PROOF OF THE TETRAHEDRAL FAMILY DEFINED BY PASCAL'S TRIANGLE:

Leo J. Borcherding

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The function *f(k,n)* is a family of series all related to, and derived from the tetrahedral numbers (**OEIS:** A000292). They are all variations of tetrahedral shapes, some of which have been classified on the OEIS and other of which have not been classified.

Formally, *f(k, n)* is defined as:

f(k,n) is the set of all series derived from the anchored series.

k = (All whole numbers (including negative values))

n = (All whole numbers >= 1)

f(0,n) is the anchored series which generates all other series.

List of sequences in *f(k,n)* which have been added to OEIS and have been applied to geometry

f(-∞,n) = 1, -∞, f(-∞,2), f(-∞,3), f(-∞,4), f(-∞,5), f(-∞,6), f(-∞,7), ...

...

f(-1,n) = 1, -1, 5, -5, 15, -15, 35, -35, 70, -70 ...	NO NAME
f(0,n) = 1, 0, 4, 0, 10, 0, 35, ...	**A000292** (variant) **(ANCHOR)**
f(1,n) = 1, 1, 4, 4, 10, 10, 35, ...	**A000292** (variant)
f(2,n) = 1, 2, 5, 8, 14, 20, 30, ...	**A006918 → A000330** (variant)
f(3,n) = 1, 3, 7, 13, 22, 34, 50, ...	**A002623**
f(4,n) = 1, 4, 10, 20, 35, 56, 84, ...	**A000292** (original)
f(5,n) = 1, 5, 14, 30, 55, 91, 140, ...	**A000330**
f(6,n) = 1, 6, 19, 44, 85, 146, 231, ...	**A005900**
f(7,n) = 1, 7, 25, 63, 129, 231, 377, ...	**A001845**
f(8,n) = 1, 8, 32, 88, 192, 360, 608, ...	**A008412**
f(9,n) = 1, 9, 40, 120, 280, 552, 968, ...	**A287324** (link to paper)
f(10,n) = 1, 10, 49, 160, 400, 832, 1520, ...	NO NAME
f(11,n) = 1, 11, 59, 209, 560, 1232, 2352, ...	NO NAME

...

f(+∞,n) = 1, + ∞, f(+∞,2), f(+∞,3), f(+∞,4), f(+∞,5), f(+∞,6), f(+∞,7), ...

k is evaluated for all real numbers; therefore, the anchor series is defined as the <u>zeroth</u> series, and can be continued in the positive and negative directions

In order to generate each series, the anchor function is put through an $f(n) + f(n+1)$ process, this can also be interpreted as $f(n) + f(n-1)$, which is evaluated as follows:

a.) $f(1,n) + f(1,n-1) = f(2,n)$		b.) $f(2,n) + f(2,n-1) = f(3,n)$
a.)		b.)
1 + 0 = 1	→	1 + 0 = 1
1 + 1 = 2	→	2 + 1 = 3
4 + 1 = 5	→	5 + 2 = 7
4 + 4 = 8	→	8 + 5 = 13
10 + 4 = 14	→	14 + 8 = 22
10 + 10 = 20	→	20 + 14 = 34
20 + 10 = 30	→	30 + 20 = 50
20 + 20 = 40	→	40 + 30 = 70

... iterate infinitely many times.

This rule can be generalized to the entire family $f(k, n)$:

$f(-k,n) = f(-k-1,n) + f(-k-1,n-1)$ (-∞ end)

...

$f(-1,n) = f(-2,n) + f(-2,n-1)$

$f(0,n) = f(-1,n) + f(-1,n-1)$

$f(1,n) = f(0,n) + f(0,n-1)$

$f(2,n) = f(1,n) + f(1,n-1)$

$f(3,n) = f(2,n) + f(2,n-1)$

$f(4,n) = f(3,n) + f(3,n-1)$

...

$f(k,n) = f(k-1,n) + f(k-1,n-1)$ (+∞ end)

This statement is similar to Leonardo Bonacci's original form for the Fibonacci sequence: $U(k+2,n) = U(k+1,n) + U(k,n)$

Most, if not all, sequences of the family $f(k,n)$, for k<=(-1), and k>=10, have not been added to the OEIS and do not have names, as of the time of publishing.

Refer to the **Figure 1 in the reading below for values of f(k,n) for all values of k**

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Figure 1: Table/Matrix of Values for *f(k,n)*, k = (all whole numbers), n = (all whole >=1)

	n	1	2	3	4	5	6	7	8	9
k										
-∞	f (-∞, n)	1	-∞	f(-∞,3)	f(-∞,4)	f(-∞,5)	f(-∞,6)	f(-∞,7)	f(-∞,8)	f(-∞,9)
…	…									
-10	f (-10, n)	1	-10	59	-260	945	-2982	8435	-21848	52615
-9	f (-9, n)	1	-9	49	-201	685	-2037	5453	-13413	30767
-8	f (-8, n)	1	-8	40	-152	484	-1352	3416	-7960	17354
-7	f (-7, n)	1	-7	32	-112	332	-868	2064	-4544	9394
-6	f (-6, n)	1	-6	25	-80	220	-536	1196	-2480	4850
-5	f (-5, n)	1	-5	19	-55	140	-316	660	-1284	2370
-4	f (-4, n)	1	-4	14	-36	85	-176	344	-624	1086
-3	f (-3, n)	1	-3	10	-22	49	-91	168	-280	462
-2	f (-2, n)	1	-2	7	-12	27	-42	77	-112	182
-1	f (-1, n)	1	-1	5	-5	15	-15	35	-35	70
0	f (0, n)	1	0	4	0	10	0	20	0	35
1	f (1, n)	1	1	4	4	10	10	20	20	35
2	f (2, n)	1	2	5	8	14	20	30	40	55
3	f (3, n)	1	3	7	13	22	34	50	70	95
4	f (4, n)	1	4	10	20	35	56	84	120	165
5	f (5, n)	1	5	14	30	55	91	140	204	285
6	f (6, n)	1	6	19	44	85	146	231	344	489
7	f (7, n)	1	7	25	63	129	231	377	575	833
8	f (8, n)	1	8	32	88	192	360	608	952	1408
9	f (9, n)	1	9	40	120	280	552	968	1560	2360
10	f (10, n)	1	10	49	160	400	832	1520	2528	3920
…	…									
+∞	f (+∞, n)	1	+∞	f(+∞,3)	f(+∞,4)	f(+∞,5)	f(+∞,6)	f(+∞,7)	f(+∞,8)	f(+∞,9)

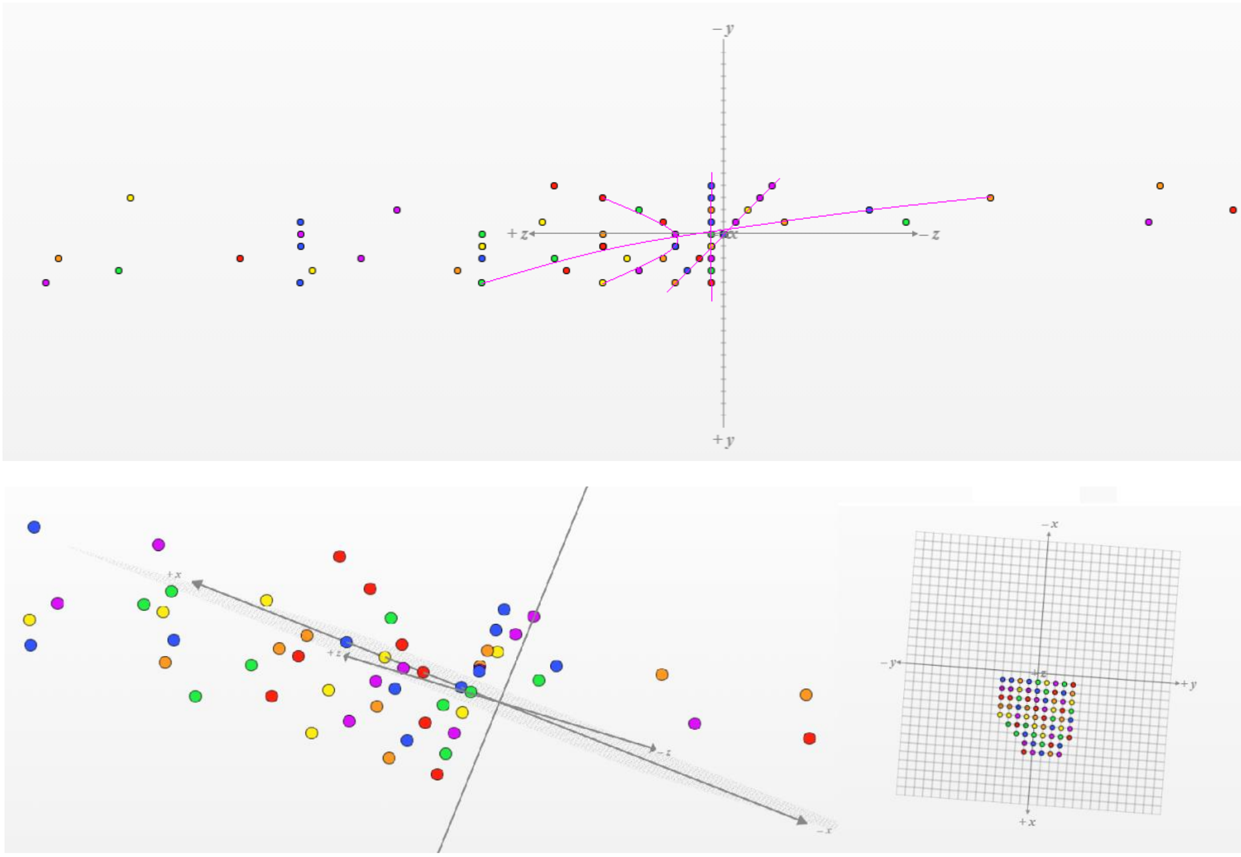
It is likely that the values of n in this table can be stretched past 0 out to -∞ through analytic continuation, the values of n should be n = (all whole numbers) rather than excluding values less than 0, as the term for all n=0, f(k,0) = 0. This can be seen in **Figure 2** below as the points plotted are limited to positive values of n.

Because of the relationship between f(0,n), f(1,n), and f(4,n) the series set is self-generated once the general rule is known, as f(0,n) will generate f(4,n) on the fourth iteration, but f(0,n) is generated at half the rate that f(4,n) is, so values from f(4,n) can be put back into f(0,n) to continually generate the entire family.

f (0, n) = 1, 0, 4, 0, 10, 0, 20, 0, 35, 0, 56, 0

f (4, n) = 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364

Figure 2: 3D Graphed points of f(k,n) of the form P(n, k, f(k,n)) → (*P(x, y, z)*)



Taking the data from **Figure 1**, the series can be graphed in 3 dimensions, revealing polynomial stripes which transverse in the varying *k* direction with constant *n*, which is opposite of that of the sequence table with constant *k* varying *n.*

By analytic continuation this graph can be stretched in the -n direction allowing insight on variables not shown in this study.

A good name for f(0,n) is the anchor of the set due to the fact that if this series is known, all other series can be derived, this is similar to the initial conditions needed to generate a given series.

This process can also be applied to other sequences using them as anchors to generate a sequence family for other existing series, this may lead to new discoveries in the relationship between varying sequences.

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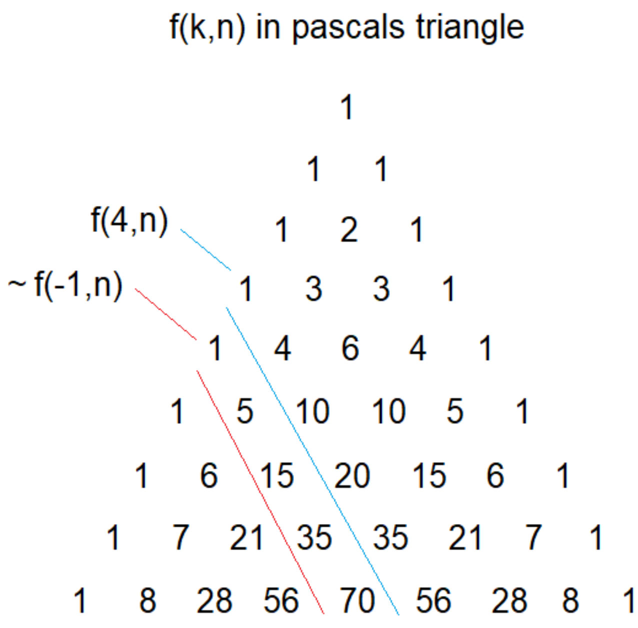
Figure 3: f(k,n)'s relation to pascals triangle.



The entire set has strong relationships to the Fibonacci sequence hence the progression of tetrahedrons as the process is iterated to create a new sequence, this creates a more complex shape based on tetrahedrons. (see f(4,n) through f(8,n) for a visualization of this complexity through stacked cannonballs).

- Leo James Borcherding

**When using the Fibonacci sequence as the anchor series, the resulting series is still the Fibonacci sequence due to the nature of how we define the Fibonacci numbers and their relationship to this family*