Karlsruhe
April 18 2019

Cycles of reduced Pell forms, general Pell equations and Pell graphs

Wolfdieter L a n g [1]

Abstract

Each unreduced *Pell* form is equivalent to its reduced principal form which generates a cycle of reduced forms. It is known that the general solution of the *Pell* form representing $+1$ is found from the trivial solution of this principal form. For the representation of nonzero integers not $+1$ the representative parallel primitive forms are instrumental. The equivalence relations between these forms can be depicted by *Pell* graphs. All proper solutions are found from the trivial solutions of these parallel forms. This memoir is based on the treatment given by *Scholz* and *Schoeneberg* [6].

1 Preliminaries and reduced Pell forms

For details on indefinite binary quadratic forms see the *Buell* [1] and *Scholz-Schoeneberg* [6] references, and also the author's paper [4] given as a link under A225953 where proofs are given of some of the later given statements.

A binary quadratic form is written as $F(\mathbf{A},\,\vec{x}) \,=\, \vec{x}^{\top}\,\mathbf{A}\,\vec{x} \,=\, a\,x^2 \,+\, b\,x\,y \,+\, c\,y^2$ with the matrix $\mathbf{A} \,=\, Matrix([[a,\,b/2],[b/2,\,c]])$ and the column vector $\vec{x} \,=\, (x, y)^{\top}$ ($\top$ for transposed). One also uses $F \,=\, [a,\,b,\,c]$ keeping $\vec{x}$ in mind. The discriminant of F is $Disc(F) \,=\, b^2 - 4\,a\,c$ which is $>\,0$ for the indefinite case.

A proper equivalence transformation (also called substitution) from a form $F(\mathbf{A},\,\vec{x})$ to a form $F(\mathbf{A}',\,\vec{x}')$ representing the same nonzero integer k is given by $\vec{x}' \,=\, \mathbf{M}\,\vec{x}$ and $\mathbf{A}' \,=\, \mathbf{M}^{-1,\top}\,\mathbf{A}\,\mathbf{M}^{-1}$, with a determinant $+1$ (unimodular) integer 2×2 matrix $\mathbf{M}$ (from $SL(2,\mathbb{Z})$). We also use $F' \,=\, [a',\,b',\,c']$ keeping $\vec{x}'$ in mind. We call this an M^{-1}-transformation from F to F' (not using a bold letter M). The invariance is $\vec{x}'^{\top}\,\mathbf{A}'\,\vec{x}' \,=\, \vec{x}^{\top}\,\mathbf{A}\,\vec{x} \,=\, k$.

In the following we consider *proper equivalence* $\mathbf{M}$ ($Det\,\mathbf{M} \,=\, +\mathbf{1}$), *primitive forms* $F \,=\, [a,\,b,\,c]$ ($\gcd(a,\,b,\,c) \,=\, 1$), and *proper solutions* $(x,\,y)^{\top}$ ($\gcd(x,\,y) \,=\, 1$). Forms with only nonpositive a, b and c are also not considered, because their representation problem is, modulo a sign change, given by treated forms.

A special equivalence transformation from $F \,=\, [a,\,b,\,c]$ to the so-called *half-reduced right neighbor form* $\widetilde{F}$ (see [6], p. 113) is given by $\widetilde{F} \,=\, [c,\,-b \,+\, 2\,c\,t,\,F(\mathbf{A},(-1,t)^{\top})]$ with $\mathbf{M}^{-1} \,=\, \mathbf{R} \,=\, \mathbf{R}(t) \,=\, Matrix([[0,-1],[1,t]])$. This form is made unique by setting $t \,=\, \left\lceil \frac{f(Disc(F)) + b}{2\,c} \,-\, 1\right\rceil$ if $c\,>\,0$ and $t \,=\, \left\lfloor 1 \,-\, \frac{f(Disc(F)) + b}{2\,|c|}\right\rfloor$ if $c\,<\,0$, where $f(Disc(F)) \,:=\, \left\lceil\sqrt{Disc(F)}\right\rceil$.

Each chain of such R-transformations of an indefinite binary quadratic (primitive) form F which is not yet reduced reaches its first reduced form FR after some number of steps (for the references for the definition see the following note). This reduced form gives rise to a cycle of (properly) equivalent forms by applying again R-transformations. It determines a sequence of parameters t for these $\mathbf{R}$ matrices, the tuple $\vec{t} \,=\, (t_1,\,t_2,\,...,\,t_P)$, where the (primitive) period P is even, $P \,=\, 2\,p$.

[1] wolfdieter.lang@partner.kit.edu, http://www.itp.kit.edu/~wl

A note on reduced principal forms $\mathbf{F_p}$ for Disc:

We use the definition of [1], p. 26, which is given explicitly in [4], Lemma 2, eq. (5) (with the formula for $b(D)$, with the discriminant $D \to Disc$). This *principal form* $F_p = [1, b(Disc), -(Disc - b^2(Disc))/4]$ is reduced. (This is in contrast to [6], p. 102, where non-reduced forms are called *Hauptform* for *Disc*.)

Proposition:

a) *The non-reduced Pell form* $F(n) = [1,\, 0,\, -D(n)]$ *of discriminant* $4\,D(n)$*, where* $D(n) =$ A000037(n) *(see Table 1), needs only two steps to reach the first reduced form*

$$FR(n) = [1,\, 2\,s(n),\, -(D(n) - s(n)^2)], \tag{1}$$

where $s(n) =$ A000194$(n) = D(n) - n$*, for* $n \geq 1$.

b) *This first reduced form* $FR(n)$ *coincides with the principal Form* $F_p(n)$ *for* $Disc(n) = 4\,D(n)$*, and with* $f(n) = \left\lceil 2\sqrt{D(n)} \right\rceil$,

$$s(n) = \frac{1}{2} \begin{cases} f(n) - 2\,, & \text{if } f(n) \text{ is even,} \\ f(n) - 1\,, & \text{if } f(n) \text{ is odd}\,. \end{cases} \tag{2}$$

Proof: a) The first R-transformation of $F(n)$ has parameter $t = s_1 = 0$ because $s_1 = \left\lceil 1 - \frac{\left\lceil 2\sqrt{D(n)} \right\rceil}{2\,D(n)} \right\rceil = 0$ (see the above given formula for t). This leads to the still unreduced form $F' = [-D(n),\, 0,\, 1]$.

The second transformation uses $t = s_2 = s(n) = \left\lceil \frac{\left\lceil 2\sqrt{D(n)} \right\rceil}{2} - 1 \right\rceil$. If one takes for $D(n)$ a (not allowed) square k^2, for $k \geq 1$, then $s_2 = k - 1$. The value of $s(n)$ will increase by 1 for the next allowed D value $D(n) = k^2 + 1$. This becomes clear after analyzing the two inequalities $k - 1 < \frac{\left\lceil 2\sqrt{k^2+1} \right\rceil}{2} - 1 < k$, *i.e.*, $2\,k < \left\lceil 2\sqrt{k^2+1} \right\rceil < 2\,(k+1)$. Hence $\left\lceil 2\sqrt{k^2+1} \right\rceil = 2\,k + 1$, and $s(n) = \lceil k - \frac{1}{2} \rceil = k$, for $D(n) = k^2 + 1$. From $D(1) = 2$ follows $s(1) = 1$, and this leads to $s(n) =$ A000194(n) (n appears n times). Because of the s jumps after the squares values of D this implies that $D(n) = n + s(n)$. The third entry of $FR(n)$ is obtained from the discriminant $4\,D(n)$.

b) Because $FR(n)$ is reduced and the first entry is 1 this is the principal form $F_p(n)$. Therefore, $b_p(n) = 2\,s(n)$ is also given by [4], eq.(5), (with $D \to 4\,D(n)$ and $f(D) \to f(n)$) and this proves the given alternative for $s(n)$ (because $Disc(n)$ is even). □

For the number of reduced primitive forms with discriminant $4\,D(n)$, with $D(n) =$ A000037(n), see $2\cdot$A307236(n).

2 The +1 Pell equation and cycles

For the *Pell* equation $x^2 - D(n)\,y^2 = +1$ the principal cycle $CR(n)$ of forms is obtained from the the reduced principal form $FR(n) = F_p(n)$. See *Table 1* for these cycles with R-transformations given by their $t-$tuples $\vec{t}(n)$. For these t-tuples see also the array A324251, and *Table 1*. The length of cycle $CR(n)$ is $LCR(n) = 2\cdot$A307372(n). The general (proper) solution $\vec{x}(n;j)$ for $j \in \mathbb{Z}$, with $x(n;j) > 0$ (obtainable always after an overall sign change of $\vec{x}(n;j)$) will be obtained with the help of the matrix $\mathbf{Auto}(n)$ for the automorphic equivalence transformation $\mathbf{Auto}(n) = \mathbf{R}(t_1(n)) \cdots \mathbf{R}(t_{2\,p(n)}(n))$.

Example: For $n = 5$, $D(5) = 7$ with the t-tuple $\vec{t}(5) = (-1,\, 1,\, -1,\, 4)$ this is

$$\mathbf{Auto}(5) = \begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}. \tag{3}$$

The solutions $\vec{x}(n;j)$ are obtained by using the matrix $\mathbf{B}(n) := \mathbf{R}(0)\,\mathbf{R}(s(n))$ needed to reach $FR(n)$ from the *Pell* form $F(n)$ (see the *Proposition*), and the fact that the first entry 1 of $FR(n)$ admits a trivial solution $\vec{x}_0 = (1,\,0)^\top$, by

$$\vec{x}(n;j) \,=\, \mathbf{B}(n)\,\mathbf{Auto}(n)^j\,\vec{x}_0\,, \quad \textit{for} \;\; j \in \mathbb{Z}\,. \tag{4}$$

Note that this formula does not always lead to solutions with positive $x(n;j)$ but by an overall sign change one ensures $x(n;j) > 0$.

There is only one j-family (family for short, also called class) of proper solutions, the so-called *ambiguous class*, meaning that there is only one trivial solution, namely $\vec{x} = (1,\,0)^\top$, and the solutions with negative $y(n;-|j|)$ are obtained from $(x(n;j),\,-y(n;j)$, for $j \geq 1$. See, *e.g.*, *Nagell* [3], pp. 195 - 200. (The name class used here is not related to the class number of binary quadratic forms, given in *Table 2* in column $h(n)$. This is why we prefer to use the notion family.)

In the example $n = 5$ from above $\mathbf{B}(5) := \mathbf{R}(0)\,\mathbf{R}(s(5) = 2) = -\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and the general proper solution of the *Pell* equation $x^2 - 7\,y^2 = +1$ is

$$\begin{pmatrix} x(j) \\ y(j) \end{pmatrix} = -\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}^j \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \textit{for} \;\; j \in \mathbb{Z}\,. \tag{5}$$

The power of the automorphic matrix can always be computed with the help of Chebyshev S polynomials, with their coefficients given in A049310.

In the example $S(n,\,x = 16)$ enters given in A077412.

$$\mathbf{Auto}(5)^j \,=\, \begin{pmatrix} S(j,\,16) - 14\,S(j-1,\,16) & 9\,S(j-1,\,16) \\ 3\,S(j-1,\,16) & S(j,\,16) \,-\, 2\,S(j-1,\,16) \end{pmatrix}, \tag{6}$$

for $j \in \mathbb{Z}$. The solutions are usually given with positive $x(j)$. Modulo the mentioned overall sign change these solutions are, for $j = -4,\,...,\,+4$,

$$\begin{pmatrix} 32257 \\ -12192 \end{pmatrix}\begin{pmatrix} 2024 \\ -765 \end{pmatrix}, \begin{pmatrix} 127 \\ -48 \end{pmatrix}\begin{pmatrix} 8 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 8 \\ 3 \end{pmatrix}, \begin{pmatrix} 127 \\ 48 \end{pmatrix}\begin{pmatrix} 2024 \\ 765 \end{pmatrix}, \begin{pmatrix} 32257 \\ 12192 \end{pmatrix}. \tag{7}$$

See A001081 and A001080 for $x(j)$ and $y(j)$, for $j \geq 0$, respectively.

For the forms involved in this example see also part of the *Pell* graph given in the *FIGURE* (erasing the links from $FPa1$ and $FPa2$ to F_p).

3 General Pell form representations and Pell graphs

The general solution of the *Pell* equation $x^2 - D(n)\,y^2 = k \neq +1$, with a nonzero integer k, is more involved.

For given k not all $D(n)$ admit solutions. *E.g.*, already $k = -1$ restricts n to be from A003814 $=$ 2, 5, 10, 13, 17, ... The problem is to find all fundamental solutions (hence the number of families) for representable k. They satisfy certain inequalities for positive and for negative k given in [3], *Theorems* 108 and 108a, respectively (k is called N there), based on the positive fundamental solution of the $+1$ Pell equation known from above for the power $j = 1$ (modulo overall sign change).

Here the so-called *parallel forms* come into play [6], p. 105, eq. (129), or [1], p. 49, f', without using the name parallel. We follow [6] but we use for a transformation matrix $\mathbf{M}$, as given above, which is the inverse of the matrix used there.

Definition: *Two equivalent forms F and F' are called parallel, with notation $F \parallel F'$, if $F = [a,\, b,\, c]$ and $F' = [a,\, b',\, c']$ with $b' \equiv b\,(mod\, 2\,a)$. c' is then determined by $Disc(F) = Disc(F')$.*

For a (primitive) form $F = [a,\, b,\, c]$ of discriminant $Disc(F)$ representing (properly) a nonzero integer k there exists a t-family of parallel forms $\{F''(t)\}_{t\in\mathbb{Z}}$ obtained in two steps from any proper solution $\vec{x}_1 = (x_1,\, y_1)^\top$ of $F = k$. In the first step the transformation from F to F' is accomplished by $\mathbf{M}^{-1} = Matrix([[x_1, v_1], [y_1, w_1]])$ with $Det\,\mathbf{M}^{-1} = x_1\, w_1 - v_1\, y_1 = +1$, because $\gcd(x_1,\, y_1) = 1$ implies the existence of such a pair $(v_1,\, w_1)$. This leads to $F' = [k,\, b',\, c']$ with some b' (in fact $b' = 2\,a\,v_1\,x_1 + b\,(w_1\,x_1 + v_1\,y_1) + 2\,c\,w_1\,y_1$), and c' is determined from $Disc(F)$. In the second step the transformation with $\mathbf{M}^{-1} = \mathbf{T}(t) = Matrix[[1, t], [0, 1]]$ is used. This leads to $F'' = [k,\, b'',\, c'']$ with $b'' = b' + 2\,k\,t$, and c'' compatible with $Disc(F)$. Hence $F' \parallel F''(t)$.

One chooses a representative of this residue class modulo $2\,|k|$ of parallel primitive forms (in the following named *rpapf*s) by fixing $t = t_0$ such that $b''(t_0) \in [0,\, 2\,|k|)$. This corresponds to the first sentence of *Theorem 73*, p. 104, of [6] (for general nonzero integer k). The second sentence is trivial: if a form F is equivalent to a form $\widehat{F}$ with first entry $a = k$ then k is represented by F because there is always the trivial solution $(1,\, 0)$ for $\widehat{F}$.

A program which computes all these representative parallel forms for given $Disc > 0$ (from A079896) and nonzero k can be given.

For even $Disc$ (later used for the *Pell* case) one has $Disc = 4\,D$. One searches all integer solutions of $c = \frac{j^2 - D}{k}$ for $j = 0,\, ...,\, |k| - 1$, because $b = 2\,j$ (from the $Disc$ formula). Thereby a set of bs is found. For odd $Disc$, *i.e.*, $Disc - 1 = 4\,D$ one has $c = \frac{j\,(j+1) - D}{k}$ for $j = 0,\, ...,\, |k| - 1$ because now $b - 1 = 2\,j$ (b is odd from the $Disc$ formula). Again all integer $c = c(j)$ qualify and this gives the representatives of the parallel forms for $Disc$ and k. The resulting forms may be imprimitive, which are not of interest, because above only primitive forms are used.

For the array of the number of *rpapf*s for discriminant $4\,D(n)$ and $|k| > 0$ see A307377, and also *Table 3* for the forms for $n = 1,\, 2,\, ...,\, 30$ and $k = 1,\, 2,\, ...,\, 10$.

Implication: From the definition and the program of representative parallel forms follows:

If a rpapf $F = [k,\, b,\, c]$ with $Disc(F)$ represents an integer $k \geq 1$ then $\widehat{F} = [-k,\, b,\, -c]$ with $Disc(\widehat{F}) = Disc(F)$ represents $-k$, and vice versa.

This sign flip of the outer entries (not the middle one) of any F will be called *outer sign flip* (an overall sign flip of F would trivially represent $-k$). Note that in general such an outer sign flip is not possible with an equivalence transformation, including improper ones. This is only possible (with a proper transformation) if both forms F and $\widehat{F}$ (the outer sign flipped one) appear in one equivalence class (*e.g.*, in the principal cycles with class number $h(n) = 1$, or if a cycle is invariant under outer sign flip, like, *e.g.*, for $n = 7$ with two such cycles, as shown in *Table 2*). Of course, a (not allowed) pure imaginary transformation could achieve this outer sign flip: $\mathbf{M}^{-1} = i\,Matrix([[1, 0], [0. -1]])$.

For the *Pell* eq. $x^2 - D(n)\,y^2 = k$, for $k \neq 0,\, +1$, the idea is to compute first the list of all representative parallel forms for $Disc(n) = 4\,D(n)$ and nonzero k, and discards the imprimitive ones.

E.g., all the representative parallel forms for $Disc(5) = 4\,D(5) = 28$ and $k = 9$ are given by $[9,\, 8,\, 1]$ and $[9,\, 10,\, 2]$. They are primitive and unreduced. Improper solutions of $x^2 - 7\,y^2 = 9$ exist also but are not of interest because they reduce to $X^2 - 7\,Y^2 = +1$ treated above.

Next one finds for each *rpapf* the first reduced form (see [6], Satz 79, p. 113, valid only for primitive forms; see p. 104 for the remark on *primitiv* and *eigentlich*).

A side remark: Note that *e.g.*, the imprimitive form $[2,\, 2,\, -2]$ is parallel for $D(3) = 5$ and $k = 2$ and unreduced. But by R-transformations no reduced form will be reached, because one finds the 2-cycle $[[2,\, 2,\, -2], [-2,\, 2,\, 2]]$ of unreduced forms.

In the example of *section 2*: $[9,\, 8,\, 1] \rightarrow [1,\, 4,\, -3] = FR(5) = F_p(5)$ with the $\mathbf{R}(6)$ matrix. Similarly, $[9,\, 10,\, 2] \rightarrow [2,\, 2,\, -3] \rightarrow [-3,\, 4,\, 1] \rightarrow [1,\, 4,\, -3] = FR(5) = F_p(5)$ with consecutive R-transformations with parameter tuple $\vec{t} = (3,\, -1,\, 4)$.

Not all such *rpapfs* need to end up, after R-transformations, in the principal cycle, called CR. This can happen if the class number $h(n)$ of discriminant $4\,D(n)$ listed in *Table 2* is not 1. There is then more than one cycle. *E.g.*, $n = 2$, $D(2) = 3$ has for $k = 2$ only one representative parallel form $[2,\, 2,\, -1]$ which is primitive and already reduced. It produces the 2-cycle $[[2,\, 2,\, -1],\, [-1,\, 2,\, 2]]$. This cycle is obtained from the principal one CR by outer sign flip, and it will therefore be called $\widehat{CR}(2)$ (see *Table 2* and *Table 3*, where this form appears in boldface because it is reduced, and it is underlined because it does not belong to the cycle $CR(2)$). Therefore, there is no solution to *Pell* equation $x^2 - 3\,y^2 = 2$ because the *Pell* form $[1,\, 0,\, -3]$ is equivalent to the principal form $FR(2) = F_p(2) = [1,\, 2,\, -2]$ of cycle $CR(2)$. This applies also for the $k = 3$ case with the one representative parallel non-reduced form $[3,\, 0,\, -1]$ which, after one step, ends up also in this non-principal cycle $\widehat{CR}(2)$.

In *Table 2* all cycles for $n = 1, 2, ..., 30$ are listed. The lengths of all cycles for discriminant $4\,D(n)$ sum up to $\Sigma L(n)$ which coincides with the number of reduced primitive forms for this discriminant given in $2\cdot$A307236(n).

Once we arrive by R-transformations from a *rpapf* with discriminant $4\,D(n)$, representing an nonzero integer $k \neq +1$, at $FR(n) = F_p$ (or any other reduced form from the principal cycle) we can give all proper solutions of $x^2 - D(n)\,y^2 = k$. If there are no such *rpapfs* there will be no solutions, and *vice versa*.

We name these *rpapfs* $FPaC(n, k; i)$, for $i \in \{1,\, 2\,...., paC(n, k)\}$ (indicating that they are connected to the principal cycle CR) and the solutions $\vec{x}(n, k; i, j)$ of $F(n)$. The case $i = 0$ can be added for the Pell $k = +1$ case from section 2: $\vec{x}(n, k; 0, j) = \vec{x}(n, k; j)$.

For example, for $D(5) = 7$ and $k = 5$ there are no such parallel forms, hence no solutions.

We start for each such parallel form $FPaC(n; k; i)$ with the trivial solution $(1,\, 0)^\top$. Then we follow the R-transformations to the principal cycle, and go from $F_p(n)$ to the *Pell* form $F(n)$ as done with the help of matric $B(n)$ above in part**1)**.

The above started example will make this clear: $n = 5$, $Disc(5) = 4\,D(5) = 28$, and $k = 9$ (only proper solutions are considered). The two *rpapfs* are $FPa(5, 9; 1) = [9,\, 8,\, 1]$ and $FPa(5, 9; 2) = [9,\, 10,\, 2]$ (we omitted the C in the notation). They are unreduced and lead to $FR(5)$ with $\vec{t}_1 = (6)$ and $\vec{t}_2 = (3,\, -1,\, 4)$ from above. The trivial solution $\vec{x}_0 = (1,\, 0)^\top$ of $[9,\, 8,\, 1]$ leads to a j-family of solutions of the *Pell* equation $x^2 - 7\,y^2 = 9$ by $\mathbf{B}(5)\,\mathbf{Auto}(5)^j\,\mathbf{R}^{-1}(6)\,\vec{x}_0$, or

$$\begin{pmatrix} x1(j) \\ y1(j) \end{pmatrix} = -\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}^j\begin{pmatrix} 6 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } \; j \in \mathbb{Z}\,. \tag{8}$$

These are (after appropriate overall sign flips, to have positive $x1(j)$) all (proper) solutions of the first j-family of solutions (called class of solutions in *e.g.*, [6] and [3])), namely, for $j = -4, ..., 0, ..., +4$

$$\begin{pmatrix} 214372 \\ -8105 \end{pmatrix}\begin{pmatrix} 13451 \\ -5084 \end{pmatrix}, \begin{pmatrix} 844 \\ -319 \end{pmatrix}\begin{pmatrix} 53 \\ -20 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$
$$\begin{pmatrix} 11 \\ 4 \end{pmatrix}, \begin{pmatrix} 172 \\ 65 \end{pmatrix}\begin{pmatrix} 2741 \\ 1036 \end{pmatrix}, \begin{pmatrix} 43684 \\ 16511 \end{pmatrix}. \tag{9}$$

The positive solutions are given in (A307168, A307169).

The second $j-$family of solutions is obtained from the trivial solution $\vec{x}_0 = (1,\, 0)^\top$ of $[9,\, 10,\, 2]$ by $\mathbf{B}(5)\,\mathbf{Auto}(5)^j\,\mathbf{R}^{-1}(4)\,\mathbf{R}^{-1}(-1)\,\mathbf{R}^{-1}(3)\,\vec{x}_0$. With $\mathbf{R}^{-1}(4)\,\mathbf{R}^{-1}(-1)\,\mathbf{R}^{-1}(3) = -Matrix([[-19, -5], [4, 1]])$ this is

$$\begin{pmatrix} x2(j) \\ y2(j) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 9 \\ 3 & 14 \end{pmatrix}^j\begin{pmatrix} -19 & -5 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } \; j \in \mathbb{Z}\,. \tag{10}$$

No overall sign flips are needed. The solutions for $j = -4, ..., 0, ..., +4$ are:

$$\begin{pmatrix} 696203 \\ -263140 \end{pmatrix} \begin{pmatrix} 43684 \\ -16511 \end{pmatrix}, \begin{pmatrix} 2741 \\ -1036 \end{pmatrix} \begin{pmatrix} 172 \\ -65 \end{pmatrix}, \begin{pmatrix} 11 \\ -4 \end{pmatrix},$$
$$\begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 20 \end{pmatrix} \begin{pmatrix} 844 \\ 319 \end{pmatrix}, \begin{pmatrix} 13451 \\ 5084 \end{pmatrix}. \qquad (11)$$

The positive solutions are given in (A307172, A307173).

A side remark: Observe that the single class of improper solutions of this *Pell* equation could also be obtained by starting with the improper trivial solution $\vec{x}1'_0 = (0, 3)^\top$ for the [9, 8, 1] case. This has fundamental positive solution $(24, 9)^\top$.

See the FIGURE for the relation between the relevant forms for the solution of $x^2 - 7\,y^2 = 9$. This is an example of a directed, node and edge labeled *Pell* graph $PG(D(5), k = 10)$. It has one loop and 10 vertices (nodes) and 10 links (edges) labeled with the forms and the t numbers for the R equivalence transformations. Because there are two cycles for discriminant $4\,D(5) = 28$ ($h(5) = 2$) there is another loop graph from the 4-cycle for the reduced form $\widehat{F}_p(5) = [-1, 4, 3]$. But this is not of direct interest for the solution of the *Pell* problem $F(5) = k$. See, however, the following remarks concerning $F(n)$ representing $-k$).

Some remarks on *Table 3* with the *rpapfs* for $n = 1, 2, ..., 30$ and $k = 1, 2, ..., 10$. As mentioned above forms in boldface are reduced. Underlined forms are not connected to the principal cycle $CR(n)$, but to either the outer sign flipped cycle $\widehat{CR}(n)$ or to other cycles which may appear and are called generically C and outer sign flipped $\widehat{C}$. Only for $n = 7$ the 6-cycle has been given separately, because it is outer sign flip invariant, as is the $CR(7)$ 2-cycle. For the cycles see *Table 2*.

From *Table 3* one can see which integers k, with $|k| \leq 10$ are representable for the *Pell* form $F(n)$. The first entries of the not underlined forms are representable, but not the first entries of the underlined forms. If they are connected to a cycle $\widehat{CR}$ then the negative of the first entry is representable by $F(n)$.

Two examples: $n = 7$, $D(7) = 10$: $F(7)$ is representable for $|k| \leq 10$ only by $k = \pm 1, \pm 6(2), \pm 9(2)$ and ± 10, with the number of proper families in brackets if it is larger than one. The negative k are also representable because $CR(7) = \widehat{CR}(7)$. $n = 8$, $D(8) = 11$: $F(8)$ is representable for $|k| \leq 10$ only by $k = +1, -2, +5(2), -7(2)$ and $-10(2)$.

Except for the cases $n = 12, 20, 25, 29$ and 30 (for $n \leq 30$) the cycles to which an underlined form is connected is obvious. The cycle connections for these cases are $n = 12$: $C, \widehat{C}, C, \widehat{CR}(12), \widehat{C}$ and $\widehat{C}$ for the six underlined forms, respectively. $n = 20$: $C, C, \widehat{C}, \widehat{C}, \widehat{C}$ and $\widehat{CR}(2)$ for the six underlined forms. $n = 25$: $C, C, \widehat{CR}(25), \widehat{C}, \widehat{C}$ and $\widehat{C}$ for the six underlined forms. $n = 29$: $\widehat{C}, C, C, \widehat{C}, C, \widehat{C}, \widehat{CR}(29), \widehat{CR}(29), C$ and $\widehat{C}$ for the ten underlined forms, and $n = 30$: $\widehat{C}, \widehat{C}, C$ and $\widehat{CR}(30)$ for the four underlined forms.

For the number of families of proper solutions of the *Pell* form $F(n)$ representing positive integers k see the array in A324252, and for negative integers k see A307303.

An aside on graphs for general form representations

In general each representation problem of a primitive binary quadratic indeterminate form F which has discriminant $Disc > 0$ from A079896 and a representable nonzero integer k leads to such a one loop graph which we call $FG(F, k)$. It may have large vertex and link numbers.

E.g., the primitive reduced principal form $F = [1, 9, -2] = F_p$ with $Disc(F) = 89$, and solution for $k = 10$, which is treated in [6], p. 116 and p. 121, has the 4 parallel forms [10, 3, −2], [10, 7, −1], [10, 13, 2] and [10, 17, 5] (in [6] two other non-parallel forms [10, −3, −2] and [10, −7, −1] are used instead of the last two ones). For them the t-tuples to reach $F = F_p$ are (−3, 9), (−8, 4, −1, 1, −1, 1, −4, 9), (5, −1, 1, −1, 1, −4, 9), and (2, −1, 1, −1, 4, −9, 4, −1, 1, −1, 1, −4, 9).

The cycle generated by $F_p = F$ has period $P = 2 \cdot 7 = 14$. The $t-$tuple is

$(-4,\, 1,\, -1,\, 1,\, -1,\, 4,\, -9,\, 4,\, -1,\, 1,\, -1,\, 1,\, -4,\, 9)$. Therefore $FG([1,\, 9,\, -2],\, 10)$ has $2+8+7+13+14 = 44$ vertices as well as links.
All four classes of proper solutions are found (modulo overall sign flips) from the trivial solution $\vec{x} = (1,0)^\top$ by $\vec{x}(Disc = 89, 10; i, j) = \mathbf{Auto}^j\, \mathbf{BPa}(i)\, \vec{x}_0$, with $i \in \{1,\, 2,\, 3,\, 4\}$ and $j \in \mathbb{Z}$.
In the graph $FG([1,\, 9,\, -2],\, 10)$ one has to go from $FPa(i)$ to F_p which is in the direction of the link arrows. Therefore the matrices $\mathbf{BPa}(i)$ acting on $\vec{x}_0$, are obtained from the inverse $\mathbf{R}$ matrices multiplied in reverse order. They are $\mathbf{BPa}(1) = Matrix([[-28, 9], [3, -1]])$, $\mathbf{BPa}(2) = Matrix([[8028, -977], [-871, 106]])$, $\mathbf{BPa}(3) = Matrix([[-1189, -212], [129, 23]])$, and $\mathbf{BPa}(4) = Matrix([[341001, 129001], [-36997, -13996]])$, respectively. The automorphic matrix uses products of $\mathbf{R}$ matrices and it becomes $\mathbf{Auto} = -Matrix([[23001, 212000], [106000, 977001]])$. The four fundamental positive solutions are then (from $j = 1$ after an overall sign flip for $i = 1$ and 4) $\vec{x}_0(1) = (8028,\, 36997)^\top$, $\vec{x}_0(2) = (28,\, 129])^\top$, $\vec{x}_0(3) = (189,\, 871)^\top$, and $\vec{x}_0(4) = (1,\, 3)^\top$.
Note that the class number for $Disc(F) = 89$ is 1, therefore there is only this principal 14-cycle.

4 Relations between solutions of different families

Because of the structure of the *Pell* graph (or also graphs $FG(F,\, k)$ for general forms) it is clear that all these primitive forms labeling the vertices are $SL(2,\, \mathbb{Z})$ equivalent due to the links standing for R-transformations or their inverses, depending on the orientation of a link. The number $paC(Disc > 0,\, k)$ of *rpapf*s reaching the cycle of the graph gives the number of fundamental solutions (using positive x by convention) of the form F with a representable nonzero k. This F is connected to its principal form F_p. Sometimes F may already be reduced and coincide with F_p. Each fundamental solution is obtained from the trivial solutions $(1,\, 0)^\top$ of each *rpapf*. It gives rise, *via* powers of the automorphic matrix, to an infinite j-family ($j \in \mathbb{Z}$) of proper solutions.
Because $FG(F,\, k)$ is connected (with one loop derived from F_p for $Disc(F)$) it is possible to go from any *rpapf* $FPa(i)$ to any other $FPa(i')$. Thus one can find a nontrivial solution for the latter from the trivial one of the first. The trivial solutions of these two parallel forms are not mapped to each other, because otherwise the matrix connecting them would have to be its own inverse (idempotent).
This shows that all *rpapf*s of $FG(F,\, k)$, and with them all other forms labeling the vertices of the graph, have the same number of different families of solutions for this k value.
In the above considered *Pell* example for $D(5) = 7$ the trivial solution $\vec{x}_0 = (1,\, 0)^\top$ of $FPa(1) = [9,\, 8,\, 1]$ for $k = 9$ maps to

$$\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \mathbf{R}(3)\, \mathbf{R}(-1)\, \mathbf{R}(4)\, \mathbf{R}^{-1}(6) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{12}$$

a solution of the second non-trivial family of $FPa(2)$ for $k = 9$. The connecting matrix is $Matrix([[1, 1], [-5, -4]])$ (not idempotent).
The other partner graph for $D(5) = 7$ with the 4-cycle $\widehat{CR}(5)$ starting with $\widehat{F}_p = [-1,\, 4,\, 3]$ (the mentioned outer sign flipped F_p) has no *rpapf*s for $k = 9$ (because the only two existing ones are connected to the given *Pell* graph). Therefore the graph $FG(\widehat{F}_p,\, 9)$ is just the 4-cycle, and there are no proper solutions for any of the forms involved in this cycle for $k = 9$. This cycle has *e.g.*, two *rpapf*s for $k = 6$, *viz* $[6,\, 2,\, -1]$ and $[6,\, 10,\, 3]$ leading to two families of proper solutions. This shows, in turn, that the *Pell* form $F(5)$ has no proper solution for $k = 6$ (and no improper ones).
Instead of going in the case *Pell* $F(5) = [1,\, 0,\, -7]$ and $k = 9$ with the trivial solutions from $FPa(1)$ to $F(5)$ finding the solution of the first family (eq. 8), one could go first from $FPa(1)$ to $FPa(2)$ (as above) finding the solution $(1,\, -5)^\top$ from its non-trivial family, and then go *via* F_p with this solution to $F(5)$, to find (after an overall sign flip) the solution $(4,\, -1)$ of the first family (the $j = 0$ solution):

$$-\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \mathbf{B}(5)\, \mathbf{R}^{-1}(4)\, \mathbf{R}^{-1}(-1))\, \mathbf{R}^{-1}(3)) \begin{pmatrix} 1 \\ -5 \end{pmatrix}. \tag{13}$$

The trivial solution of $Pa2$ leads to the second family for $F(5)$ representing $k = 9$, as given above in eq. (10).

In a case with more than one family of solutions on cannot, of course, obtain all solutions from one *rpapf* FPa without knowing besides the trivial solution also a solution for each other family. This show the importance of knowing all *rpapf*s in order to use only the trivial solutions to find all solutions.

5 Conclusion

The search for fundamental solutions for each family, done *e.g.*, by scanning the inequalities of *Nagell* mentioned above, has in this paper, based on the *Scholz-Schoeneberg* reference, replaced by the search of all representative parallel primitive forms (*rpapf*s) for $Disc > 0, k \in \mathbb{Z} \setminus \{0\}$ with the above given program.

References

[1] D. A. Buell, *Binary Quadratic Forms*, Springer, 1989.

[2] Maple^{TM}, http://www.maplesoft.com/.

[3] T. Nagell, *Introduction to Number Theory*, Chelsea Publishing Company, New York, 1964.

[4] Wolfdieter Lang, A2225953: Periods of Indefinite Binary Quadratic Forms, Continued Fractions and the Pell ±4 Equations, 2013, https://oeis.org/A225953/a225953_2.pdf.

[5] The On-Line Encyclopedia of Integer Sequences^{TM}, published electronically at http://oeis.org. 2010.

[6] A. Scholz and B. Schoeneberg, *Einführung in die Zahlentheorie*, Sammlung Göschen Band 5131, Walter de Gruyter, 1973.

OEIS [5] A-numbers: A000037, A000194, A001080, A001081, A003814, A049310, A077412, A079896, A082174, A087048, A225953, A307168, A307169, A307172, A307173, A307236, A307303, A307359, A307372, A307377, A324251, A345252.

Keywords: Binary quadratic forms, general Pell equation, Pell graphs. MSC-Numbers: 11E16 05C38

Legends:

FIGURE:*Pell* graph for $F(5) = [1, 0, -7]$, with a principal cycle of length 4, and connections of twp parallel forms $FPa1$ and $FPa2$ to the principal form F_p.

Table 1: One fourth of discriminant $D(n)$ =A000037(n), $s(n)$ = A000194(n). $FR(n) = F_p(n)$ is the reduced principal form of $4 \cdot D(n)$, and the length $LCR(n)$ =A307372(n) of the principal cycle $CR(n)$.

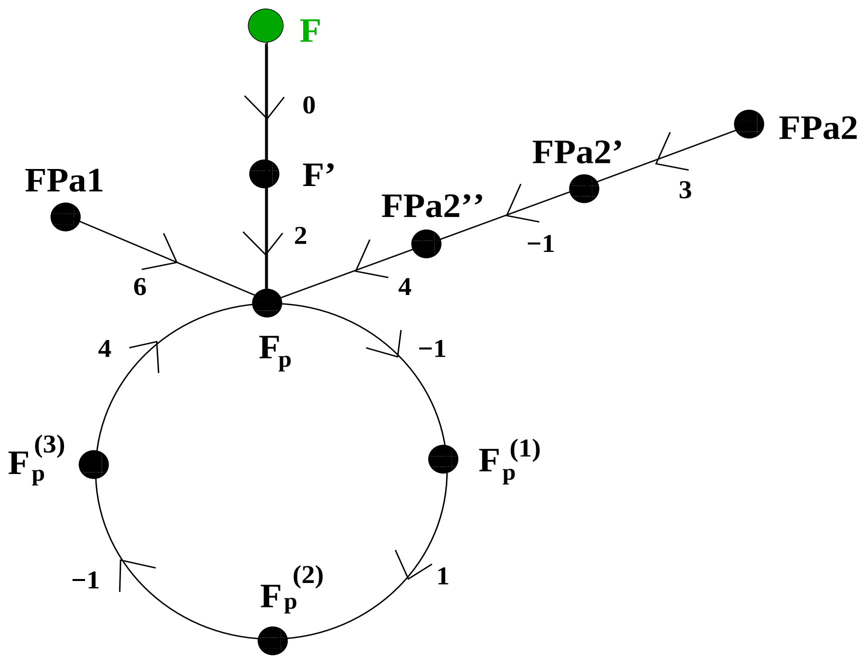
Table 2: Class number $h(n)$ =A307359(n). $L(n)$ gives the lengths of the cycles. CR is the cycle starting with FR given in *Table 1*. $\widehat{\mathrm{CR}}$ is the cycle with the signs of all first and last entries of the forms of the cycle CR changed. The same rule applies to the cycles C defined for some rows and $\widehat{\mathrm{C}}$. $\mathbf{\Sigma}L(n)$ is the sum of the total lengths of the cycles (A307236).

Table 3: Representative parallel primitive forms *rpapf*s, for discriminant $4\,D(n)$, for $n = 1, 2, ..., 30$, and representation of $k = 1, 2, ..., 10$. The first forms are the *Pell* forms $F(n)$. Boldface forms are already reduced, hence members of cycles. Underlined forms do not reach the principal cycle $CR(n)$, so there is no proper solution for the *Pell* form for positive k in the considered range. If they are connected to cycle $\widehat{CR}(n)$ then $-k$ is represented by $F(n)$. These forms with outer signs flipped represent $-k$, and they reach other cycles listed in *Table 2*. The cases $n = 12, 20, 25, 29$ and 30 have four cycles, and are considered separately in the text.

FIGURE

All solutions for Pell x^2 − 7 y^2 = 9

Disc(5) = 4 D(5) = 4*7



F = [1, 0, −7] **F' = [−7, 0, 1]**

F_p = [1, 4, −3] **$F_p^{(1)}$ = [−3, 2, 2]** **$F_p^{(2)}$ = [2, 2, −3]** **$F_p^{(3)}$ = [−3, 4, 1]**

FPa1 = [9, 8, 1]

FPa2 = [9, 10, 2] **FPa2' = [2, 2, −3]** **FPa2'' = [−3, 4, 1]**

Table 1: First reduced forms FR(n) for Pell forms $[1, 0, -D(n)]$ and their cyles CR(n)

n	D(n)	s(n)	FR(n)	$\vec{t}$(n)	cycles(n) (starting with FR(n))	LCR(n)
1	2	1	$[1,2,-1]$	$(-2,2)$	$[-1,2,1]$	2
2	3	1	$[1,2,-2]$	$(-1,2)$	$[-2,2,1]$	2
3	5	2	$[1,4,-1]$	$(-4,4)$	$[-1,4,1]$	2
4	6	2	$[1,4,-2]$	$(-2,4)$	$[-2,4,1]$	2
5	7	2	$[1,4,-3]$	$(-1,1,-1,4)$	$[-3,2,2]$, $[2,2,-3]$, $[-3,4,1]$	4
6	8	2	$[1,4,-4]$	$(-1,4)$	$[-4,4,1]$	2
7	10	3	$[1,6,-1]$	$(-6,6)$	$[-1,6,1]$	2
8	11	3	$[1,6,-2]$	$(-3,6)$	$[-2,6,1]$	2
9	12	3	$[1,6,-3]$	$(-2,6)$	$[-3,6,1]$	2
10	13	3	$[1,6,-4]$	$(-1,1,-1,1,-6,1,-1,1,-1,6)$	$[-4,2,3]$, $[3,4,-3]$, $[-3,2,4]$, $[4,6,-1]$, $[-1,6,4]$, $[4,2,-3]$, $[-3,4,3]$, $[3,2,-4]$, $[-4,6,1]$	10
11	14	3	$[1,6,-5]$	$(-1,2,-1,6)$	$[-5,4,2]$, $[2,4,-5]$, $[-5,6,1]$	4
12	15	3	$[1,6,-6]$	$(-1,6)$	$[-6,6,1]$	2
13	17	4	$[1,8,-1]$	$(-8,8)$	$[-1,8,1]$	2
14	18	4	$[1,8,-2]$	$(-4,8)$	$[-2,8,1]$	2
15	19	4	$[1,8,-3]$	$(-2,1,-3,1,-2,8)$	$[-3,4,5]$, $[5,6,-2]$, $[-2,6,5]$, $[5,4,-3]$, $[-3,8,1]$	6
16	20	4	$[1,8,-4]$	$(-2,8)$	$[-4,8,1]$	2
17	21	4	$[1,8,-5]$	$(-1,1,-2,1,-1,8)$	$[-5,2,4]$, $[4,6,-3]$, $[-3,6,4]$, $[4,2,-5]$, $[-5,8,1]$	6
18	22	4	$[1,8,-6]$	$(-1,2,-4,2,-1,8)$	$[-6,4,3]$, $[3,8,-2]$, $[-2,8,3]$, $[3,4,-6]$, $[-6,8,1]$	6
19	23	4	$[1,8,-7]$	$(-1,3,-1,8)$	$[-7,6,2]$, $[2,6,-7]$, $[-7,8,1]$	4
20	24	4	$[1,8,-8]$	$(-1,8)$	$[-8,8,1]$	2
21	26	5	$[1,10,-1]$	$(-10,10)$	$[-1,10,1]$	2
22	27	5	$[1,10,-2]$	$(-5,10)$	$[-2,10,1]$	2
23	28	5	$[1,10,-3]$	$(-3,2,-3,10)$	$[-3,8,4]$, $[4,8,-3]$, $[-3,10,1]$	4
24	29	5	$[1,10,-4]$	$(-2,1,-1,2,-10,2,-1,1,-2,10)$	$[-4,6,5]$, $[5,4,-5]$, $[-5,6,4]$, $[4,10,-1]$, $[-1,10,4]$, $[4,6,-5]$, $[-5,4,5]$ $[5,6,-4]$, $[-4,10,1]$	10
25	30	5	$[1,10,-5]$	$(-2,10)$	$[-5,10,1]$	2
26	31	5	$[1,10,-6]$	$(-1,1,-3,5,-3,1,-1,10)$	$[-6,2,5]$, $[5,8,-3]$, $[-3,10,2]$, $[2,10,-3]$, $[-3,8,5]$, $[5,2,-6]$, $[-6,10,1]$	8
27	32	5	$[1,10,-7]$	$(-1,1,-1,10)$	$[-7,4,4]$, $[4,4,-7]$, $[-7,10,1]$	4
28	33	5	$[1,10,-8]$	$(-1,2,-1,10)$	$[-8,6,3]$, $[3,6,-8]$, $[-8,10,1]$	4
29	34	5	$[1,10,-9]$	$(-1,4,-1,10)$	$[-9,8,2]$, $[2,8,-9]$, $[-9,10,1]$	4
30	35	5	$[1,10,-10]$	$(-1,10)$	$[-10,10,1]$	2

Table 2: Class number h(n) and all cycles for $4 \cdot D(n)$ for n = 1..30

n	D(n)	h(n)	L(n)	cycles	ΣL(n)
1	2	1	(2)	[CR(1)]	2
2	3	2	(2, 2)	[CR(2), $\widehat{\mathrm{CR}}$(2)]	4
3	5	1	(2)	[CR(3)]	2
4	6	2	(2, 2)	[CR(4), $\widehat{\mathrm{CR}}$(4)]	4
5	7	2	(4, 4)	[CR(5), $\widehat{\mathrm{CR}}$(5)]	8
6	8	2	(2, 2)	[CR(6), $\widehat{\mathrm{CR}}$(6)]	4
7	10	2	(6, 2)	[[[3, 2, -3], [-3, 4, 2], [2, 4, -3], [-3, 2, 3], [3, 4, -2], [-2, 4, 3]], CR(7)]	8
8	11	2	(2, 2)	[CR(8), $\widehat{\mathrm{CR}}$(8)]	4
9	12	2	(2, 2)	[CR(9), $\widehat{\mathrm{CR}}$(9)]	4
10	13	1	(10)	[CR(10)]	10
11	14	2	(4, 4)	[CR(11), $\widehat{\mathrm{CR}}$(11)]	8
12	15	4	(2, 2, 2, 2)	[CR(12), $\widehat{\mathrm{CR}}$(12), C = [[2, 6, -3], [-3, 6, 2]], $\widehat{\mathrm{C}}$	8
13	17	1	(2)	[CR(13)]	2
14	18	2	(2, 2)	[CR(14), $\widehat{\mathrm{CR}}$(14)]	4
15	19	2	(6, 6)	[CR(15), $\widehat{\mathrm{CR}}$(15)]	12
16	20	2	(2, 2)	[CR(16), $\widehat{\mathrm{CR}}$(16)]	4
17	21	2	(6, 6)	[CR(17), $\widehat{\mathrm{CR}}$(17)]	12
18	22	2	(6, 6)	[CR(18), $\widehat{\mathrm{CR}}$(18)]	12
19	23	2	(4, 4)	[CR(19), $\widehat{\mathrm{CR}}$(19)]	8
20	24	4	(4, 4, 2, 2)	[C = [[4,4,-5],[-5,6,3],[3,6,-5],[-5,4,4]], $\widehat{\mathrm{C}}$, CR(20), $\widehat{\mathrm{CR}}$(20)]	12
21	26	2	(6, 2)	[[[5, 2, -5], [-5, 8, 2], [2, 8, -5], [-5, 2, 5], [5, 8, -2], [-2, 8, 5]], CR(21)]	8
22	27	2	(2, 2)	[CR(22), $\widehat{\mathrm{CR}}$(22)]	4
23	28	2	(4, 4)	[CR(23), $\widehat{\mathrm{CR}}$(23)]	8
24	29	1	(10)	[CR(24)]	10
25	30	4	(4, 4, 2, 2)	[C = [[3, 6, -7], [-7, 8, 2], [2, 8, -7], [-7, 6, 3]], $\widehat{\mathrm{C}}$, CR(25), $\widehat{\mathrm{CR}}$(25)]	12
26	31	2	(8, 8)	[CR(26), $\widehat{\mathrm{CR}}$(26)]	16
27	32	2	(4, 4)	[CR(27), $\widehat{\mathrm{CR}}$(27)]	8
28	33	2	(4, 4)	[CR(28), $\widehat{\mathrm{CR}}$(28)]	8
29	34	4	(6, 6, 4, 4)	[C = [[5, 4, -6], [-6, 8, 3], [3, 10, -3], [-3, 8, 6], [6, 4, -5], [-5, 6, 5]], $\widehat{\mathrm{C}}$, CR(29), $\widehat{\mathrm{CR}}$(29)]	20
30	35	4	(2, 2, 2, 2)	[CR(30), $\widehat{\mathrm{CR}}$(30), C = [[2, 10, -5], [-5, 10, 2]], $\widehat{\mathrm{C}}$]	8

Table 3: Representative parallel primitive forms (rpapfs) for D(n), for n = 1..30, and k = 1..10

n	**D(n)**	**rpapfs**
1	2	[1, 0, -2], [**2, 0, -1**], [[7, 6, 1], [7, 8, 2]]
2	3	[1, 0, -3], [2, 2, -1], [3, 0, -1], [6, 6, 1]
3	5	[1, 0, -5], [[4, 2, -1], [4, 6, 1]], [5, 0, -1]
4	6	[1, 0, -6], [2, 0, -3], [3, 0, -2], [[5, 2, -1], [5, 8, 2]], [6, 0, -1], [[10, 8, 1] , [10, 12, 3]]
5	7	[1, 0, -7], [**2, 2, -3**], [[**3, 2, -2**], [**3, 4, -1**]], [[6, 2, -1], [6, 10, 3]], [7, 0, -1] [[9, 8, 1], [9, 10, 2]]
6	8	[1, 0, -8], [**4, 4, -1**], [[7, 2, -1], [7, 12, 4]], [[8, 0, -1], [8, 8, 1]]
7	10	[1, 0, -10], [2, 0, -5], [[**3, 2, -3**], [**3, 4, -2**]], [5, 0, -2], [[6, 4, -1], [6, 8, 1]], [[9, 2, -1], [9, 16, 6]], [10, 0, -1]
8	11	[1, 0, -11], [2, 2, -5], [[5, 2, -2], [5, 8, 1]], [[7, 4, -1], [7, 10, 2]], [[10, 2, -1], [10, 18, 7]]
9	12	[1, 0, -12], [3, 0, -4], [4, 0, -3], [[8, 4, -1], [8, 12, 3]]
10	13	[1, 0, -13], [[**3, 2, -4**], [**3, 4, -3**]], [[**4, 2, -3**], [**4, 6, -1**]], [[9, 4, -1], [9, 14, 4]]
11	14	[1, 0, -14], [2, 0, -7], [[**5, 4, -2**], [**5, 6, -1**]], [7, 0, -2], [[10, 4, -1], [10, 16, 5]]
12	15	[1, 0, -15], [2, 2, -7], [3, 0, -5], [5, 0, -3], [**6, 6, -1**], [[7, 2, -2], [7, 12, 3]], [10, 10, 1]
13	17	[1, 0, -17], [[8, 6, -1], [8, 10, 1]]
14	18	[1, 0, -18], [2, 0, -9], [[7, 4, -2], [7, 10, 1]], [[9, 0, -2], [9, 6, -1], [9, 12, 2]],
15	19	[1, 0, -19], [2, 2, -9], [[3, 2, -6], [**3, 4, -5**]], [[**5, 4, -3**], [**5, 6, -2**]], [[6, 2, -3], [6, 10, 1]], [[9, 2, -2], [9, 16, 5]], [[10, 6, -1], [10, 14, 3]]
16	20	[1, 0, -20], [4, 0, -5], [5, 0, -4]
17	21	[1, 0, -21], [3, 0, -7], [[**4, 2, -5**], [**4, 6, -3**]], [[**5, 2, -4**], [**5, 8, -1**]], [7, 0, -3]
18	22	[1, 0, -22], [2, 0, -11], [[3, 2, -7], [**3, 4, -6**]], [[**6, 4, -3**], [**6, 8, -1**]], [[7, 2, -3], [7, 12, 2]], [[9, 4, -2], [9, 14, 3]]
19	23	[1, 0, -23], [2, 2, -11], [[**7, 6, -2**], [**7, 8, -1**]]
20	24	[1, 0, -24], [3, 0, -8], [**4, 4, -5**], [[**5, 4, -4**], [**5, 6, -3**]], [[8, 0, -3], [**8, 8, -1**]]
21	26	[1, 0, -26], [2, 0, -13], [[**5, 2, -5**], [**5, 8, -2**]], [[10, 8, -1], [10, 12, 1]]
22	27	[1, 0, -27], [2, 2, -13], [[9, 6, -2], [9, 12, 1]]
23	28	[1, 0, -28], [[3, 2, -9], [3, 4, -8]], [4, 0, -7], [7, 0, -4] [[8, 4, -3], [8, 12, 1]], [[9, 2, -3], [9, 16, 4]]
24	29	[1, 0, -29], [[4, 2, -7], [**4, 6, -5**]], [[**5, 4, -5**], [**5, 6, -4**]], [[7, 2, -4], [7, 12, 1]]
25	30	[1, 0, -30], [2, 0, -15], [3, 0, -10], [5, 0, -6], [6, 0, -5], [[**7, 6, -3**], [**7, 8, -2**]], [10, 0, -3]
26	31	[1, 0, -31], [2, 2, -15], [[3, 2, -10], [3, 4, -9]], [[**5, 2, -6**], [**5, 8, -3**]], [[**6, 2, -5**], [**6, 10, -1**]], [[9, 4, -3], [9, 14, 2]], [[10, 2, -3], [10, 18, 5]]
27	32	[1, 0, -32], [**4, 4, -7**], [[**7, 4, -4**], [**7, 10, -1**]]
28	33	[1, 0, -33], [3, 0, -11], [[**8, 6, -3**], [**8, 10, -1**]]
29	34	[1, 0, -34], [2, 0, -17], [[3, 2, -11], [3, 4, -10]], [[**5, 4, -6**], [**5, 6, -5**]], [[**6, 4, -5**], [**6, 8, -3**]], [[**9, 8, -2**], [**9, 10, -1**]], [[10, 4, -3], [10, 16, 3]]
30	35	[1, 0, -35], [2, 2, -17], [5, 0, -7], [7, 0, -5], [**10, 10, -1**]