Census Function for a Regular Language

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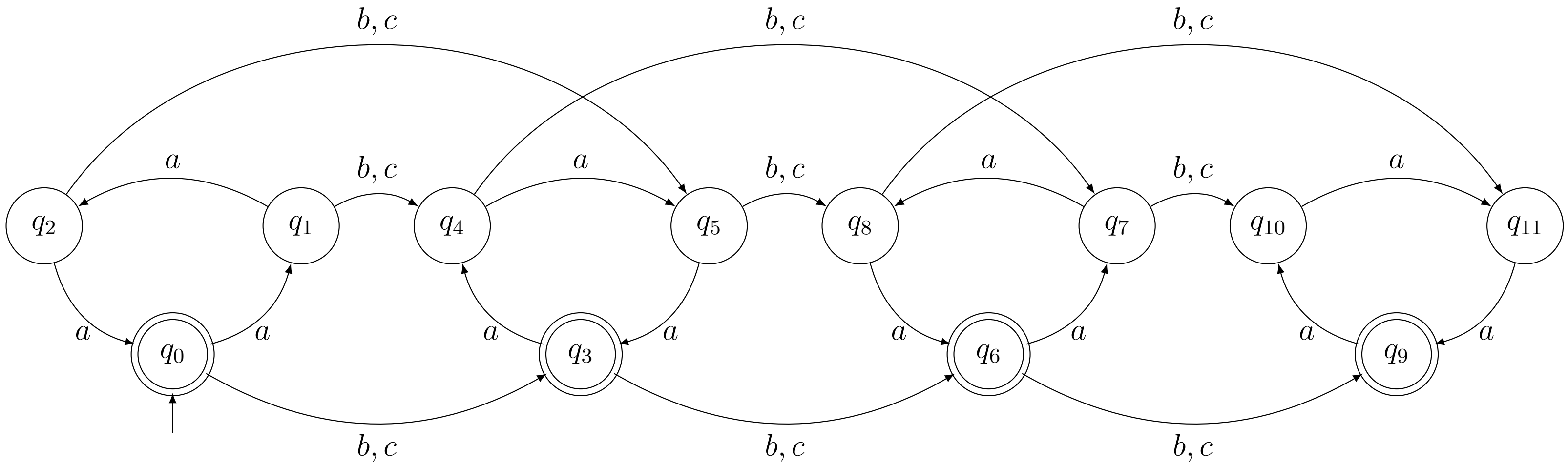
The languages is the strings over $\Sigma = \{a, b, c\}$, with the number of a's is divisible by 3 and the number of b's y c's is at the most 3.

$$\mathcal{L} = \{w \in \Sigma^* \mid |w|_a \equiv 0 \pmod 3 \ \wedge \ |w|_b + |w|_c \leq 3\}$$

We want to find a formula for $|\mathcal{L}_n|$, donde $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$, for this we consider the automata for the language:



With state initial q_0 and accepting states q_0, q_3, q_6, y q_9, the adajacency matrix is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Understing that $|\mathcal{L}_n|$ is the sum of the entrances $(1,1)$, $(1,4)$ $(1,7)$, y $(1,10)$ of the nth power of A, and calculating the powers, we obtain:

n	$\lvert\mathcal{L}_n\rvert$
0	1
1	2
2	4
3	9
4	8
5	40
6	161
7	14
8	112
9	673
10	20
11	220

Now we consider the characteristic polynomial of the matrix A:

$$p(\lambda) = \lambda^{12} - 4\lambda^9 + 6\lambda^6 - 4\lambda^3 + 1$$

For the Cayley-Hamilton theorem:

$$\begin{aligned} \mathcal{O}_{12} &= A^{12} - 4A^9 + 6A^6 - 4A^3 + 1 \\ A^{12} &= 4A^9 - 6A^6 + 4A^3 - 1 \\ A^{n+12} &= 4A^{n+9} - 6A^{n+6} + 4A^{n+3} - A^n \end{aligned}$$

And considering that this is also fulfilled by the elements of the matrix and any linear combination of these, we have:

$$|\mathcal{L}_{n+12}| = 4|\mathcal{L}_{n+9}| - 6|\mathcal{L}_{n+6}| + 4|\mathcal{L}_{n+3}| - |\mathcal{L}_n|$$

For solve this recurrence ecuation, we will use the method ilustrated in [2], the associated polynomial is:

$$r^{12} = 4r^9 - 6r^6 + 4r^3 - r$$

With roots:

$$r_1 = 1$$
$$r_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$r_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

All with multiplicity 4, so the census function formula has the form:

$$|\mathcal{L}_n| = \alpha_1+\alpha_2 n+\alpha_3 n^2+\alpha_4 n^3+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n(\alpha_5+\alpha_6 n+\alpha_7 n^2+\alpha_8 n^3)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^n(\alpha_9+\alpha_{10}n+\alpha_{11}n^2+\alpha_{12}n^3)$$

Where $\alpha_i \in \mathbb{C}$, using the values of $|\mathcal{L}_n|$ earned previusly, we have a system 12x12, solving:

$\alpha_1 = \frac{1}{3}$

$\alpha_2 = \frac{8}{9}$

$\alpha_3 = -\frac{2}{3}$

$\alpha_4 = \frac{4}{9}$

$\alpha_5 = \frac{1}{3}$

$\alpha_6 = \frac{8}{9} - \frac{2\sqrt{3}}{3}i$

$\alpha_7 = -\frac{5}{3} + \frac{\sqrt{3}}{3}i$

$\alpha_8 = \frac{4}{9}$

$\alpha_9 = \frac{1}{3}$

$\alpha_{10} = \frac{8}{9} + \frac{2\sqrt{3}}{3}i$

$\alpha_{11} = -\frac{5}{3} - \frac{\sqrt{3}}{3}i$

$\alpha_{12} = \frac{4}{9}$

So the formula for our census function is:

$$|\mathcal{L}_n| = \tfrac{1}{3} + \tfrac{8}{9}n - \tfrac{2}{3}n^2 + \tfrac{4}{9}n^3 + \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)^n \left(\tfrac{1}{3} + \left(\tfrac{8}{9} - \tfrac{2\sqrt{3}}{3}i\right)n + \left(-\tfrac{5}{3} + \tfrac{\sqrt{3}}{3}i\right)n^2 + \tfrac{4}{9}n^3\right)$$
$$+ \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^n \left(\tfrac{1}{3} + \left(\tfrac{8}{9} + \tfrac{2\sqrt{3}}{3}i\right)n + \left(-\tfrac{5}{3} - \tfrac{\sqrt{3}}{3}i\right)n^2 + \tfrac{4}{9}n^3\right)$$

What's more, we have that $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$, $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$ and $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3=1$ So we have the next cases:

1. $n \equiv 0 \pmod 3$

$$\begin{aligned}|\mathcal{L}_n| = \tfrac{1}{3}+\tfrac{8}{9}n-\tfrac{2}{3}n^2+\tfrac{4}{9}n^3+\tfrac{1}{3}+\left(\tfrac{8}{9}-\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}+\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3 \\ +\tfrac{1}{3}+\left(\tfrac{8}{9}+\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}-\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\end{aligned}$$

$$|\mathcal{L}_n| = 1+\frac{8}{3}n-4n^2+\frac{4}{3}n^3$$

2. $n \equiv 1 \pmod 3$

$$\begin{aligned}|\mathcal{L}_n| &= \tfrac{1}{3}+\tfrac{8}{9}n-\tfrac{2}{3}n^2+\tfrac{4}{9}n^3+\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{3}+\left(\tfrac{8}{9}-\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}+\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ &\quad +\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{3}+\left(\tfrac{8}{9}+\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}-\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ |\mathcal{L}_n| &= \tfrac{8}{9}n-\tfrac{2}{3}n^2+\tfrac{4}{9}n^3+\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(\left(\tfrac{8}{9}-\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}+\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ &\quad +\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\left(\left(\tfrac{8}{9}+\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}-\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ |\mathcal{L}_n| &= \tfrac{8}{9}n-\tfrac{2}{3}n^2+\tfrac{4}{9}n^3+\left(\tfrac{5}{9}+\tfrac{7\sqrt{3}}{9}i\right)n+\left(\tfrac{1}{3}-\sqrt{3}i\right)n^2+\left(-\tfrac{2}{9}+\tfrac{2\sqrt{3}}{9}i\right)n^3 \\ &\quad +\left(\tfrac{5}{9}-\tfrac{7\sqrt{3}}{9}i\right)n+\left(\tfrac{1}{3}+\sqrt{3}i\right)n^2+\left(-\tfrac{2}{9}-\tfrac{2\sqrt{3}}{9}i\right)n^3 \\ |\mathcal{L}_n| &= 2n\end{aligned}$$

3. $n \equiv 2 \pmod 3$

$$\begin{aligned}|\mathcal{L}_n| &= \tfrac{1}{3}+\tfrac{8}{9}n-\tfrac{2}{3}n^2+\tfrac{4}{9}n^3+\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{3}+\left(\tfrac{8}{9}-\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}+\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ &\quad +\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{1}{3}+\left(\tfrac{8}{9}+\tfrac{2\sqrt{3}}{3}i\right)n+\left(-\tfrac{5}{3}-\tfrac{\sqrt{3}}{3}i\right)n^2+\tfrac{4}{9}n^3\right) \\ |\mathcal{L}_n| &= -2n+2n^2\end{aligned}$$

References

[1] De Castro, Rodrigo. (2004). *Teoría de la Computación. Lenguajes, autómatas, gramáticas*. Bogotá: UNILIBROS.

[2] Gutiérrez, José. & Lanchares, Víctor. (2010). *Elementos de Matemática discreta*. España: Universidad de la Rioja. Servicio de Comunicación.

[3] Grossmam, Stanley. & Flores, José. (2012). *Álgebra Lineal*. 7a ed. México: McGraw-Hill.

[4] Isaacs, Rafael. (2021). *Función de censo para algunos lenguajes regulares*.