The area of the symmetric representation of σ(n), i.e. the area between the (n-1)-st and the n-th Dyck paths, equals σ(n).

Hartmut F. W. Höft, 2018-01-30

Notations, Definitions and Equations

$n = 2^m \times q$, $m \geq 0$, q odd.

$r = \text{row}[n] = \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor$

s(n) = area of the symmetric representation of σ(n).

For $1 \leq i \leq r$ and $r < j \leq n$:

$\text{A235791}(n,i) = a(i) = \left\lceil \frac{n+1}{i} - \frac{i+1}{2} \right\rceil$ and $a(r+1) = 0$,

$\text{A237591}(n,i) = a(i) - a(i+1)$,

$a(i) = \frac{n}{i} - \frac{i+1}{2} + 1 = 2^m \times \frac{q}{i} - \frac{i+1}{2} + 1$ for $i|n$, $i \leq r$ and i odd,

$a(\frac{2 \times n}{j}) = a(2^{m+1} \times i) = \frac{j-1}{2} - 2^m \times \frac{q}{j} + 1$ for $j|n$, $r < j$, j odd, $q = i \times j$, and $2^{m+1} \times i \leq r$.

$\text{A237048}(n,i) = b(i) = \begin{cases} 1 & \text{if } n - \frac{i}{2}(i+1) \equiv 0 \bmod i \\ 0 & \text{otherwise} \end{cases}$,

in other words, $b(i) = 1$ if $i \mid q$ and $b(2^{m+1} \times i) = 1$ if $i \times j = q$, and 0 otherwise.

$\text{A249223}(n,i) = c(i) = \sum_{j=1}^{i} (-1)^{(j+1)} \times b(j)$

Note that c(r) is the difference between the odd divisors d of n with $d \leq r$, and the odd divisors d of n with $d > r$.

Proof

The area s(n) is twice the sum of the products of the length of the legs and their width accumulated to the diagonal of the symmetric representation of σ(n) subtracting once the width at the diagonal, i.e.,

$s(n) = 2 \times \sum_{i=1}^{r} \text{A237591}(n,i) \times \text{A249223}(n,i) - \text{A249223}(n,r)$

$= 2 \times \sum_{i=1}^{r} a(i) \times c(i) - c(r)$

$= 2 \times \sum_{i=1}^{r} \left((a(i) - a(i+1)) \times \left(\sum_{j=1}^{i} \left((-1)^{j+1} \times b(j)\right)\right) - c(r)\right.$

Simplifying the nested alternating sums reduces to

$= 2 \times \sum_{i=1}^{r} \left((-1)^{i+1} \times a(i) \times b(i)\right) - a(r+1) \times \sum_{j=1}^{r} \left((-1)^{j+1} b(j)\right) - c(r)$

and using the equations above, the middle term vanishes and the first term simplifies to

$= 2 \times \sum_{i|q \,\&\, i \leq r} a(i) - 2 \times \sum_{j|q \,\&\, j > r} a(j) - c(r)$

$= 2 \times \sum_{i|q \,\&\, i \leq r} \left(2^m \times \frac{q}{i} - \frac{i+1}{2} + 1\right) - 2 \times \sum_{j|q \,\&\, j > r} \left(\frac{j-1}{2} - 2^m \times \frac{q}{j} + 1\right) - c(r)$

$= 2^{m+1} \times \sum_{i|q \,\&\, i \leq r} \frac{q}{i} + 2^{m+1} \times \sum_{j|q \,\&\, j > r} \frac{q}{j} - \sum_{i|q \,\&\, i \leq r} i - \sum_{j|q \,\&\, j > r} j$

$\quad + \sum_{i|q \,\&\, i \leq r} 1 - \sum_{j|q \,\&\, j > r} 1 - c(r)$

$= \left(2^{m+1} - 1\right) \times \rho(q) + c(r) - c(r)$

$= \sigma(n)$