A subpart of the area of the symmetric representation of σ(n) at the diagonal has size 1 exactly when n is a hexagonal number.

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Notations and Definitions

$n = 2^m \times q$, $m \geq 0$, q odd.

$r(n) = \left\lfloor \frac{1}{2}\left(\sqrt{8 \times n + 1} - 1\right)\right\rfloor$, the length of the nth row in the irregular triangles of A235791 and A237591.

$a(n, i) = \left\lceil \frac{n+1}{i} - \frac{i+1}{2} \right\rceil$, $1 \leq i \leq r(n)$, are the entries in the nth row of the irregular triangle in A235791;

$a(n, r(n) + 1) = 0$ by definition.

$b(n, i) = a(n, i) - a(n, i+1)$, $1 \leq i \leq r(n)$, are the entries in the nth row of the irregular triangle in A237591; they are the lengths of the legs of the Dyck path, the upper boundary of the symmetric representation of sigma(n), from the y-axis at (0,n) to the diagonal at (A240542(n), A240542(n)).

$h(n) = n \times (2 n - 1)$ = A000384(n) is the nth hexagonal number.

Statements

THEOREM 1

A subpart of size 1 in the symmetric representation of sigma(n) occurs only at the diagonal.

THEOREM 2 Equivalent are:

(1) n is a hexagonal number,

(2) the size of the smallest subpart in the symmetric representation of sigma(n) equals 1 and its only instance occurs at the diagonal.

Proofs

PROOF of THEOREM 1

Let $k \leq r(n)$ be the position of the subpart of size 1. Then k is odd and k|n since the width of the symmetric representation of sigma(n) increases by 1 at the kth leg. Furthermore, the length of that leg equals 1. Thus,

$b(n, k) = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil - \left\lceil \frac{n+1}{k+1} - \frac{k+2}{2} \right\rceil = 1$ so that $\frac{n}{k} = \left\lceil \frac{n+1}{k+1} - \frac{1}{2} \right\rceil$, i.e., $\frac{n-k}{k} < \frac{2n+2-k-1}{2 \times (k+1)} \leq \frac{n}{k}$.

The left inequality is $2 n < k^2 + 3 k$ which results in $\frac{1}{2} \times \sqrt{8 n + 9} - \frac{3}{2} < k$ so that

$r(n) - 1 \leq \frac{1}{2} \times \sqrt{8 \times n + 1} - \frac{3}{2} < \frac{1}{2} \times \sqrt{8 \times n + 9} - \frac{3}{2} \leq k \leq \left\lfloor \frac{1}{2} \times \sqrt{8 \times n + 1} - \frac{1}{2} \right\rfloor = r(n)$

implies that $k = r(n)$, proving the claim.

PROOF of THEOREM 2

(1) ⇒ (2):

First, r(h(n)) = 2 × n - 1 so that r(h(n)) | h(n) since

$$r(h(n) = \left\lfloor \frac{1}{2} \times \left(\sqrt{8 \times n \times (2 \times n - 1) + 1} - 1 \right) \right\rfloor = \left\lfloor \frac{1}{2} \times \left(\sqrt{(4 \times n - 1)^2} - 1 \right) \right\rfloor = 2 \times n - 1$$

Second, the length of leg r(h(n)) in the Dyck path ending at the diagonal equals 1 since

$$b(h(n), r(h(n))) = a(h(n), r(h(n))) = \left\lceil \frac{h(n)+1}{r(h(n))} - \frac{r(h(n))+1}{2} \right\rceil = \left\lceil n + \frac{1}{2n-1} - n \right\rceil = 1$$

Therefore, the width of the symmetric representation of sigma(h(n)) at the diagonal is increased by 1 and its top level is a subpart of size 1. By Theorem 1 this is the only subpart of size 1.

(2) ⇒ (1):

By hypothesis, the width of the symmetric representation of sigma(n) increases by 1 at the diagonal so that r(n) is an odd divisor of n. Furthermore, the length of the leg of the Dyck path at the diagonal is 1 since the subpart at the diagonal has size 1, so that

$$b(n,\ r(n)) = 1 = \left\lceil \frac{n+1}{r(n)} - \frac{r(n)+1}{2} \right\rceil = \frac{n}{r(n)} - \frac{r(n)+1}{2} + \left\lceil \frac{1}{r(n)} \right\rceil.$$

Therefore, 2 × n - r(n) × (r(n) + 1) = 0, and writing r(n) as odd number 2 × m - 1, m≥1, we get n = m × (2 × m - 1), i.e., n is a hexagonal number.