w_7 = 1-\sum_{j=1}^{\infty}\frac{28jq^{7j}}{1-q^{7j}}+\sum_{j=1}^{\infty}\frac{4jq^j}{1-q^j}.$$

Now using (2.14) gives us

$$q\frac{d}{dq}\log w_7 = \frac{1}{6}(7P(q^7)-P(q)).$$

Using Equation (3.14) and Theorem 3.7 completes the proof □

We give the derivative for X_7.

Theorem 4.7. *Let Z_7 and X_7 be defined by* (3.14) *and* (3.15). *Then*

$$q\frac{d}{dq}\log X_7 = Z_7\sqrt{1-26X_7-27X_7^2}.$$

Proof. By Theorem 3.10 we have

$$X_7 = \frac{w_7}{1+13w_7+49w_7^2},$$

where w_7 is defined by (3.12). By logarithmic differentiation

$$\begin{aligned} q\frac{d}{dq}\log X_7 &= q\frac{d}{dq}\log\left(\frac{w_7}{1+13w_7+49w_7^2}\right) \\ &= q\frac{d}{dq}\log w_7-\frac{w_7(13+98w_7)}{1+13w_7+49w_7^2}q\frac{d}{dq}\log w_7. \end{aligned}$$

By Theorem 4.6, this simplifies to

$$q\frac{d}{dq}\log X_7 = Z_7\left(\frac{(1+7w_7)(1-7w_7)}{1+13w_7+49w_7^2}\right).$$

Using (3.18) and (3.19) we obtain

$$q\frac{d}{dq}\log X_7 = Z_7\sqrt{1-26X_7-27X_7^2}.$$

This completes the proof. □

Our next derivative is for y_7.

Theorem 4.8. *Let* P *and* y_7 *be defined by* (2.14) *and* (3.11). *Then*

$$q\frac{d}{dq}\log y_7 \;=\; \frac{7}{24}\left(P(q)-P(q^7)\right).$$

Proof. Using (3.11)

$$y_7 \;=\; \prod_{j=1}^{\infty}\frac{(1-q^j)^7}{(1-q^{7j})}$$

and logarithmic differentiation gives us

$$\begin{aligned} q\frac{d}{dq}\log y_7 &= q\frac{d}{dq}\log\left(\prod_{j=1}^{\infty}\frac{(1-q^j)^7}{(1-q^{7j})}\right) \\ &= \sum_{j=1}^{\infty}\frac{7jq^j}{1-q^j}-\sum_{j=1}^{\infty}\frac{7jq^{7j}}{1-q^{7j}}. \end{aligned}$$

Using (2.14) we obtain

$$q\frac{d}{dq}\log y_7 \;=\; \frac{7}{24}(P(q)-P(q^7))$$

which completes the proof. □

We give the derivative for z_7.

Theorem 4.9. *Let* P*,* w_7 *and* z_7 *be defined in* (2.14), (3.12) *and* (3.13), *respectively. Then*

$$q\frac{d}{dq}\log z_7 \;=\; \frac{7}{72}(P(q)-P(q^7))+\frac{1}{3}\frac{w_7(13+98w_7)}{(1+13w_7+49w_7^2)}z_7^2.$$

Proof. Using Theorem 3.8, that is

$$z_7^3=(1+13w_7+49w_7^2)y_7$$

and applying logarithmic differentiation gives us

$$3q\frac{d}{dq}\log z_7 \;=\; q\frac{d}{dq}\log y_7+q\frac{d}{dq}\log(1+13w_7+49w_7^2).$$

Now using Theorem 4.8 and rearranging the second term on the right hand side we obtain

$$3q\frac{d}{dq}\log z_7 \quad = \quad \frac{7}{24}(P(q)-P(q^7))+\frac{w_7(13+98w_7)}{1+13w_7+49w_7^2}q\frac{d}{dq}\log(w_7).$$

Use Theorem 4.6 to get

$$3q\frac{d}{dq}\log z_7 \quad = \quad \frac{7}{24}(P(q)-P(q^7))+\frac{w_7(13+98w_7)}{1+13w_7+49w_7^2}z_7^2$$

then division by 3 completes the proof. □

Chapter 5

Differential equations

As we mentioned in Chapter 3, modular forms satisfy linear differential equations. The weight now becomes an important factor in determining the order of a particular differential equation. Zagier [65, Sec. 5.4 Proposition 21] says if our function f is modular form with integral positive weight k and we have a modular function t then f satisfies a linear differential equation of order $k+1$ with respect to t. In this section we derive differential equations for z_3, Z_3 and z_7, Z_7. We use techniques from calculus such as the chain rule, the product and quotient rules and differential operators and rely on the properties of Eisenstein series and theta functions.

5.1 Level 3 differential equations

We want to show that z_3, a modular form of weight one, satisfies a second order linear differential equation with respect to x_3, a modular form of weight zero.

Theorem 5.1. *Let z_3 and x_3 be defined by* (3.1) *and* (3.3). *Then the following differential equation for z_3 with respect to x_3 holds:*

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{2}{9}z_3. \tag{5.1}$$

Proof. By Theorem 4.2 we have

$$q\frac{dx_3}{dq} = z_3^2 x_3(1-x_3).$$

Using the chain rule we obtain the differential operator

$$\frac{d}{dx_3} = \frac{1}{z_3^2 x_3(1-x_3)} q\frac{d}{dq}$$

where rearranging gives

$$x_3(1-x_3)\frac{d}{dx_3} = \frac{1}{z_3^2} q\frac{d}{dq}. \tag{5.2}$$

Two applications of (5.2) give

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{1}{z_3^2 x_3(1-x_3)} q\frac{d}{dq}\left(\frac{1}{z_3^2} q\frac{dz_3}{dq}\right). \tag{5.3}$$

Using (4.1) we have

$$\begin{aligned}\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) &= \frac{1}{z_3^2x_3(1-x_3)}q\frac{d}{dq}\left(\frac{P(q)-(1-4x_3)z_3^2}{12z_3}\right)\\ &= \frac{1}{12z_3^2x_3(1-x_3)}q\frac{d}{dq}\left(\frac{P(q)}{z_3}-(1-4x_3)z_3\right).\end{aligned}$$

We now calculate the derivative of the right hand side. Let

$$A=\frac{1}{12z_3^2x_3(1-x_3)}.$$

Then

$$\begin{aligned}\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) &= Aq\frac{d}{dq}\left(\frac{P(q)}{z_3}-(1-4x_3)z_3\right)\\ &= A\left(q\frac{d}{dq}\frac{P(q)}{z_3}-q\frac{d}{dq}(1-4x_3)z_3\right)\end{aligned}$$

by separating the sum. Then using Theorems 4.1, 4.2, Equation (4.1) and the chain rule and product and quotient rules of calculus we obtain

$$\begin{aligned}&\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right)\\ &= A\left(\frac{P^2(q)-Q(q)}{12z_3}-\frac{P(q)z_3P(q)-(1-4x_3)^2z_3^2}{12z_3^2}+4z_3^3x_3(1-x_3)\right)\\ &= A\left(\frac{P^2(q)-Q(q)-P^2(q)+(1-4x_3)^2z_3^2}{12z_3}+4z_3^3x_3(1-x_3)\right)\\ &= A\left(\frac{-Q(q)+(1-4x_3)^2z_3^2)}{12z_3}+4z_3^3x_3(1-x_3)\right). \qquad (5.4)\end{aligned}$$

Now substituting A back into (5.4) to obtain

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{(1-4x_3)^2z_3^4-Q(q)}{144z_3^3x_3(1-x_3)}+\frac{z_3}{3}. \qquad (5.5)$$

In a similar manner we can use (4.2) in (5.3) to obtain

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{1}{12z_3^2x_3(1-x_3)}q\frac{d}{dq}\left(\frac{3P(q^3)}{z_3}-(3-4x_3)z_3\right)$$

and the derivative of the right hand side can be found using Theorems 4.1, Equations (4.1) and (4.2) as follows:

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{(3-4x_3)^2z_3^4-9Q(q^3)}{144z_3^3x_3(1-x_3)}+\frac{z_3}{3}. \tag{5.6}$$

Adding (5.5) and (5.6) gives us

$$2\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{(1-4x_3)^2z_3^4-Q(q)}{144z_3^3x_3(1-x_3)}+\frac{z_3}{3}+\frac{(3-4x_3)^2z_3^4-9Q(q^3)}{144z_3^3x_3(1-x_3)}+\frac{z_3}{3}.$$

After rearranging and dividing by 2 we get

$$\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = \frac{2z_3}{9}+\frac{10z_3^4-Q(q)-9Q(q^3)}{288z_3^3x_3(1-x_3)}.$$

Then using (3.6) completes the proof. □

In our next theorem we change variables from x_3 to X_3.

Theorem 5.2. *Let* z_3 *and* X_3 *be defined by* (3.1) *and* (3.4). *Then the following differential equation for* z_3 *with respect to* X_3 *holds:*

$$X_3(1-108X_3)\frac{d^2z_3}{dX_3^2}+(1-162X_3)\frac{dz_3}{dX_3}=6z_3. \tag{5.7}$$

Proof. We begin with Equation (3.10)

$$x_3=\frac{1}{2}-\frac{\sqrt{1-108X_3}}{2}.$$

We differentiate with respect to X_3 and rearrange to get

$$\frac{dX_3}{dx_3} = \frac{\sqrt{1-108X_3}}{27}.$$

We now take an arbitrary function F. By using the chain rule we can work out the derivative of F in terms of x.

$$\frac{dF}{dX_3}\frac{dX_3}{dx_3}=\frac{\sqrt{1-108X_3}}{27}\frac{dF}{dX_3}.$$

We can now use the resulting operator

$$\frac{d}{dx_3}=\frac{\sqrt{1-108X_3}}{27}\frac{d}{dX_3}. \tag{5.8}$$

Multiplying both sides of (5.8) by $x_3(1-x_3)$ and using Theorem 3.5 we obtain

$$\begin{aligned} x_3(1-x_3)\frac{d}{dx_3} &= x_3(1-x_3)\frac{\sqrt{1-108X_3}}{27}\frac{d}{dX_3} \\ &= X_3\sqrt{1-108X_3}\frac{d}{dX_3}. \end{aligned}$$

Now let us multiply both sides of (5.1) by $x_3(1-x_3)$ to obtain

$$x_3(1-x_3)\frac{d}{dx_3}\left(x_3(1-x_3)\frac{dz_3}{dx_3}\right) = x_3(1-x_3)\frac{2z_3}{9}.$$

Using the operator (5.8) we change variables from x_3 to X_3 to get

$$\begin{aligned} X_3\sqrt{1-108X_3}\frac{d}{dX_3}\left(X_3\sqrt{1-108X_3}\frac{dz_3}{dX_3}\right) &= 27X_3\frac{2z_3}{9} \\ &= 6X_3z_3. \end{aligned} \tag{5.9}$$

Let

$$B_3 = \sqrt{1-108X_3}.$$

We can now abbreviate (5.9) as follows:

$$X_3B_3\frac{d}{dX_3}\left(X_3B_3\frac{dz_3}{dX_3}\right) = 6X_3z_3.$$

We have now established the differential equation in self adjoint form. We can obtain (5.7) from that. □

To prove the differential equation for Z_3 with respect to X_3 we need the following standard technique.

Lemma 5.3. *Suppose y is a solution of a second order linear differential equation*

$$y'' + f_1y' + f_2y = 0. \tag{5.10}$$

Then $Y = y^2$ is a solution to the third order ordinary linear differential equation

$$Y''' + 3f_1Y'' + (f_1' + 4f_2 + 2f_1^2)Y' + (2f_2' + 4f_1f_2)Y = 0, \tag{5.11}$$

where the primes denote differentiation with respect to x, and f_1 and f_2 are functions of x.

Proof. The essence of the proof is that Y, Y', Y'' and Y''' are all linear combinations of y^2, yy' and $(y')^2$, hence they are linearly dependent. The details are as follows. If

$$Y = y^2$$

then

$$Y' = 2yy'$$

and

$$Y'' = 2(y')^2 + 2yy''.$$

We can now substitute for y'' from the differential equation (5.10) so Y'' is a linear combination of y^2, yy' and $(y')^2$. The result is

$$Y'' \;=\; 2(y')^2 - 2f_1yy' - 2f_2y^2.$$

Similarly Y''' can also be written as a a linear combination of y^2, yy' and $(y')^2$. We find that

$$Y''' \;=\; 2f_1^2yy' + 2f_1f_2y^2 - 2f_1'yy' - 2f_2'y^2 - 6f_1(y')^2 - 8f_2yy'.$$

Now we can substitute these values for Y, Y', Y'' and Y''' into (5.11)

$$\begin{aligned}
&Y''' + 3f_1Y'' + (f_1' + 4f_2 + 2f_1^2)Y' + (2f_2' + 4f_1f_2)Y \\
&\quad= 2f_1^2yy' + 2f_1f_2y^2 - 2f_1'yy' - 2f_2'y^2 - 6f_1(y')^2 - 8f_2yy' \\
&\qquad - 6f_1^2yy' - 6f_1f_2y^2 + 6f_1(y')^2 \\
&\qquad + 2f_1'yy' + 8f_2yy' + 4f_1^2yy' \\
&\qquad + 2f_2'y^2 + 4f_1f_2y^2 \\
&\quad= 0
\end{aligned}$$

and Y is indeed a solution to Equation (5.11) completing the proof. □

We now show Z_3, a modular form of weight two, satisfies a third order linear differential equation with respect to X_3, a modular form of weight zero .

Theorem 5.4. *Let Z_3 and X_3 be defined as* (3.2) *and* (3.4)*. Then the following differential equation holds:*

$$X_3^2(1-108X_3)\frac{d^3Z_3}{dX_3^3} + 3X_3(1-162X_3)\frac{d^2Z_3}{dX_3^2} + (1-348X_3)\frac{dZ_3}{dX_3} = 12Z_3. \quad (5.12)$$

Proof. Use Theorem 5.2 to let

$$\begin{aligned} f_1 &= \frac{1-162X_3}{X_3(1-108X_3)} \\ f_2 &= \frac{-6}{X_3(1-108X_3)}. \end{aligned}$$

Now use Lemma 5.3 and substitute f_1 and f_2 into

$$Z_3''' + 3f_1 Z_3'' + (f_1' + 4f_2 + 2f_1^2)Z_3' + (2f_2' + 4f_1 f_2)Z_3 = 0$$

and our theorem is proved. □

5.2 Level 7 differential equations

We begin by finding a differential equation for z_7, a modular form of weight one, with respect to w_7. We move through a change of variables to find a second order linear differential equation for z_7 with respect to X_7, a modular form of weight zero and a third order differential equation for Z_7, weight two with respect to X_7. The following theorem is also found in [25].

Theorem 5.5. *Let w_7 and z_7 be defined by* (3.12) *and* (3.13). *Then the following differential equation for z_7 with respect to w_7 holds:*

$$\frac{d}{dw_7}\left(w_7 \frac{dz_7}{dw_7}\right) = 2z_7\left(\frac{1+16w_7+49w_7^2}{(1+13w_7+49w_7^2)^2}\right). \tag{5.13}$$

Proof. Using Theorems 4.6, 4.9 and the chain rule we obtain two equations

$$\frac{q\frac{d}{dq}\log z_7}{q\frac{d}{dq}\log w_7} = \frac{\frac{1}{z_7}\times\frac{dz_7}{dq}}{\frac{1}{w_7}\times\frac{dw_7}{dq}} = \frac{w_7}{z_7}\times\frac{dz_7}{dw_7}$$

and

$$\frac{q\frac{d}{dq}\log z_7}{q\frac{d}{dq}\log w_7} = \frac{1}{z_7^2}\times\left(\frac{7}{72}(P(q)-P(q^7)) + \left(\frac{w_7}{3}\times\frac{13+98w_7}{1+13w_7+49w_7^2}\right)z_7^2\right).$$

Equating and multiplying by z_7 gives us

$$w_7\frac{dz_7}{dw_7} = \frac{7}{72z_7}\times(P(q)-P(q^7)) + \left(\frac{w_7}{3}\times\frac{13+98w_7}{1+13w_7+49w_7^2}\right)z_7. \tag{5.14}$$

After manipulating Theorem 4.6 we obtain the operator identity

$$\frac{d}{dw_7} = \frac{1}{w_7 z_7^2} q \frac{d}{dq}.$$

We now apply this operator to (5.14) and use the product rule to obtain

$$\begin{aligned}\frac{d}{dw_7}\left(w_7\frac{dz_7}{dw_7}\right) &= \frac{7}{72 w_7 z_7^3} q\frac{d}{dq}(P(q)-P(q^7)) - \frac{7}{72 z_7^2}\left(\frac{dz_7}{dw_7}\right)(P(q)-P(q^7)) \\ &+ \frac{w_7}{3}\left(\frac{13+98w_7}{1+13w_7+49w_7^2}\right)\frac{dz_7}{dw_7} + \frac{z_7}{3}\left(\frac{13+196w_7+637w_7^2}{(1+13w_7+49w_7^2)^2}\right).\end{aligned}$$

We compute the derivatives of $P(q)$ and $P(q^7)$ using Theorem 4.1. Then by Theorem 3.6 we can express $P(q)$, $P(q^7)$, $Q(q)$ and $Q(q^7)$ in terms of w_7, z_7 and dz_7/dw_7. With the help of a computer this simplifies to

$$\frac{d}{dw_7}\left(w_7\frac{dz_7}{dw_7}\right) = 2z_7\left(\frac{1+16w_7+49w_7^2}{(1+13w_7+49w_7^2)^2}\right)$$

and our theorem is proved. □

The following theorem, that also appears in [25], is found by changing the variable from w_7 to X_7 we get a second order differential equation for z_7 with respect to X_7 as follows:

Theorem 5.6. *Let z_7 and X_7 be defined by* (3.13) *and* (3.15). *Then the following differential equation for z_z with respect to X_7 holds:*

$$X_7(1-26X_7-27X_7^2)\frac{d^2 z_7}{dX_7^2} + (1-39X_7-54X_7^2)\frac{dz_7}{dX_7} - 2(1+3X_7)z_7 = 0. \quad (5.15)$$

In self adjoint form this becomes

$$B_7 X_7 \frac{d}{dX_7}\left(B_7 X_7 \frac{dz_7}{dX_7}\right) = 2X_7(1+3X_7)z_7 \quad (5.16)$$

where

$$B_7 = \sqrt{1-26X_7-27X_7^2}.$$

Proof. From the derivatives for w_7 and X_7 in Theorems 4.6 and 4.7 we obtain two equations

$$\frac{q\frac{d}{dq}\log X_7}{q\frac{d}{dq}\log w_7} = \frac{\frac{1}{X_7}\times\frac{dX_7}{dq}}{\frac{1}{w_7}\times\frac{dw_7}{dq}} = \frac{w_7}{X_7}\times\frac{dX_7}{dw_7}$$

and

$$\frac{q\frac{d}{dq}\log X_7}{q\frac{d}{dq}\log w_7} = \frac{Z_7\sqrt{1-26X_7-27X_7^2}}{Z_7}$$
$$= \sqrt{1-26X_7-27X_7^2}$$
$$= B_7.$$

Equating gives us

$$\frac{w_7}{X_7} \times \frac{dX_7}{dw_7} = B_7.$$

Rearranging gives the differential operator

$$w_7\frac{d}{dw_7} \quad = \quad X_7B_7\frac{d}{dX_7}. \tag{5.17}$$

Now multiplying (5.13) by w_7

$$w_7\frac{d}{dw_7}\left(w_7\frac{dz_7}{dw_7}\right) \quad = \quad 2z_7w_7\left(\frac{1+16w_7+49w_7^2}{(1+13w_7+49w_7^2)^2}\right). \tag{5.18}$$

We now use (5.17) in (5.18) to show

$$\begin{aligned}
B_7X_7\frac{d}{dX_7}\left(B_7X_7\frac{dz_7}{dX_7}\right) &= 2z_7w_7\left(\frac{1+16w_7+49w_7^2}{(1+13w_7+49w_7^2)^2}\right) \\
&= 2z_7\left(\frac{w_7}{1+13w_7+49w_7^2}\right)\left(\frac{1+16w_7+49w_7^2}{1+13w_7+49w_7^2}\right) \\
&= 2z_7X_7\left(\frac{1+16w_7+49w_7^2}{1+13w_7+49w_7^2}\right) \\
&= 2z_7X_7\left(1+\frac{3w_7}{(1+13w_7+49w_7^2)}\right) \\
&= 2z_7X_7(1+3X_7).
\end{aligned}$$

We have established the differential equation in self adjoint form, completing the proof. □

We now change the variable from z_7 to Z_7 to find a third order differential equation for Z_7 with respect to X_7. This differential equation also appears in [24].

Theorem 5.7. *Let Z_7 and X_7 be defined by* (3.14) *and* (3.15)*. Let* $B_7 = \sqrt{1-26X_7-27X_7^2}$ *and* $H_7 = 4X_7(1+3X_7)$*. Then the following differential equation for Z_7 with respect to X_7 holds:*

$$\frac{d}{dX_7}\left(B_7X_7\frac{d}{dX_7}\left(B_7X_7\frac{dZ_7}{dX_7}\right)\right) = 2H_7\frac{dZ_7}{dX_7} + \frac{dH_7}{dX_7}Z_7. \tag{5.19}$$

In expanded form this becomes

$$\begin{aligned} X_7^2(1-26X_7-27X_7^2)\frac{d^3Z_7}{dX_7^3} &+ 3X_7(1-39X_7-54X_7^2)\frac{dZ_7^2}{dX_7^2} \\ &+ (1-86X_7-186X_7^2)\frac{dZ_7}{dX_7} - 4(1+6X_7)Z_7 = 0. \end{aligned} \tag{5.20}$$

Proof. To find the third order differential equation for Z_7 with respect to X_7 we use Equation (5.15) to obtain

$$f_1 = \frac{(1-39X_7-54X_7^2)}{X_7(1-26X_7-27X_7^2)}$$

and

$$f_2 = \frac{-2(1+3X_7)}{X_7(1-26X_7-27X_7^2)}.$$

Now use Lemma 5.3 and substituting f_1 and f_2 into

$$Z_7''' + 3f_1Z_7'' + (f_1' + 4f_2 + 2f_1^2)Z_7' + (2f_2' + 4f_1f_2)Z_7 = 0$$

gives (5.20) . □

Chapter 6

Main results

In this chapter we give the main results of this thesis. That is, we find two new integer sequences $\{c_7(n)\}$ and $\{u_7(n)\}$ as a result of solving differential equations. We start with an outline of the method of solution and give examples from the classical case and level 3 before moving on to the level 7 results.

The technique we use to solve differential equations of the type in the previous chapter is called the method of Frobenius; see [3, p. 180] or [69, p. 251]. These second and third order differential equations will have two or three linearly independent solutions respectively, however only one solution is analytic at $x = 0$ as required by the initial conditions. The form of solution depends on the level. The classical case, due to Jacobi [41], is now classified as level 4 and expressed as a hypergeometric function which we define as

$$ {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!} \tag{6.1} $$

where the shifted factorial $(a)_n$ is given by:

$$ (a)_n = \begin{cases} a(a+1)(a+2)\ldots(a+n-1) & \text{if } n \in \mathbb{Z}^+ \\ 1 & \text{if } n = 0. \end{cases} $$

Ramanujan changed parameters in Jacobi's hypergeometric functions. In [52] he came up corresponding theories now known as Ramanujan's theories of elliptic functions to alternative bases, see [8, Ch. 33]. They correspond to levels 1, 2 and 3. Further functions at higher levels, that is 5, 6, 7, 8, 9 and higher, are found in [11], [16], [17], [20], [21], [23], [22], [26], [27] etc. and the solutions are expressed as power series. The coefficients in the expansion of these series give us interesting integer sequences and that is the focus of this chapter.

6.1 Classical case: level 4

Hypergeometric functions that are solutions to second order differential equations were studied by Euler in 1769, Gauss in 1812 and others; see [1, Sec. 2.3]. They showed that

$$z = {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} x^n \tag{6.2}$$

was the only analytic solution that satisfied

$$x(1-x)\frac{d^2z}{dx^2} + (1-2x)\frac{dz}{dx} - \frac{1}{4}z = 0$$

when the initial conditions $z = 1$ when $x = 0$ were taken into account. Jacobi [41] gives a parametrization of z and x in terms of modular forms and theta functions; this is described in Chapter 7.

6.2 Level 3

We now consider solutions to the differential equations in Theorems 5.1 and 5.6. The first, Equation (5.1) is of second order. Let z_3 and x_3 be as defined in (3.1) and (3.3). If we take as initial conditions $z_3 = 1$ when $x_3 = 0$, then the only solution that is analytic is the hypergeometric function in x_3

$$z_3 = {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix}; x_3\right). \tag{6.3}$$

Proofs can be found in [8, Ch. 33], [10] and [22].

The second, Equation (5.12) is third order. Let Z_3 and X_3 be defined by (3.2) and (3.4). This time the analytic solution is found by letting the initial conditions be $Z_3 = 1$ when $X_3 = 0$ and gives a hypergeometric function in X_3

$$Z_3 = {}_3F_2\left(\begin{matrix} \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1 \end{matrix}; 108X_3\right). \tag{6.4}$$

A proof can be found in [1, Ch. 2].

6.3 Level 7

In the previous chapter we solved three level 7 differential equations to give two power series in X_7 and one in w_7. The coefficients satisfy recurrence relations which give us integer sequences.

6.3.1 A new sequence $\{c_7(n)\}$

If we expand z_7 as a power series in X_7 then the following three-term quadratic recurrence relation occurs. This is one of the main results of this thesis.

Theorem 6.1. *Let z_7 and X_7 be defined by* (3.13) *and* (3.15) *and let $\{c_7(n)\}$ be the sequence defined by the recurrence relation*

$$(n+1)^2 c_7(n+1) = (26n^2+13n+2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1) \quad \text{for } n \geq 0 \tag{6.5}$$

with initial conditions

$$c_7(-1) = 0, \quad c_7(0) = 1.$$

Then in a neighbourhood of $q = 0$ the generating function is

$$z_7 = \sum_{n=0}^{\infty} c_7(n) X_7^n. \tag{6.6}$$

Proof. In Equation (5.15) we showed that z_7 satisfied a second order linear differential equation with respect to X_7. Using (6.6) we find that the first and second derivatives of z_7 are as follows:

$$z_7' = \sum_{n=0}^{\infty} n c_7(n) X_7^{n-1}$$

and

$$z_7'' = \sum_{n=0}^{\infty} n(n-1) c_7(n) X_7^{n-2}.$$

Substituting these into the differential equation (5.15) gives

$$X_7(1-26X_7-27X_7^2)\sum_{n=0}^{\infty} n(n-1)c_7(n)X_7^{n-2} + (1-39X_7-54X_7^2)\sum_{n=0}^{\infty} nc_7(n)X_7^{n-1} - 2(1+3X_7)\sum_{n=0}^{\infty} c_7(n)X_7^n = 0.$$

After some computation we deduce the recurrence relation

$$[n(n+1)+(n+1)]c_7(n+1) - [26n(n-1)+39n+2]c_7(n) - [27(n-1)(n-2)+54(n-1)+6]c_7(n-1) = 0.$$

This simplifies to (6.5) and our proof is complete.

□

The values of $c_7(n)$ for $0 \leq n \leq 10$ are

$$\begin{gathered} 1 \\ 2 \\ 22 \\ 336 \\ 6006 \\ 117348 \\ 2428272 \\ 52303680 \\ 1160427510 \\ 26337699740 \\ 608642155660. \end{gathered}$$

As confirmation of this result, the X_7 and z_7 in the power series (6.6) can be expanded respectively in q series as follows:

$$\begin{aligned} X_7 = {} & q - 9\,q^2 + 30\,q^3 - 15\,q^4 - 240\,q^5 + 978\,q^6 - 1463\,q^7 - 2361\,q^8 \\ & + 18201\,q^9 - 42800 q^{10} + O\left(q^{11}\right) \end{aligned} \tag{6.7}$$

and

$$z_7 = 1 + 2\,q + 4\,q^2 + 6\,q^4 + 2\,q^7 + 8\,q^8 + 2\,q^9 + O\left(q^{11}\right). \tag{6.8}$$

Substituting the q-expansions into (6.6) and comparing expressions as far as q^{10} gives us a check. Experimental results showed that the terms of the sequence are integers but this does not constitute a proof. However, we will now give a simple proof for this.

Theorem 6.2. *The sequence $\{c_7(n)\}$ defined by the recurrence relation* (6.5)*, that is*

$$\begin{aligned} (n+1)^2 c_7(n+1) = {} & (26n^2 + 13n + 2) c_7(n) + 3(3n-1)(3n-2) c_7(n-1) \\ & \text{for } n \geq 0, \end{aligned}$$

with initial conditions

$$c_7(-1) = 0, \quad c_7(0) = 1,$$

takes only integer values.

Proof. If we expand (6.6) in powers of X_7 we have

$$z_7 = \sum_{n=0}^{\infty} c_7(n) X_7^n = c_7(0) + c_7(1)X_7 + c_7(2)X_7^2 + \cdots . \tag{6.9}$$

We can substitute (6.7) into (6.9) to obtain

$$z_7 = c_7(0) + c_7(1)(q - 9q^2 + 30q^3 - \cdots) + c_7(2)(q - 9q^2 + 30q^3 - \cdots)^2 + \cdots . \tag{6.10}$$

But we have a q-expansion of z_7 (6.8) hence equating (6.8) and (6.10) gives us

$$\begin{aligned} &1 + 2q + 4q^2 + 6q^4 + 2q^7 + 8q^8 + 2q^9 + \cdots \\ &\qquad = c_7(0) + c_7(1)(q - 9q^2 + 30q^3 - \cdots) + c_7(2)(q - 9q^2 + 30q^3 - \cdots)^2 + \cdots . \end{aligned}$$

We equate coefficients as follows: the coefficient for the constant term is

$$c_7(0) = 1$$

so $c_7(0)$ is an integer. The coefficient the q term is

$$c_7(1) = 2.$$

Now the coefficients of q^2, q^3 and q^4 are found by

$$\begin{aligned} 4 &= -9c_7(1) + c_7(2) \\ 0 &= 30c_7(1) - 18c_7(2) + c_7(3) \\ 6 &= -15c_7(1) + 141c_7(2) - 27c_7(3) + c_7(4) \end{aligned}$$

giving

$$\begin{aligned} c_7(2) &= 22 \\ c_7(3) &= 336 \\ c_7(4) &= 6006. \end{aligned}$$

In general, equating the coefficients of q^n produces an integer relation among $c_7(1), c_7(2), \ldots, c_7(n)$ where the coefficient of $c_7(n)$ is 1. Then using induction on n the terms $c_7(n)$ will be integers. □

The significance of this result is twofold. First, it is not obvious from the recurrence relation that we would see an integer sequence. Equation (6.5) gives

$$c_7(n+1) = \frac{1}{(n+1)^2}\left((26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1)\right).$$

Since $(n+1)^2$ is in the denominator, we would expect $c_7(n+1)$ to be a rational number with denominator something like $((n+1)!)^2$ but we find integers. The proof that the sequence must be integers is short but there was a lot of background involved. We needed Ramanujan's Eisenstein series, theta functions and differential equations. Secondly, recurrence relations with polynomial coefficients that are integer valued are fairly rare. The ones that are known have interesting properties particularly congruences and parametrization by modular forms. So there is interest in finding new integer sequences of this type.

If we expand Z_7 as a power series in X_7 using (3.16) the three-term cubic recurrence relation discovered by S. Cooper [24] is revealed.

Theorem 6.3. *Let Z_7 and X_7 be defined by* (3.14) *and* (3.15)*. Let $\{t_7(n)\}$ be the sequence defined by the recurrence relation*

$$(n+1)^3 t_7(n+1) = (2n+1)(13n^2+13n+4)t_7(n) + 3n(9n^2-1)t_7(n-1) \quad \text{for } n \geq 0 \tag{6.11}$$

with initial conditions $t_7(0) = 1$. Alternatively, let $\{t_7(n)\}$ be the sequence defined by the binomial sum

$$t_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} \binom{n+k}{k}. \tag{6.12}$$

In a neighbourhood of $q = 0$ the generating function is

$$Z_7 = \sum_{n=0}^{\infty} t_7(n) X_7^n. \tag{6.13}$$

Proof. The proof is similar to Theorem 6.1. We substitute (6.13) and its derivatives into (5.19). The details can be found in [24]. As a check we substitute q expansions for X_7^n and Z_7 into (6.13) and compare expressions. □

The numbers in $\{t_7(n)\}$ are all integers which again is surprising as the recurrence relation would suggest rational numbers since there is division by $(n+1)^3$. Two proofs are given in [24]. The first proof is similar to the proof in Theorem 6.2. The second proof relies on the fact that $\{t_7(n)\}$ has a sum of binomial coefficients which by definition are integers. Since no binomial sum has yet been found for the sequence $\{c_7(n)\}$ this short confirming proof was not available to us.

We state without proof that the generating function satisfies an interesting functional equation.

Theorem 6.4. *Let $\{c_7(n)\}$ and $\{t_7(n)\}$ be sequences defined by* (6.5) *and* (6.11) *and let g and f be the generating functions given by*

$$g(y) \quad = \quad \sum_{n=0}^{\infty} t_7(n) y^n$$

and

$$f(y) \quad = \quad \sum_{n=0}^{\infty} c_7(n) y^n.$$

Then in a neighbourhood of $y = 0$

$$\frac{1}{(1+4y)^2} g\left(\frac{y}{(1+4y)^3}\right) \quad = \quad \frac{1}{(1+2y)^2} g\left(\frac{y^2}{(1+2y)^3}\right). \tag{6.14}$$

By taking the square root of $g(y)$ we obtain

$$\frac{1}{1+4y} f\left(\frac{y}{(1+4y)^3}\right) \quad = \quad \frac{1}{1+2y} f\left(\frac{y^2}{(1+2y)^3}\right). \tag{6.15}$$

Proof. The functional equation (6.14) was established by S. Cooper and D. Ye [28] and the functional equation for (6.15) is obtained by taking square roots of (6.14). This is possible since $Z_7 = z_7^2$ by Theorem 3.7 where z_7 and Z_7 are the power series (6.6) and (6.13), respectively.

□

6.3.2 A second new sequence $\{u_7(n)\}$

Our last series expansion in this section expands w_7 as a power series in y_7. This is again a new result.

For comparison we state the corresponding level 5 results found in Table 2 of [18] as follows:

$$\frac{\eta_1^5}{\eta_5} = \sum_{n=-\infty}^{\infty} u_5(n) \left(\frac{\eta_5^6}{\eta_1^6}\right)^n$$

where η_m is defined by (2.13). The power series expansion satisfies a three-term recurrence relation with cubic polynomials given by

$$(n+1)^3 u_5(n+1) = -(2n+1)(11n^2+11n+5)u_5(n) - 125n^3 u_5(n-1)$$
$$for \;\; n \geq 0$$

with initial condition $u_5(0) = 1$. A solution to the recurrence relation in terms of binomial coefficients is given by

$$u_5(n) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}^3 \binom{4n-5k}{3n}.$$

Motivated by the results of level 5 we looked to see if there was a similar relationship at level 7. We did find a relationship. However, it is a five-term quartic recurrence relation, and no binomial sum has yet been found.

Theorem 6.5. *Let y_7 and w_7 be defined by* (3.11) *and* (3.12). *Let $\{u_7(n)\}$ be the sequence defined by the five-term quartic recurrence relation*

$$\begin{aligned}(n+1)^4 u_7(n+1) = &-(26n^4 + 52n^3 + 58n^2 + 32n + 7)u_7(n)\\ &- (267n^4 + 268n^2 + 18)u_7(n-1)\\ &- (1274n^4 - 2548n^3 + 2842n^2 - 1568n + 343)u_7(n-2)\\ &- 2401(n-1)^4 u_7(n-3) \qquad (6.16)\\ \text{for } n \geq 0 &\end{aligned}$$

with initial conditions $u_7(0) = 1, u_7(-1) = u_7(-2) = u_7(-3) = 0$. Then in a neighbourhood of $q = 0$ the generating function is

$$y_7 = \sum_{n=-\infty}^{\infty} u_7(n) w_7^n. \qquad (6.17)$$

Using eta product notation we can write (6.17) *as*

$$\frac{\eta_1^7}{\eta_7} = \sum_{n=-\infty}^{\infty} u_7(n) \left(\frac{\eta_7^4}{\eta_1^4}\right)^n.$$

Proof. The variables y_7 and z_7 are algebraically related and the variables w_7 and X_7 are also algebraically related as we see from Theorems 3.8 and 3.10. Therefore, y_7 satisfies a differential equation with respect to w_7, has a power series solution and gives rise to the recurrence relation satisfied by $u_7(n)$. □

The values of $u_7(n)$ for $0 \le n \le 17$ are as follows:

$$\begin{gathered} 1 \\ -7 \\ 42 \\ -231 \\ 1155 \\ -4998 \\ 15827 \\ -791 \\ -566244 \\ 6506955 \\ -53524611 \\ 369879930 \\ -2218053747 \\ 11306008875 \\ -43772711220 \\ 55203364377 \\ 172838094533 \\ -16542312772356. \end{gathered}$$

As confirmation the y_7 and w_7 in the power series (6.17) can be expanded respectively in q series as follows:

$$\begin{aligned} y_7 = {} & q + 4q^2 + 14q^3 + 40q^4 + 105q^5 + 252q^6 + 574q^7 + 1236q^8 + 2564q^9 + 5124q^{10} + \\ & + 9948q^{11} + 18788q^{12} + 34685q^{13} + 62664q^{14} + 111132q^{15} + 193672q^{16} + 332325q^{17} \\ & + 561996\,q^{18} + O(q^{19}) \end{aligned}$$

and

$$\begin{aligned} w_7 = {} & 1 - 7\,q + 14\,q^2 + 7\,q^3 - 49\,q^4 + 21\,q^5 + 35\,q^6 + 42\,q^7 - 56\,q^8 - 119q^9 + 105q^{10} \\ & - 70q^{11} + 147q^{12} + 147q^{13} - 133q^{14} - 168q^{15} - 231q^{16} + 252q^{17} - 154q^{18} + O(q^{19}). \end{aligned}$$

Substituting q-expansions of y_7 and w_7 into (6.17) and comparing expressions up to q^{10} gives us a check. As with the previous two sequences, this sequence is integer valued and for the same reasons as we have explained before. However, we notice the erratic size and irregular sign change. These will be explained by the asymptotic formulae in Chapter 8.

The weight 2 sequence $\{t_7(n)\}$ is found as A183204 in the On–Line Encyclopedia of Integer Sequences (OEIS) [57]. At the time of writing neither the weight 1 sequence $\{c_7(n)\}$ nor the weight 3 sequence $\{u_7(n)\}$ are found in OEIS.

Chapter 7

Clausen's identity

An interesting identity attributed to the Danish mathematician T.Clausen in 1828 is that the square of a particular ${}_2F_1$ hypergeometric function with certain parameters, defined in Equation (6.1), can be expressed in terms of a ${}_3F_2$ hypergeometric function. This can be found in Question 13 of [1, Ch. 2] as follows:

$$\left\{ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix}; x \right) \right\}^2 = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix}; x \right). \tag{7.1}$$

Clausen's identity can be combined with the quadratic transformation formula

$${}_2F_1\left(\begin{matrix} 2a,\ 2b \\ a+b+\frac{1}{2} \end{matrix}; x \right) = {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix}; 4x(1-x) \right) \tag{7.2}$$

to give

$$\left\{ {}_2F_1\left(\begin{matrix} 2a,\ 2b \\ a+b+\frac{1}{2} \end{matrix}; x \right) \right\}^2 = {}_3F_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix}; 4x(1-x) \right). \tag{7.3}$$

There are many applications of Clausen's identity. For example in 1914 S.Ramanujan used it to derive 17 series for $\frac{1}{\pi}$; see [52]. In this thesis we compare the classical case, known as level 4, with level 3 and just note that we have a Clausen analogue for level 7.

7.1 Classical case: level 4

The classical case, now known as level 4, was used by Jacobi [41] in 1829. The functions z and x are given in terms of q namely,

$$z = {}_2F_1\left(\begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{matrix}; x \right) = \varphi^2(q)$$

and

$$x = \frac{16q\psi^4(q^2)}{\varphi^4(q)}$$

where φ and ψ are defined by (2.7) and (2.9). See [9], [10] and [22] for proofs of this result.

7.2 Level 3 case

In a similar fashion we look at a Clausen type analogue for Z_3. Using Equations (3.10), (6.3) and (6.4), that is

$$z_3 = {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}; x\right)$$

and

$$Z_3 = {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, 1\end{matrix}; 4x(1-x)\right)$$

and using the identity in Equation (3.5) we obtain

$$\left\{{}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}; x_3\right)\right\}^2 = {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}, \frac{1}{2} \\ 1, 1\end{matrix}; 4x_3(1-x_3)\right). \tag{7.4}$$

Where (7.4) is an instance of (7.3) with $a = \frac{1}{6}$ and $b = \frac{1}{3}$.

7.3 Level 7 case

At level 7 we express z_7 and Z_7 in terms of a power series (6.6) and (6.13). That is

$$Z_7 = \sum_{n=0}^{\infty} t_7(n) X_7^n$$

and

$$z_7 = \sum_{n=0}^{\infty} c_7(n) X_7^n.$$

We recall that the recurrence relations of Equations (6.5) and (6.11) are

$$(n+1)^2 c_7(n+1) = (26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1)$$
$$\text{for } n \geq 0,$$

with initial conditions

$$c_7(-1) = 0, \quad c_7(0) = 1$$

and

$$(n+1)^3 t_7(n+1) = (2n+1)(13n^2 + 13n + 4)t_7(n) + 3n(9n^2-1)t_7(n-1)$$

with initial conditions

$$t_7(0) = 1.$$

To show a Clausen type analogue we use (3.16)

$$z_7^2 = Z_7$$

to obtain

$$\left(\sum_{n=0}^{\infty} c_7(n) X_7^n\right)^2 = \sum_{n=0}^{\infty} t_7(n) X_7^n.$$

In this case it is interesting that the square of one series is equal to another series. It is not obvious. Levels 3 and 4 are special cases of Clausen's formula in which we can vary the parameter. Levels 5 and 6 also have Clausen analogues that include parameters so we were interested to know if this is the case for level 7. A question for further research is:
Are we able to generalize the level 7 case to include a parameter?

Chapter 8

Asymptotics

8.1 Asymptotics

In this section we look at the asymptotic behaviour of our recurrence relations. Asymptotic analysis is a method of describing limiting behaviour. We want to find a function that best approximates a term in a recurrence relation at a particular large value of n. We start by looking at a second order recurrence relation with constant coefficients of the form

$$s_{n+1} = \alpha s_n + \beta s_{n-1}. \tag{8.1}$$

The standard method is to try a solution of the form

$$s_n = r^n. \tag{8.2}$$

When we substitute Equation (8.2) into Equation (8.1) we obtain

$$r^{n+1} = \alpha r^n + \beta r^{n-1}.$$

Now we divide both sides by r^{n-1} giving

$$r^2 = \alpha r + \beta.$$

We solve the characteristic equation to give two roots r_1 and r_2. Assuming $r_1 \neq r_2$ the general solution to the recurrence relation is given by

$$s_n = \gamma_1 r_1^n + \gamma_2 r_2^n$$

where γ_1 and γ_2 are constant coefficients. We can find the values of γ_1 and γ_2 by using the initial conditions which are the first two terms in the sequence. That is either s_0 and s_1 or s_{-1} and s_0. In the case of repeated roots we find the solution is

$$s_n = \gamma_1 r_1^n + \gamma_2 n r_1^n.$$

If we consider our prototype, the Fibonacci sequence where

$$F_{n+1} = F_n + F_{n-1}, \qquad F_{-1} = 0,\ F_0 = 1$$

then solve the quadratic equation

$$r^2 = r + 1.$$

We determine the two roots

$$r_1 = \frac{1+\sqrt{5}}{2}$$

and

$$r_2 = \frac{1-\sqrt{5}}{2}.$$

We find γ_1 and γ_2 by using the initial conditions so

$$\frac{\gamma_1}{r_1} + \frac{\gamma_2}{r_2} = 0$$

and

$$\gamma_1 + \gamma_2 = 1.$$

Solving for γ_1 and γ_2 gives

$$\gamma_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

and

$$\gamma_2 = \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right).$$

So the solution to the recurrence relation is

$$\begin{aligned} F_n &= \gamma_1 r_1^n + \gamma_2 r_2^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

If we just take the dominant term we see F_n grows exponentially and has an asymptotic relationship

$$F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \qquad \text{as } n \to \infty.$$

More generally we write

$$f(n) \sim g(n) \ \text{ as } \ n \to \infty$$

to mean

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

8.2 Birkhoff-Trjitzinsky method for asymptotic expansions

So far we have talked about recurrence relations with constant coefficients. We are now going to look at more general recurrence relations where the coefficients are algebraic functions or special functions such as the hypergeometric function. Wimp and Zeilberger [61] give a good account of the development of a technique for finding the asymptotic expansion of a recurrence relation for a sequence by the Birkhoff–Trjitzinsky method. This method is to try a solution in the form

$$s_n \sim C n^{\alpha} r^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots \right).$$

This is a generalization of our prototype example for the Fibonacci sequence above where we raised the root to the power of n and in that case we had constant coefficients. We look at several asymptotic expansions. First, the sequence $\{t_7(n)\}$ is used as an example of the Birkhoff–Trjitzinsky method. This will be followed by our best approximations for the new sequences $\{c_7(n)\}$ and $\{u_7(n)\}$.

8.3 Asymptotic behaviour of the sequence $\{t_7(n)\}$

In his paper [38], Hirschhorn determined the asymptotic behaviour for $\{t_7(n)\}$ in the three-term cubic recurrence relation from Equation (6.11). Hirschhorn's proof relies on the fact that this equation has a binomial sum

$$t_7(n) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} \binom{n+k}{k}.$$

He showed that

$$t_n \sim \frac{1}{4}\left(\frac{3}{\pi n}\right)^{\frac{3}{2}} 27^n \left(1 - \frac{65}{144n} + \frac{3865}{41472n^2} - \frac{111727}{17915904n^3} + \cdots\right), \quad \text{as } n \longrightarrow \infty$$

where the dominant term is given by

$$t_n \sim \frac{1}{4}\left(\frac{3}{\pi n}\right)^{\frac{3}{2}} 27^n \quad \text{as} \quad n \longrightarrow \infty.$$

8.4 Asymptotic behaviour of the sequence $\{c_7(n)\}$

We have a similar type of recurrence relations in Equation (6.5). However, we have no binomial sum but we consider whether there might be asymptotic behaviour for $\{c_7(n)\}$ in this equation of the form

$$c_n \sim Cn^\alpha r^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right)$$

where C, α and r are constants to be determined, the correction term is $\left(1 + \frac{a_1}{n} + \dots\right)$ and $n \geq 1$.

The following conjecture is the asymptotic expansion we found for the sequence $\{c_7(n)\}$ and in the following subsection we are going to explain how we found these parameters.

Conjecture 8.1. *Let n be a positive integer*

$$c_n \quad \sim Cn^{-\frac{3}{2}} 27^n \left(1 - \tfrac{215}{1008n} - \tfrac{1265}{290304n^2} + \tfrac{4683055}{877879296n^3} + \cdots\right)$$

as $n \to \infty$, where $C \approx 0.09552$.

8.4.1 Determining r and α

We determine r and α by the Birkhoff–Trjitzinsky method as outlined by Wimp and Zeilberger in [61]. We start with recurrence relation (6.5)

$$(n+1)^2 c_7(n+1) = (26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1) \qquad (8.3)$$

with initial conditions $c_7(-1) = 0, c_7(0) = 1$ and assume the asymptotic formula

$$c_n \sim Cn^\alpha r^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right). \qquad (8.4)$$

We substitute the expansion of the ansatz (8.4), that is the starting equation, into the recurrence relation (8.3) to obtain

$$\begin{aligned} & (n+1)^2 C(n+1)^{\alpha} r^{n+1} \left(1 + \frac{a_1}{n+1} + \cdots\right) \\ - & (26n^2 + 13n + 2) C n^{\alpha} r^{n} \left(1 + \frac{a_1}{n} + \cdots\right) \\ - & 3(3n-1)(3n-2) C (n-1)^{\alpha} r^{n-1} \left(1 + \frac{a_1}{n-1} + \cdots\right) = 0 \end{aligned}$$

and divide through by Cr^{n-1}

$$\begin{aligned} & (n+1)^{\alpha+2} r^2 \left(1 + \frac{a_1}{n+1} + \cdots\right) \\ - & (26n^2 + 13n + 2) n^{\alpha} r \left(1 + \frac{a_1}{n} + \cdots\right) \\ - & 3(3n-1)(3n-2)(n-1)^{\alpha} \left(1 + \frac{a_1}{n-1} + \cdots\right) = 0. \end{aligned}$$

Then divide by $n^{2+\alpha}$ and put $x = \frac{1}{n}$ to obtain

$$\begin{aligned} & (1+x)^{\alpha+2} r^2 \left(1 + \frac{a_1 x}{1+x} + \cdots\right) \\ - & (26 + 13x + 2x^2) r \left(1 + a_1 x + \cdots\right) \\ - & 3(-3+x)(-3+2x)(1-x)^{\alpha} \left(1 + \frac{a_1 x}{1-x} + \cdots\right) = 0. \end{aligned}$$

We expand this in powers of x to give

$$(r^2 - 26r - 27) + (r^2 a_1 + (\alpha+2) r^2 - 26 r a_1 - 13r + 27\alpha + 27 - 27a_1)x + O(x^2) = 0.$$

Now working with the constant term we solve the quadratic equation

$$r^2 - 26r - 27 = 0 \tag{8.5}$$

to get

$$r = 27 \ \text{ or } \ r = -1.$$

Now to determine α we set the coefficient of x to zero

$$(r^2 - 26r - 27)a_1 + (\alpha+2)r^2 - 13r + 27\alpha + 27 = 0$$

and using (8.5) we find

$$(\alpha + 2)r^2 - 13r + 27\alpha + 27 = 0.$$

Now using (8.5) again

$$\begin{aligned}
\alpha &= -\frac{2r^2 - 13r + 27}{r^2 + 27} \\
&= -\frac{2(26r + 27) - 13r + 27}{26r + 27 + 27} \\
&= -\frac{3(13r + 27)}{2(13r + 27)} \\
&= -\frac{3}{2}.
\end{aligned}$$

8.4.2 Numerical confirmation of α

We want a quick numerical verification of the α that we found in the previous section. Using the dominant term from Equation (8.4)

$$\frac{c_n}{27^n} \approx C n^{\alpha}$$

we proceed by taking logarithms of both sides to obtain

$$\log \frac{c_n}{27^n} \approx \log C + \alpha \log n.$$

Now replace n with $2n$ and subtract the old equation from the new

$$\log \frac{c_{2n}}{27^{2n}} - \log \frac{c_n}{27^n} \approx \log C + \alpha \log 2n - \log C - \alpha \log n.$$

This simplifies to

$$\log \frac{c_{2n}}{27^n c_n} \approx \alpha \log \frac{2n}{n}.$$

Solving for α gives

$$\alpha \approx \frac{\log \frac{c_{2n}}{27^n c_n}}{\log 2}.$$

A numerical calculation in Maple taking $n = 2000$ gives $\alpha = -1.49992$ which is in strong agreement with $\alpha = -\frac{3}{2}$.

8.4.3 Determining the constant C

We want to find an estimate for the constant term C. This is hard but we follow the approach taken by Wimp and Zeilberger in [61] using

$$C \approx \frac{c_n n^{\frac{3}{2}}}{27^n}. \tag{8.6}$$

However, this method is only accurate to $O(\frac{1}{n})$. A better method to use is with finite differences. We define a difference operator.

Definition 8.2. *The difference operator $\triangle$, where $f : \mathbb{R} \to \mathbb{R}$, is defined by*

$$\triangle f(n) = f(n+1) - f(n).$$

By mathematical induction we can apply the operator k times to give

$$\triangle^k f(n) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(n+k-j).$$

Suppose we have a function g with an asymptotic expansion given by

$$g(n) \sim c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots \text{ as } n \to \infty.$$

If we multiply both sides by n^2 we get

$$n^2 g(n) \sim c_0 n^2 + c_1 n + c_2 + \frac{c_3}{n} + \cdots \text{ as } n \to \infty.$$

Applying the difference operator $\triangle^2$ gives us

$$\begin{aligned} \frac{\triangle^2(n^2 g(n))}{2!} &\sim c_0 + \frac{c_3}{n(n+1)(n+2)} + \cdots \\ &\sim c_0 + O\left(\frac{1}{n^3}\right) \quad \text{as } n \to \infty. \end{aligned}$$

This gives a better approximation to C than (8.6) because of the smaller error term. We can estimate c_0 to high accuracy more generally by

$$\frac{\triangle^k(n^k g(n))}{k!} \sim c_0 + O\left(\frac{1}{n^{k+1}}\right) \text{ as } n \to \infty.$$

Now using Equation (8.6) we estimate the constant

$$\triangle^k \left(\frac{n^{\frac{3}{2}+k} c_n}{27^n k!}\right) \sim C\left(1 + O\left(\frac{1}{n^{k+1}}\right)\right) \text{ as } n \to \infty. \tag{8.7}$$

Zeilberger has the following to say about the method we used [68]:

> The Birkhoff–Trjitzinsky method suffers from one drawback. It only does the asymptotics up to a *multiplicative* constant C. But **nowadays** this is hardly a problem. Just crank-out the first ten thousand terms of the sequence using the very recurrence whose asymptotics you are trying to find, not forgetting to furnish the few necessary *initial conditions*, and then *estimate* the constant **empirically**. If you are lucky, then Maple can recognize it in terms of "famous" constants like e and π, by typing `identify(C)`.

By taking $n = 10,000$ and $k = 40$ we find

$$
\begin{aligned}
C =\,& 0.0955223052681267146513079107870296256727946666510071798669 9\\
& 4823491765926452453145767581209397218144601400343835459078543\\
& 716038032131869546216944
\end{aligned}
$$

to 144 decimal places. How do we know this is correct? We look at the error for different values of n. Let us denote the error when the k^{th} order difference is used to approximate C in (8.7) by

$$E_{n,k} = O\left(\frac{1}{n^{k+1}}\right).$$

We now compare errors when the value of n is doubled.

$$
\begin{aligned}
\frac{E_{2n,k}}{E_{n,k}} &\approx \frac{\frac{b}{(2n)^{k+1}}}{\frac{b}{n^{k+1}}}, \qquad \text{where } b \text{ is a constant of proportionality}\\
&= \frac{1}{2^{k+1}}.
\end{aligned}
$$

If we let $k = 40$ then

$$\frac{1}{2^{k+1}} = \frac{1}{2^{41}} \approx \frac{1}{10^{12}}.$$

We expect the decimal places to increase by about 12 places as n doubles. We found

n	decimal places in agreement	difference
1,250	107	
2,500	119	12
5,000	132	13
10,000	144	12

For $n = 10,000$ we checked for 170 decimal places and on the basis of the preceding results we conclude that 144 decimal places in agreement for $n = 10,000$ is correct.

Hirschhorn in [38] found the asymptotic behaviour of the sequence $\{t_7(n)\}$ (6.11). He was able to prove a nice explicit value for C of $\frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{3}{2}}$. We have tried a number of things to identify C for $\{c_7(n)\}$ (6.5). We have tried the Maple function "identify", scaled by π and tried to see if it was an algebraic number. So far it has resisted all attempts at identification. So for now with reference to the above quote by Zeilberger we have not been "lucky" and Maple is unable to recognize C. We tried to find an explicit value for C in Wolfram Alpha [62] without success. We also did an advanced search on the inverse calculator at `http://isc.carma.newcastle.edu.au/advanced` that produced the result "Wow, really found nothing".

8.4.4 The correction term

We are going to assume there is a correction term since other similar sequences have them. Now $c_7(n)$ satisfies the recurrence relation (6.5)

$$(n+1)^2 c_7(n+1) - (26n^2 + 13n + 2)c_7(n) - (27n^2 - 27n + 6)c_7(n-1) = 0.$$

which we can rewrite as

$$\left(1+\frac{1}{n}\right)^2 c_7(n+1) - \left(26 + \frac{13}{n} + \frac{2}{n^2}\right) c_7(n) - \left(27 - \frac{27}{n} + \frac{6}{n^2}\right) c_7(n-1) = 0. \tag{8.8}$$

Now let us suppose

$$c_n = Cn^{-\frac{3}{2}} 27^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right).$$

We can now substitute c_n into Equation (8.8) to obtain

$$\begin{aligned}
&\left(1+\frac{1}{n}\right)^2 C(n+1)^{-\frac{3}{2}} 27^{n+1} \left(1 + \frac{a_1}{n+1} + \frac{a_2}{(n+1)^2} + \frac{a_3}{(n+1)^3} + \cdots\right) \\
- &\left(26 + \frac{13}{n} + \frac{2}{n^2}\right) Cn^{-\frac{3}{2}} 27^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right) \\
- &\left(27 - \frac{27}{n} + \frac{6}{n^2}\right) C(n-1)^{-\frac{3}{2}} 27^{n-1} \left(1 + \frac{a_1}{n-1} + \frac{a_2}{(n-1)^2} + \frac{a_3}{(n-1)^3} + \cdots\right) = 0.
\end{aligned}$$

We now divide by $C27^n$ and multiply by $n^{3/2}$ to get

$$\begin{aligned}
& \left(1+\frac{1}{n}\right)^{\frac{1}{2}} 27\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+\frac{a_3}{(n+1)^3}+\cdots\right) \\
- & \left(26+\frac{13}{n}+\frac{2}{n^2}\right)\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\cdots\right) \\
- & \left(1-\frac{1}{n}\right)^{-\frac{3}{2}} \frac{1}{27}\left(27-\frac{27}{n}+\frac{6}{n^2}\right)\left(1+\frac{a_1}{n-1}+\frac{a_2}{(n-1)^2}+\frac{a_3}{(n-1)^3}+\cdots\right) \\
= & \ 0. \qquad (8.9)
\end{aligned}$$

Now we make a change of variable. Let

$$\frac{1}{n}=u$$

so

$$\frac{1}{n \pm 1}=\frac{u}{1 \pm u}.$$

Since the following expansions hold

$$(1+u)^{\frac{1}{2}}=1+\frac{u}{2}-\frac{u^2}{8}+\frac{u^3}{16}-\cdots$$

$$1+\frac{a_1 u}{1+u}+\frac{a_2 u^2}{(1+u)^2}+\frac{a_3 u^3}{(1+u)^3}+\cdots=1+a_1 u+(-a_1+a_2) u^2+(a_1-2 a_2) u^3+\cdots$$

and

$$(1-u)^{-\frac{3}{2}}=1+\frac{3 u}{2}+\frac{15 u^2}{8}+\frac{35 u^3}{16}+\cdots$$

$$1+\frac{a_1 u}{1-u}+\frac{a_2 u^2}{(1-u)^2}+\frac{a_3 u^3}{(1-u)^3}+\cdots=1+a_1 u+(a_1+a_2) u^2+(a_1+2 a_2+a_3) u^3+\cdots .$$

Equation (8.9) can be expanded to

$$\begin{aligned}
& 27(1+\frac{u}{2}-\frac{u^2}{8}+\frac{u^3}{16}+\cdots)(1+a_1 u+(-a_1+a_2) u^2+(a_1-2 a_2) u^3+\cdots) \\
- & \left(26+13 u+2 u^2\right)\left(1+a_1 u+a_2 u^2+a_3 u^3+\cdots\right) \\
- & (1+\frac{3 u}{2}+\frac{15 u^2}{8}+\frac{35 u^3}{16}+\cdots)\left(1-u+\frac{6}{27} u^2\right) \\
\times & (1+a_1 u+(a_1+a_2) u^2+(a_1+2 a_2+a_3) u^3+\cdots)=0. \qquad (8.10)
\end{aligned}$$

Equation (8.10) is a linear triangular system for $a_1,\ a_2,\ a_3\ldots.$ We solve this by setting the coefficients of the powers of u equal to zero.

$$-28a_1 - \frac{215}{36} = 0$$
$$a_1 = -\frac{215}{1008}.$$

Similarly, considering coefficients of u^2 and u^3 we find respectively

$$a_2 = -\frac{1265}{290304},$$
$$a_3 = \frac{4683055}{877879296}.$$

We are able to calculate as many terms as we want. This works because C cancels out.

8.5 Asymptotic behaviour of the sequence $\{u_7(n)\}$

The recurrence relation for Equation (6.16) gives us an integer sequence that moves in a seemingly haphazard fashion. We looked at the sign change by looking at the asymptotics to see if there was a pattern but no pattern was found.

8.5.1 Analytic determination of r and α

We start with our recurrence relation

$$(n+1)^4 u_7(n+1) = -Pu_7(n) - Qu_7(n-1) - Ru_7(n-2) - Su_7(n-3)$$

where

$$\begin{aligned} P &= 26n^4 + 52n^3 + 58n^2 + 32n + 7, \\ Q &= 267n^4 + 268n^2 + 18, \\ R &= 1274n^4 - 2548n^3 + 2842n^2 - 1568n + 343, \\ S &= 2401(n-1)^4 \end{aligned} \tag{8.11}$$

with initial conditions $u_7(0) = 1$ and $u_7(-1) = u_7(-2) = u_7(-3) = 0$.
We use the ansatz

$$u_7(n) \sim Cn^{\alpha} r^n \tag{8.12}$$

and substitute the expansion of (8.12) into the recurrence relation (8.11) to give us a polynomial in α having four complex repeated roots

$$(n+1)^4 C(n+1)^{\alpha} r^{n+1} + PCn^{\alpha} r^n + QC(n-1)^{\alpha} r^{n-1} \\ + RC(n-2)^{\alpha} r^{n-2} + SC(n-3)^{\alpha} r^{n-3} = 0. \quad (8.13)$$

Now dividing (8.13) through by Cr^{n-3} gives us

$$(n+1)^{\alpha+4} r^4 + Pn^{\alpha} r^3 + Q(n-1)^{\alpha} r^2 + R(n-2)^{\alpha} r + S(n-3)^{\alpha} = 0. \quad (8.14)$$

Now we divide (8.14) by $n^{4+\alpha}$ and put $x = \frac{1}{n}$ to obtain

$$(1+x)^{4+\alpha} r^4 + \left(7x^4 + 32x^3 + 58x^2 + 52x + 26\right) r^3 + \left(18\,x^4 + 268\,x^2 + 267\right) r^2 (1-x)^{\alpha} \\ + 49\left(7x^4 - 32x^3 + 58x^2 - 52x + 26\right) r\,(1-2x)^{\alpha} + 2401\,(-1+x)^4 (1-3x)^{\alpha} = 0.$$

Expand in powers of x to give

$$r^4 + 26r^3 + 267r^2 + 1274r + 2401 \\ + \left((4+\alpha)\,r^4 + 52r^3 - 267r^2\alpha - 2548r\alpha - 2548r - 7203\alpha - 9604\right) x \\ + \left(\left(\frac{\alpha^2}{2} + \frac{7\alpha}{2} + 6\right) r^4 + 58r^3 + 267r^2 \left(\frac{\alpha^2}{2} - \frac{\alpha}{2}\right) + 268r^2 + 1274r\left(2\alpha^2 - 2\alpha\right) \right. \\ \left. + 5096r\alpha + 2842r + \frac{21609\alpha^2}{2} + \frac{36015\alpha}{2} + 14406\right) x^2 + O\left(x^3\right).$$

Now working with the constant term

$$r^4 + 26r^3 + 267r^2 + 1274r + 2401 \quad = \quad 0$$

and factoring gives

$$(r^2 + 13r + 49)^2 = 0. \quad (8.15)$$

Solving the quartic equation gives

$$r = -\frac{13}{2} \pm \frac{3i}{2}\sqrt{3} \quad (8.16)$$

with multiplicity 2.
Next we set the coefficient of x to zero:

$$(4+\alpha)\,r^4 + 52r^3 - 267r^2\alpha - 2548r\alpha - 2548r - 7203\alpha - 9604 = 0.$$

Factoring gives

$$\left(r^2+13r+49\right)\left(\alpha r^2-13\alpha r+4r^2-147\alpha-196\right).$$

Using (8.15) we see the coefficient of x is indeed zero.

Similarly, we set the coefficient of x^2 to zero:

$$\left(\frac{\alpha^2}{2}+\frac{7\alpha}{2}+6\right)r^4+58r^3+267r^2\left(\frac{\alpha^2}{2}-\frac{\alpha}{2}\right)+268r^2+1274r\left(2\alpha^2-2\alpha\right)$$
$$+5096r\alpha+2842r+\frac{21609\alpha^2}{2}+\frac{36015\alpha}{2}+14406=0.$$

We have a quartic in r and since by (8.15) $r^2=-13r-49$ we can rewrite

$$\begin{aligned}r^2&=-13r-49\\ r^3&=rr^2=r(-13r-49)\\ r^4&=(-13r-49)^2.\end{aligned}$$

Substituting for r^4, r^3, r^2 and collecting terms leads to a quadratic in terms of α where

$$\left(\frac{1}{2}\left(-13r-49\right)^2+\frac{1625r}{2}+4263\right)\alpha^2+\left(\frac{7}{2}\left(-13r-49\right)^2+\frac{8567r}{2}+24549\right)\alpha$$
$$+6\left(-13r-49\right)^2+58r\left(-13r-49\right)-642r+1274=0. \tag{8.17}$$

Substituting the complex conjugate roots (8.16) into (8.17) and factoring gives us

$$\frac{3}{2}\left(71\pm 39i\sqrt{3}\right)\left(3\alpha+5\right)\left(3\alpha+4\right)=0.$$

Solving for α we obtain

$$\alpha_1=-\frac{5}{3}\qquad\text{and}\qquad\alpha_2=-\frac{4}{3}. \tag{8.18}$$

8.5.2 Numerical investigation

Now $u_7(n)$ is a liner combination of the terms of the form

$$u_7(n)\sim n^\alpha r^n\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\cdots\right). \tag{8.19}$$

We used Kauers' software program for Mathematica [42] to find the possibilities for (8.19) and to compute further terms. This program uses the Birkhoff–Trjitzinsky method. We used the command **Asymptotics** which takes the recurrence relation as the input and returns the dominant term and all its asymptotic solutions. We have also computed the first correction term.

$$n^{-\frac{5}{3}}\left(\frac{1}{2}(-13-3i\sqrt{3})\right)^n\left(1+\frac{5(-10359i+2579\sqrt{3})}{162(-71i+39\sqrt{3})n}+\frac{4(-125526767i+56300217\sqrt{3})}{15309(-239i+2769\sqrt{3})n^2}\right),$$

$$n^{-\frac{4}{3}}\left(\frac{1}{2}(-13-3i\sqrt{3})\right)^n\left(1+\frac{14(-1413i+409\sqrt{3})}{81(-71i+39\sqrt{3})n}+\frac{4(-52615987i+32152017\sqrt{3})}{15309(-239i+2769\sqrt{3})n^2}\right),$$

$$n^{-\frac{5}{3}}\left(\frac{1}{2}(-13+3i\sqrt{3})\right)^n\left(1+\frac{5(10359i+2579\sqrt{3})}{162(71i+39\sqrt{3})n}+\frac{4(125526767i+56300217\sqrt{3})}{15309(239i+2769\sqrt{3})n^2}\right),$$

$$n^{-\frac{4}{3}}\left(\frac{1}{2}(-13+3i\sqrt{3})\right)^n\left(1+\frac{14(1413i+409\sqrt{3})}{81(71i+39\sqrt{3})n}+\frac{4(52615987i+32152017\sqrt{3})}{15309(239i+2769\sqrt{3})n^2}\right).$$

The numerical results are in agreement with the previous subsection. The terms $n^{-5/3}$ and $n^{-4/3}$ are in agreement with α_1 and α_2 in (8.18). The term $\frac{1}{2}(-13\pm 3i\sqrt{3})$ is the value for r in (8.16) and is responsible for the sign changes in the sequence $\{u_7(n)\}$.

8.5.3 Determining the constant C

We use the ansatz (8.4) again to try to determine the constant C. However, the terms of the sequence alternate in sign erratically

$$\{1, -7, 42, -231, 1155, -4998, 15827, -791, -566244, 6506955, -53524611, 369879930, -2218053747, 11306008875, -43772711220, 55203364377, 172838094533, -16542312772356, \ldots\}$$

and involve complex conjugates. We know α and r from using Kauer's software but determining C is difficult. We are unable to use the method of difference operators that we used in the three-term quadratic recurrence relation in (6.5) since the asymptotic expansion involves fractional powers. So we tried a graphical approach using roots r^n where

$$r_1, r_2 = \frac{1}{2}(-13 \pm 3i\sqrt{3}).$$

Polar form gives us

$$r = 7e^{\pm i\alpha}$$

where

$$\alpha = \arctan\left(\frac{3\sqrt{3}}{13}\right) \pm \pi.$$

Both roots occur in the asymptotic formula where the dominant term is

$$u_n \sim \frac{c_1}{n^{\frac{4}{3}}} 7^n e^{ni\alpha} + \frac{c_2}{n^{\frac{4}{3}}} 7^n e^{-ni\alpha}. \tag{8.20}$$

We must have c_1 and c_2 as complex conjugates to produce real values so

$$c_1 = ce^{i\beta} \quad \text{and} \quad c_2 = ce^{-i\beta}.$$

We want to remove the growth factors for our graphical analysis so we multiply (8.20) by $n^{\frac{4}{3}}/7^n$ to give

$$\frac{n^{\frac{4}{3}} u_n}{7^n} \sim c\left(e^{ni(\alpha+i\beta)} + e^{-ni(\alpha-i\beta)}\right).$$

Using the trigonometric identity

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

we obtain

$$\frac{n^{\frac{4}{3}} u_n}{7^n} \sim A\cos(n(\alpha+\beta))$$

where $A = 2c$.

Using Maple we plotted the terms for n from 900 to 910, scaled by $n^{\alpha} r^n$ to remove the growth factor. In figure 8.1, n is represented by dots. We want to find the best curve to fit these points to give us a rough estimate for the amplitude A and the phase β. We were able to use the method of least squares to find the values of A and β that minimizes the expression

$$f = \sum_{n=900}^{950} \left(A\cos\left(n\left(\arctan\left(\frac{3\sqrt{3}}{13}\right) - \pi\right) + \beta\right) - n^{\frac{4}{3}} 7^{-n} u_7(n)\right)^2.$$

The result is $A = 6.502807770$ and $\beta = -1.083913253$. Our best estimate for the asymptotics for the sequence $\{u_7(n)\}$ is then

$$u_n \sim 6.502807770 n^{-\frac{4}{3}} 7^n \cos\left(n\left(\alpha - 1.083913253\right)\right).$$

The amplitude A represents the constant C by the graphical approach. We were unable to obtain an analytical proof for the constant C.

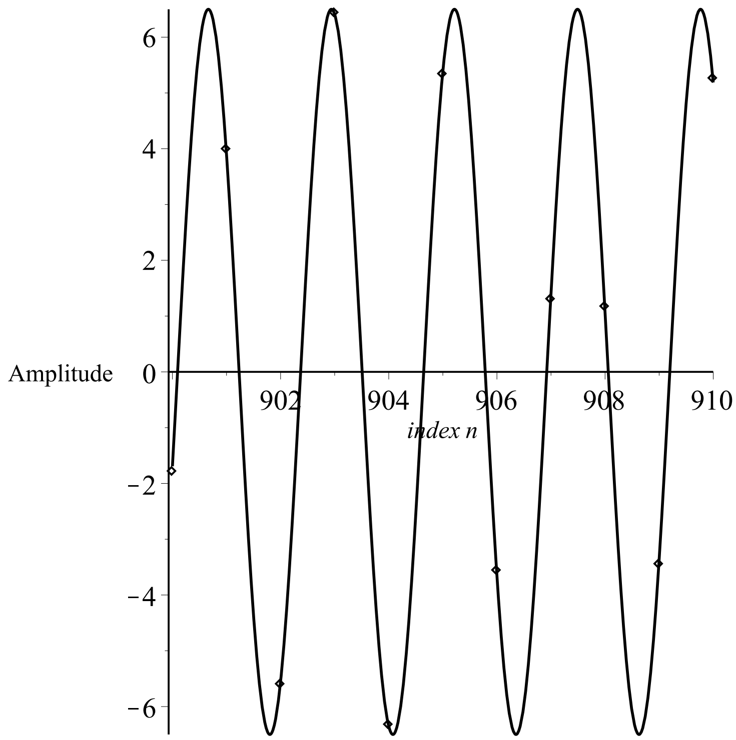


Figure 8.1: Determination of Amplitude A and phase shift β

8.6 Some references to relevant asymptotic methods in the literature

An excellent guide to methods of asymptotics is given by Odlyzko [50]. Of particular interest is section 9 where he discusses some limitations of the Birkhoff–Trjitzinsky method. In another paper Wong and Li [63] discuss the lack of knowledge of the asymptotic theory of second-order linear difference equations. They give reasons for this, one being a criticism of Birkhoff's work as being long and complicated. They also give references to works that make this method more accessible including their own work. However, Wimp and Zeilberger [61] give an extensive coverage of the Birkhoff–Trjitzinsky method with a number of useful examples including the asymptotics for Apéry's sequence. Zeilberger has developed a software package for Maple that calculates asymptotics [68]. Kauers also developed a software package but this time for Mathematica that computes the asymptotic expansion of solutions for recurrence equations [42]. Apéry's work on the irrationality of $\zeta(3)$ is listed as an application when Kooman and Tijdeman [45] discuss Kooman's thesis "The survey of convergence properties of linear recurrence sequences".

Chapter 9

Congruences

We look at congruences associated with modular forms of level 7. First, in 1878 Édouard Lucas [48] showed a way to express binomial coefficients $\binom{m}{n}$ modulo a prime in terms of the base p expansion of the integers m and n where p is prime. This is know as Lucas' Congruence.

Theorem 9.1. *If*

$$m = m_0 + m_1 p + \cdots + m_r p^r \pmod{p}$$

and

$$n = n_0 + n_1 p + \cdots + n_r p^r \pmod{p}$$

are the base p expansions of m and n, where p is a prime, then

$$\binom{m}{n} \equiv \binom{m_0}{n_0}\binom{m_1}{n_1}\cdots\binom{m_r}{n_r} \pmod{p}.$$

Proof. Proofs can be found in [31, p. 271] and [48]. □

A similar congruence for the sequence $\{t_7(n)\}$ in Equation (6.12) was given by Malik and Straub [49]. They showed that

$$t(n) \equiv t(n_0)t(n_1)\ldots t(n_r) \pmod{p} \tag{9.1}$$

where

$$n = n_0 + n_1 p + \cdots + n_r p^r \pmod{p}$$

is the expansion of n base p.
A congruence of the form (9.1) will be said to be a Lucas-type congruence.

We consider an example. In base 7

$$89 = 1 \times 7^2 + 5 \times 7^1 + 5 \times 7^0 = 155_7.$$

So the number 89 has a base 7 expansion of 155. Hence

$$t(89) - t(1)t(5)t(5) \equiv 0 \pmod 7.$$

where

$$\begin{aligned} t(89) &= 0 \pmod 7 \\ t(1) &= 4 \pmod 7 \\ t(5) &= 0 \pmod 7 \end{aligned}$$

hence

$$0 - 4 \times 0 \times 0 \equiv 0 \pmod 7.$$

We considered the sequence $\{c_7(n)\}$ in (6.5) to ascertain whether it satisfies a Lucas-type congruence like (9.1). We make the following conjecture:

Conjecture 9.2. *Let* $\{c_7(n)\}$ *defined by*

$$(n+1)^2 c_7(n+1) = (26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1) \tag{9.2}$$

with initial conditions

$$c_7(-1) = 0, \quad c_7(0) = 1$$

be a sequence of integers. Let $n = n_0 + n_1 p + \cdots + n_r p^r \pmod p$ *be the base* p *expansion of* n. *Then*

$$c(n) \equiv c(n_0)c(n_1)\ldots c(n_r) \pmod p$$

for all integers n *if and only if* $p = 0$ *or the primes* p *are congruent to* $1, 2$ *or* 4 *modulo 7.*

As evidence for this claim, the primes up to 200 were checked numerically. Those that showed evidence of Lucas-type congruence were checked for all integers n up to $2,000$ and the congruence was found to hold in all cases. The congruence was found not to hold for any of the other primes.

The consequences of Conjecture (9.2) are twofold. First, the work of Malik and Straub [49] shows that for certain sequences, where the coefficients are a binomial sum, Lucas-type congruences are satisfied for all primes. We do not know of a formula for Equation (9.2) as the sum of binomial coefficients so the technique used by Malik and Straub [49] is not available to prove this example. Secondly, the sequences studied by Malik and Straub [49] have Lucas-type congruences for every prime whereas the Equation (9.2) satisfies Lucas-type congruences for only certain primes so this may point to a reason we have not been able to find a binomial sum for the Equation (9.2), indeed there might not be one.

A second type of congruence was conjectured by Chan, Cooper and Sica in [19] where the coefficients of certain power series provide numbers that satisfy congruences modulo certain powers of primes. They proposed a number of conjectures one of which is related to Equation (6.6). Using z_7 defined as (3.13) and x_7 that we define as (3.15) with identity (3.16) they deduced (6.6). They conjectured the following congruence on the coefficients of (6.6).

Conjecture 9.3. *Let* $\{c_7(n)\}$ *be defined in Equation* (6.5)*, then*

$$c_7(np) \equiv c_7(n) \pmod{p^2}$$

if and only if $p = 0$ *or the primes* p *are congruent to* $1, 2$ *or* 4 *modulo 7.*

We notice in both the conjectures that the primes p that are congruent to $1, 2$ or 4, modulo 7 are the squares, that is the quadratic residues, modulo 7.

We looked at the sequence $\{u_7(n)\}$ (6.16) where the primes up to 200 were checked numerically and none had a Lucas-type congruence of the form (9.1).

We conclude this section by saying Conjectures 9.2 and 9.3 are topics for further investigation.

Chapter 10

Conclusions and further work

10.1 Conclusion

This thesis examined two new integer sequences at level 7. The background needed to establish these results is given in Chapters 2–5. We began with an historical overview of theta functions where we looked at the earliest known use of theta functions by Bernoulli. This was followed by the theta functions that would be needed in this thesis including Ramanujan's Eisenstein series and theta functions, Dedekind's eta function and the Borweins' theta functions. A number of identities were used such as Euler's product identity and Jacobi's triple product then a brief description of the classification of modular forms for weight and level was given. Following this background we gave definitions for cubic and septic theta functions and some of their properties which led on to their derivatives and finally to differential equations and proofs. In Chapter 6 we gave the main results and proofs and solved differential equations to obtain power series which are holonomic functions. That is they are solutions of linear differential equations where coefficients of these generating functions satisfy linear recurrence relations of polynomials. It was a surprise that the recurrence relation produced two integer sequences $\{c_7(n)\}$ and $\{u_7(n)\}$. In the next three chapters we examined some of the properties of these sequences. We noted that our new sequence $\{c_7(n)\}$ and S. Cooper's sequence $\{t_7(n)\}$ are related by Clausen's identity. We looked at the asymptotic behaviour of each sequence using the Birkhoff–Trjitzinsky method employing analytical and numerical processes. Finally, we looked at congruences and conjectured that the sequence $\{c_7(n)\}$ satisfied a Lucas congruence for primes p that are congruent to 0,1,2 or 4, modulo 7.

10.2 Further work

A number of questions arose in the course of our research. We will restate these together as a summary in one place. The first question that remains unanswered was raised in Chapter 6.

Question 10.1. *Is there an explicit formula as sums of binomial coefficients for $c_7(n)$ similar to the binomial sum for $t_7(n)$ found by Zudilin?*

The second question that arose was part of our asymptotic investigation of $\{c_7(n)\}$ in chapter 7.

Question 10.2. *Is there a nice explicit value for the constant C in terms of π or other known mathematical constants, similar to the constant for $t_7(n)$ found by Hirschhorn.*

The next question we have regards Clausen's analogue. We saw in the classical case for level 4 and at level 3 there is a general formula for Clausen's identity that contains additional parameters.

Question 10.3. *We have shown at level 7 that two series z_7 and Z_7 are related by a Clausen-type analogue but question is: Can we generalize this to include a parameter?*

In chapter 9 we gave two conjectures with regard to congruences which have yet to be proven. These are

Conjecture 10.4. *Let $\{c_7(n)\}$ defined by*

$$(n+1)^2 c_7(n+1) = (26n^2 + 13n + 2)c_7(n) + 3(3n-1)(3n-2)c_7(n-1)$$

with initial conditions

$$c_7(-1) = 0, \quad c_7(0) = 1$$

be a sequence of integers. Then $\{c_7(n)\}$ satisfies a Lucas-type congruence for all integers n if and only if $p = 0$ or the prime p is congruent to $1, 2$ or 4 modulo 7.

and

Conjecture 10.5. *Let $\{c_7(n)\}$ be defined in Equation (6.5), then*

$$c_{np} \equiv c_n \pmod{p^2}$$

if and only if $p = 0$ or the prime p is congruent to $1, 2$ or 4 modulo 7.

REFERENCES

[1] Andrews, G. E., Askey, R., Roy, R. (1999). Special Functions. Cambridge University Press, Cambridge.

[2] Andrews, G. E., Berndt, B. C. (2005). Ramanujan's Lost Notebook, Part 1, Springer-Verlag, New York.

[3] Apostol, T. M. (1969). Calculus, Vol. II. John Wiley and Sons, New York.

[4] Apostol, T. M. (2012). Modular Functions and Dirichlet Series in Number Theory. Springer-Verlag, New York.

[5] Apostol, T. M. (2013). Introduction to Analytic Number Theory. Springer-Verlag, New York.

[6] Berndt, B. C. (1991). Ramanujan's Notebooks, Part III. Springer-Verlag, New York.

[7] Berndt, B. C. (1992). On a certain theta-function in a letter of Ramanujan from Fitzroy House. Ganita, 43(1–2), 33–43.

[8] Berndt, B. C. (1998). Ramanujan's Notebooks, Part V. Springer-Verlag, New York.

[9] Berndt, B. C. (2006). Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence.

[10] Berndt, B. C., Bhargava, S., Garvan, F. G. (1995). Ramanujan's theories of elliptic functions to alternative bases. Transactions of the American Mathematical Society, 347, 4163–4244.

[11] Berndt, B. C., Chan, H. H., Sohn, J., Son, S. H. (2000). Eisenstein series in Ramanujan's lost notebook. The Ramanujan Journal, 4(1), 81–114.

[12] Bernoulli, J. (1713). Ars conjectandi. Impensis Thurnisiorum, fratrum, Basel.

[13] Borwein, J. M., Borwein, P. B. (1991). A cubic counterpart of Jacobi's identity and the AGM. Transactions of the American Mathematical Society, 323(2), 691–701.

[14] Borwein, J. M., Borwein, P. B., Bailey, D. H. (1989). Ramanujan, modular equations, and approximations to pi or how to compute one billion digits of pi. The American Mathematical Monthly, 96(3), 201–219.

[15] Borwein, J. M., Borwein, P. B., Garvan, F. G. (1994). Some cubic modular identities of Ramanujan. Transactions of the American Mathematical Society,343, 35–47.

[16] Chan, H. H. (1995). On Ramanujan's cubic continued fraction. Acta Arithmetica, 73, 343–355.

[17] Chan, H. H., Cooper, S. (2008). Eisenstein series and theta functions to the septic base. Journal of Number Theory, 128(3), 680–699.

[18] Chan, H. H., Cooper, S. (2012). Rational analogues of Ramanujan's series for $1/\pi$. In Mathematical Proceedings of the Cambridge Philosophical Society, 153(2), 361–383.

[19] Chan, H. H., Cooper, S., Sica, F. (2010). Congruences satisfied by Apéry-like numbers. International Journal of Number Theory, 6(01), 89–97.

[20] Chan, H. H., Liu, Z. G. (2006). Elliptic functions to the quintic base. Pacific Journal of Mathematics, 226(1), 53–64.

[21] Chan, H. H., Ong, Y. L. (1999). On Eisenstein Series and $\sum_{m,n}^{\infty} q^{m^2+mn+2n^2}$. Proceedings of the American Mathematical Society, 127(6), 1735–1744.

[22] Cooper, S. (2009). Inversion formulas for elliptic functions. Proceedings of the London Mathematical Society, 99, 461–483.

[23] Cooper, S. (2009). On Ramanujan's function $k(q) = r(q)r^2(q^2)$. The Ramanujan Journal, 20(3), 311–328.

[24] Cooper, S. (2012). Sporadic sequences, modular forms and new series for $1/\pi$. The Ramanujan Journal, 29(1-3), 163–183.

[25] Cooper, S. (2016). Ramanujan's theta functions. Preprint

[26] Cooper, S., Lam, H. Y. (2011). Eisenstein series and elliptic functions on $\Gamma_0(10)$. Advances in Applied Mathematics, 46(1), 192–208.

[27] Cooper, S., Toh, P. C. (2009). Quintic and septic Eisenstein series. The Ramanujan Journal, 19(2), 163–181.

[28] Cooper, S., Ye, D. (2015). Level 14 and 15 analogues of Ramanujan's elliptic functions to alternative bases. Transactions of the American Mathematical Society, DOI: http://dx.doi.org/10.1090/tran6658.

[29] Dedekind, R. (1877). Schreiben an Herrn Borchardt über die Theorie der elliptischen Modul-Functionen. Journal für die Reine und Angewandte Mathematik, 83, 265–292.

[30] Diamond, F., Shurman, J. (2006). A first course in modular forms (Vol. 228). Springer-Verlag, New York.

[31] Dickson, L. E. (1952). History of the theory of numbers, vol. I. Chelsea, New York.

[32] Franklin, F. (1881). Sur le développement du produit infini $(1-x)(1-x^2)(1-x^3)$. Comptes Rendus, 82, 448–450.

[33] Gessel, I. (1982). Some congruences for Apéry numbers. Journal of Number Theory, 14(3), 362–368.

[34] Hald, A. (2003). A History of Probability and Statistics and their Applications before 1750 (Vol. 501). John Wiley and Sons, New Jersey.

[35] Hardy, G. H., Wright, E. M. (1979). An introduction to the theory of numbers. Oxford University Press, New York.

[36] Hirschhorn, M. D. (1987). A simple proof of Jacobi's four-square theorem. Proceedings of the American Mathematical Society, 101(3), 436–438.

[37] Hirschhorn, M. D. (2012). Estimating the Apéry numbers. Fibonacci Quarterly, 50, 129–131.

[38] Hirschhorn, M. D. (2013). A sequence considered by Shaun Cooper. New Zealand Journal of Mathematics, 43, 37–42.

[39] Hirschhorn, M. D. (2013) Estimating the Apéry numbers II. Fibonacci Quarterly, 51(3), 215–217.

[40] Hirschhorn, M. D., Garvan, F. G., Borwein, J. M. (1993). Cubic analogues of the Jacobian theta function (z; q). Canadian Journal of Mathematics, 45(4), 673–694.

[41] Jacobi, C. G. J. (1881). Gesammelte Werke, Bande 1–8. Kgl. Preuss. Akad. Wiss., Berlin.

[42] Kauers, M. (2011). A Mathematica package for computing asymptotic expansions of solutions of P-finite recurrence equations. Technical Report Research Institute for Symbolic Computing 11–04, Johannes Kepler University Linz. `http://www.risc.jku.at/research/combinat/software/`(webpage accessed January 2016).

[43] Kilford, L. J. P. (2008). Modular Forms: A Classical and Computational Introduction. Imperial College Press, London.

[44] Koblitz, N. I. (2012). Introduction to elliptic curves and modular forms (Vol. 97). Springer-Verlag, New York.

[45] Kooman, R. J., Tijdeman, R. (1990). Convergence properties of linear recurrence sequences. Nieuw Arch. Wisk., **8**, 13–25.

[46] Krattenthaler, C., Müller, T. W. (2015). Generalised Apéry numbers modulo 9. Journal of Number Theory, 147, 708–720.

[47] Liu, Z. G. (2003). Some Eisenstein series identities related to modular equations of the seventh order. Pacific Journal of Mathematics, 209(1), 103–130.

[48] Lucas, É. (1878). Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques suivant un module premier. Bulletin de la Société Mathématique de France, 6, 49–54.

[49] Malik, A., Straub, A. (2016). Divisibility properties of sporadic Apéry-like numbers. Research in Number Theory, 2(1), 1–26.

[50] Odlyzko, A. M. (1995). Asymptotic enumeration methods. Handbook of Combinatorics, 2, 1063–1229. Elsiver, Amsterdam.

[51] Ono, K. (2004). The web of modularity: arithmetic of the coefficients of modular forms and q-series (No. 102). American Mathematical Soc..

[52] Ramanujan, S. (1914). Modular equations and approximations to π. Quart. J. Math, 45, 350–372.

[53] Ramanujan, S. (1916). On certain arithmetical functions. Trans. Cambridge Philos. Soc, 22(9), 159–184.

[54] Ramanujan, S. (1957). Notebooks (2 volumes). Tata Institute of Fundamental Research, Bombay.

[55] Sawyer, W. W. (1955). Prelude to mathematics. Pelican Books, London.

[56] Serre, J. P. (1973). A Course in Arithmetic. Springer-Verlag, New York.

[57] Sloane, N. J. (2007). The on-line encyclopedia of integer sequences, `https://oeis.org/`(webpage accessed July 2016).

[58] Van der Poorten, A. (1979). A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$. The Mathematical Intelligencer, 1(4), 195–203.

[59] Venkatachaliengar, K. (2012). Development of Elliptic Functions According to Ramanujan. Edited and revised by S. Cooper. World Scientific Publishing Company, Singapore.

[60] Whittaker, E. T., Watson, G. N. (1996). A course of modern analysis. Cambridge University Press, Cambridge.

[61] Wimp, J., Zeilberger, D. (1985). Resurrecting the asymptotics of linear recurrences. Journal of Mathematical Analysis and Applications, 111(1), 162–176.

[62] Wolfram Alpha, `https://www.wolframalpha.com/examples/Math.html` (webpage accessed July 2016).

[63] Wong, R., Li, H. (1992). Asymptotic expansions for second-order linear difference equations. Journal of Computational and Applied Mathematics, 41(1), 65–94.

[64] Zagier, D. (1992). Introduction to modular forms. In: From Number Theory to Physics 238–291. Springer-Verlag, Berlin.

[65] Zagier, D. (2008). Elliptic modular forms and their application, In: The 1-2-3 of Modular Forms, 1–123. Springer-Verlag, Berlin.

[66] Zagier, D. (2009). Integral solutions of Apéry-like recurrence equations. Groups and Symmetries: from Neolithic Scots to John McKay, CRM Proc. Lecture Notes, Am. Math. Soc., Providence, 47, 349–366.

[67] Zagier, D. (2015). Don Zagier Lecture: Day 1. `https://www.youtube.com/watch?v=zKt5L0ggZ3o`. (webpage accessed July 2016)

[68] Zeilberger, D. (2008). Asyrec: A Maple package for computing the asymptotics of solutions of linear recurrence equations with polynomial coefficients. The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger. `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/asy.html` (webpage accessed March 2016).

[69] Zill, D. G., Cullen, M. R. (2005). Differential Equations with Boundary-Value Problems. Brooks/Cole, London.