Background notes for slides presented at
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Patras, Greece, 2008: https://www.mathstat.dal.ca/fibonacci/thirteenth.html
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Abstract: https://www.mathstat.dal.ca/fibonacci/abstracts.pdf

Counting *i*-paths

Notation

i-path: *i* non-intersecting paths on a lattice using only vertical and horizontal moves [i.e. monotonic non-decreasing], beginning (LHS) on *i* points on a descending diagonal and ending (RHS) on *i* consecutive points on a descending diagonal

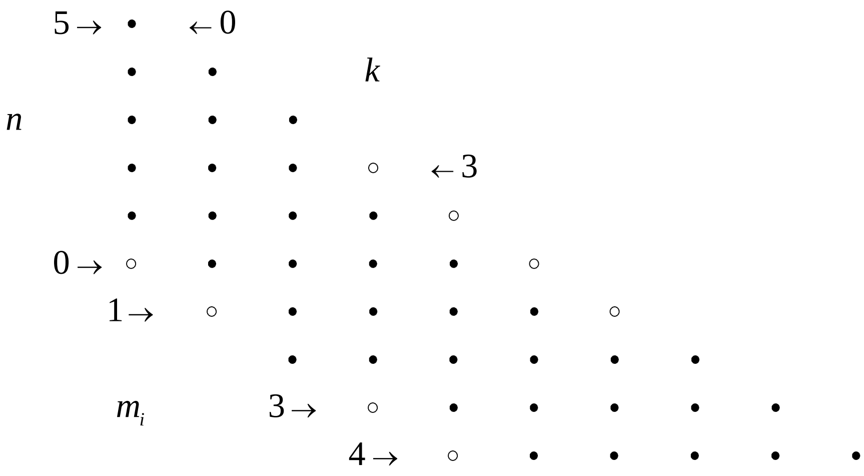
$\mathbf{m}_i = \{0, m_1, m_2, ..., m_{i-1}\}$: positions on 0$^{\text{th}}$ descending diagonal of initial points of an *i*-path

$p(\mathbf{m}_i, n, k)$: number of *i*-paths (of length $n \geq 0$) from $\mathbf{m}_i$ to *i* consecutive points on the n^{th} descending diagonal beginning at column $k \geq 0$

$a(\mathbf{m}_i, n)$: total number of *i*-paths (of length $n \geq 0$) from $\mathbf{m}_i$ to *i* consecutive points on the n^{th} descending diagonal

$$a(\mathbf{m}_i, n) = \sum_k p(\mathbf{m}_i, n, k)$$

Example:



Start and end points for the 4-paths counted by $p(\{0,1,3,4\},5,3)$

Note that for consecutive m_i this is equivalent to beginning from points on the LH vertical in the same rows as the starting points as there is only one possible set of non-intersecting paths between them and those on the LH diagonal.

1-paths (easy, included for completeness)

$$\begin{array}{llllll}
\bullet 1 \\
\bullet 1 & \bullet 5 \\
\bullet 1 & \bullet 4 & \bullet 10 \\
\bullet 1 & \bullet 3 & \bullet 6 & \bullet 10 \\
\bullet 1 & \bullet 2 & \bullet 3 & \bullet 4 & \bullet 5 \\
\circ 1 & \bullet 1 & \bullet 1 & \bullet 1 & \bullet 1 & \bullet 1
\end{array}$$

$p(\{0\},n,k)$ - the number of 1-paths from one LH point $(\circ)$ to the point (n up, k across), $n,k \geq 0$

Formulae:

(Arrows show direction of movement from a general LH point to obtain a similar term of reduced index)

For paths to an individual point (n,k), moving $\circ$ up one reduces n by one, moving $\circ$ across one reduces k by one:

$$\begin{array}{ccccc} & & \uparrow & & \rightarrow \\ p(\{0\},n,k) & = & p(\{0\},n-1,k) & + & p(\{0\},n,k-1) \end{array}$$

$$\text{BCs:} \quad p(\{0\},n,k) = \begin{cases} 0 & n<0 \text{ or } k<0 \\ 1 & n=0 \ \& \ k=0 \end{cases}$$

$$p(\{0\},n,k) = \binom{n+k}{k} = \binom{n+k}{n}$$

For paths to all points on the n^{th} diagonal, moving $\circ$ up or across one reduces n by one:

$$\begin{array}{ccccc} & & \uparrow & & \rightarrow \\ a(\{0\},n) & = & a(\{0\},n-1) & + & a(\{0\},n-1) \end{array}$$

$$\text{BCs:} \quad a(\{0\},n) = \begin{cases} 0 & n<0 \\ 1 & n=0 \end{cases}$$

$$\begin{aligned} a(\{0\},n) &= 2a(\{0\},n-1) \\ &= \sum_k \binom{n}{k} = 2^n = \{1,2,4,8,16,32,\ldots\} \end{aligned}$$

2-paths

•1						
•1	•15					
•1	•10	•50				
•1	•6	•20	•50			
•1	•3	•6	•10	•15		
○1	•1	•1	•1	•1	•1	
	○	•	•	•	•	•

$p(\{0,1\},n,k)$ - the number of 2-paths from two consecutive diagonal LH points $(\circ)$ to the two consecutive diagonal points with top left point at (n up, k across), $n,k \geq 0$

Formulae:

(Arrows show direction of movement from a general LH pair of points to obtain a similar term of reduced index)

$$p(\{0,m_1\},n,k) = \overset{\uparrow\uparrow}{p(\{0,m_1\},n-1,k)} + \overset{\rightarrow\uparrow}{p(\{0,m_1-1\},n,k-1)} + \overset{\uparrow\rightarrow}{p(\{0,m_1+1\},n-1,k)} + \overset{\rightarrow\rightarrow}{p(\{0,m_1\},n,k-1)}$$

$$\text{BCs:} \quad p(\{0,m_1\},n,k) = \begin{cases} 0 & m_1 = 0 \text{ or } n < 0 \text{ or } k < 0 \\ 1 & m_1 = 1 \ \& \ n = 0 \ \& \ k = 0 \\ 0 & m_1 > 1 \ \& \ n = 0 \ \& \ k = 0 \end{cases}$$

(New?)

$$\begin{aligned} p(\{0,1\},n,k) &= p(\{0,1\},n-1,k) + p(\{0,0\},n,k-1) + p(\{0,2\},n-1,k) + p(\{0,1\},n,k-1) \\ &= p(\{0,1\},n-1,k) + p(\{0,2\},n-1,k) + p(\{0,1\},n,k-1) \end{aligned}$$

$$a(\{0,m_1\},n) = \overset{\uparrow\uparrow}{a(\{0,m_1\},n-1)} + \overset{\rightarrow\uparrow}{a(\{0,m_1-1\},n-1)} + \overset{\uparrow\rightarrow}{a(\{0,m_1+1\},n-1)} + \overset{\rightarrow\rightarrow}{a(\{0,m_1\},n-1)}$$

$$= 2a(\{0,m_1\},n-1) + a(\{0,m_1-1\},n-1) + a(\{0,m_1+1\},n-1)$$

$$\text{BCs:} \quad a(\{0,m_1\},n) = \begin{cases} 0 & m_1 = 0 \text{ or } n < 0 \\ 1 & m_1 = 1 \ \& \ n = 0 \\ 0 & m_1 > 1 \ \& \ n = 0 \end{cases}$$

(Given by Shapiro and shown to be Catalan, although the BCs are incomplete therein.)

$$\begin{aligned} a(\{0,1\},n) &= a(\{0,1\},n-1)+a(\{0,0\},n-1)+a(\{0,2\},n-1)+a(\{0,1\},n-1) \\ &= 2a(\{0,1\},n-1)+a(\{0,2\},n-1) \\ &= \{1,2,5,14,42,132,\ldots\} \quad \text{(Catalan, A000108)} \end{aligned}$$

$$\begin{aligned} a(\{0,2\},n) &= a(\{0,2\},n-1)+a(\{0,1\},n-1)+a(\{0,3\},n-1)+a(\{0,2\},n-1) \\ &= 2a(\{0,2\},n-1)+a(\{0,1\},n-1)+a(\{0,3\},n-1) \\ &= \{0,1,4,14,48,165,572,\ldots\} \quad \text{(4}^{\text{th}}\text{ convolution of Catalan, A002057)} \end{aligned}$$

$$a(\{0,3\},n) = \{0,0,1,6,27,110,429,1638,\ldots\} \quad \text{(6}^{\text{th}}\text{ convolution of Catalan, A003517)}$$

$$a(\{0,4\},n) = \{0,0,0,1,8,44,208,910,\ldots\} \quad \text{(8}^{\text{th}}\text{ convolution of Catalan, A003518)}$$

Matrices (new?):

$$\begin{pmatrix} a(\{0,1\},n) \\ a(\{0,2\},n) \\ \\ \vdots \\ \\ a(\{0,n-1\},n) \\ a(\{0,n\},n) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & & \cdots & & 0 \\ 1 & 2 & 1 & 0 & & & 0 \\ 0 & 1 & 2 & 1 & 0 & & 0 \\ & & \cdots & & & & \vdots \\ \vdots & & 0 & 1 & 2 & 1 & 0 \\ & & & 0 & 1 & 2 & 1 \\ 0 & & \cdots & & 0 & 1 & 2 \end{pmatrix}_{n\times n} \begin{pmatrix} a(\{0,1\},n-1) \\ a(\{0,2\},n-1) \\ \\ \vdots \\ \\ a(\{0,n-1\},n-1) \\ a(\{0,n\},n-1)=1 \end{pmatrix}$$

For example,

$$\begin{pmatrix} a(\{0,1\},4) \\ a(\{0,2\},4) \\ a(\{0,3\},4) \\ a(\{0,4\},4) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 42 \\ 48 \\ 27 \\ 8 \end{pmatrix}$$

(see $n = 4$ terms of the Catalan and Catalan convolution sequences above).

3-paths

•1							
•1	•35						
•1	•20	•175					
•1	•10	•50	•175				
•1	•4	•10	•20	•35			
○1	•1	•1	•1	•1	•1		
	○	•	•	•	•	•	
		○	•	•	•	•	•

$p(\{0,1,2\},n,k)$ - the number of 3-paths from three consecutive diagonal LH points $(\circ)$ to the three consecutive diagonal points with top left point at (n up, k across), $n,k \geq 0$

Formulae:

(Arrows show direction of movement from a general LH triple of points to obtain a similar term of reduced index)

$$\begin{aligned}
p(\{0,m_1,m_2\},n,k) \quad = \quad & \overset{\uparrow\uparrow\uparrow}{p(\{0,m_1,m_2\},n-1,k)} && + \overset{\rightarrow\uparrow\uparrow}{p(\{0,m_1-1,m_2-1\},n,k-1)} \\
+ \quad & \overset{\uparrow\rightarrow\uparrow}{p(\{0,m_1+1,m_2\},n-1,k)} && + \overset{\rightarrow\rightarrow\uparrow}{p(\{0,m_1,m_2-1\},n,k-1)} \\
+ \quad & \overset{\uparrow\uparrow\rightarrow}{p(\{0,m_1,m_2+1\},n-1,k)} && + \overset{\rightarrow\uparrow\rightarrow}{p(\{0,m_1-1,m_2\},n,k-1)} \\
+ \quad & \overset{\uparrow\rightarrow\rightarrow}{p(\{0,m_1+1,m_2+1\},n-1,k)} && + \overset{\rightarrow\rightarrow\rightarrow}{p(\{0,m_1,m_2\},n,k-1)}
\end{aligned}$$

$$\text{BCs:} \quad p(\{0,m_1,m_2\},n,k) = \begin{cases} 0 & m_1 = 0 \text{ or } m_2 = m_1 \text{ or } n < 0 \text{ or } k < 0 \\ 1 & m_1 = 1 \;\&\; m_2 = 2 \;\&\; n = 0 \;\&\; k = 0 \\ 0 & (m_1 > 1 \text{ or } m_2 > 2) \;\&\; n = 0 \;\&\; k = 0 \end{cases}$$

(New?)

$$\begin{aligned}
p(\{0,1,2\},n,k) &= p(\{0,1,2\},n-1,k) + p(\{0,0,1\},n,k-1) + p(\{0,2,2\},n-1,k) + p(\{0,1,1\},n,k-1) \\
&\quad + p(\{0,1,3\},n-1,k) + p(\{0,0,2\},n,k-1) + p(\{0,2,3\},n-1,k) + p(\{0,1,2\},n,k-1) \\
&= p(\{0,1,2\},n-1,k) + p(\{0,1,3\},n-1,k) + p(\{0,2,3\},n-1,k) + p(\{0,1,2\},n,k-1)
\end{aligned}$$

$$
\begin{array}{lclcl}
 & & \uparrow\uparrow\uparrow & & \rightarrow\uparrow\uparrow \\
a(\{0,m_1,m_2\},n) & = & a(\{0,m_1,m_2\},n-1) & + & a(\{0,m_1-1,m_2-1\},n-1) \\
 & & \uparrow\rightarrow\uparrow & & \rightarrow\rightarrow\uparrow \\
 & + & a(\{0,m_1+1,m_2\},n-1) & + & a(\{0,m_1,m_2-1\},n-1) \\
 & & \uparrow\uparrow\rightarrow & & \rightarrow\uparrow\rightarrow \\
 & + & a(\{0,m_1,m_2+1\},n-1) & + & a(\{0,m_1-1,m_2\},n-1) \\
 & & \uparrow\rightarrow\rightarrow & & \rightarrow\rightarrow\rightarrow \\
 & + & a(\{0,m_1+1,m_2+1\},n-1) & + & a(\{0,m_1,m_2\},n-1)
\end{array}
$$

$$
\text{BCs:}\quad a(\{0,m_1,m_2\},n) = \begin{cases} 0 & m_1 = 0 \text{ or } m_2 = m_1 \text{ or } n < 0 \\ 1 & m_1 = 1 \ \& \ m_2 = 2 \ \& \ n = 0 \\ 0 & (m_1 > 1 \text{ or } m_2 > 2) \ \& \ n = 0 \end{cases}
$$

(New? Chung et al give an alternative? system derived from a tree interpretation.)

$$
\begin{aligned}
a(\{0,1,2\},n) &= a(\{0,1,2\},n-1) + a(\{0,0,1\},n-1) + a(\{0,2,2\},n-1) + a(\{0,1,1\},n-1) \\
&\quad + a(\{0,1,3\},n-1) + a(\{0,0,2\},n-1) + a(\{0,2,3\},n-1) + a(\{0,1,2\},n-1) \\
&= 2a(\{0,1,2\},n-1) + a(\{0,1,3\},n-1) + a(\{0,2,3\},n-1) \\
&= \{1,2,6,22,92,422,\ldots\} \quad \text{(Baxter, A001181)}
\end{aligned}
$$

(The correspondence between 3-paths and Baxter numbers was established by Dulucq & Guibert by use of two bijections.)

$$
a(\{0,1,3\},n) = a(\{0,2,3\},n) = \{0,\ 1,\ 5,\ 24,\ 119,\ 615,\ 3303,\ldots\} \quad \text{(Not in Sloane)}
$$

(Baxter convolution?)

Matrices:

$$
\begin{pmatrix}
a(\{0,1,2\},n) & a(\{0,1,3\},n) & \cdots & a(\{0,1,n-1\},n) & a(\{0,1,n\},n) \\
0 & a(\{0,2,3\},n) & \cdots & a(\{0,2,n-1\},n) & a(\{0,2,n\},n) \\
\vdots & 0 & & \vdots & \vdots \\
 & & 0 & a(\{0,n-2,n-1\},n) & a(\{0,n-2,n\},n) \\
0 & \cdots & & 0 & a(\{0,n-1,n\},n)
\end{pmatrix} = \ldots
$$

(work in progress)

4-paths

•1								
•1	•70							
•1	•35	•490						
•1	•15	•105	•490					
•1	•5	•15	•35	•70				
○1	•1	•1	•1	•1	•1			
	○	•	•	•	•	•		
		○	•	•	•	•	•	
			○	•	•	•	•	•

$p(\{0,1,2,3\},n,k)$ - the number of 4-paths from four consecutive diagonal LH points $(\circ)$ to the four consecutive diagonal points with top left point at (n up, k across), $n,k \geq 0$

Recurrences for p and a each have 16 terms and appear to be new recurrence systems for these triangles and sequences.

$$a(\{0,1,2,3\},n) = \{1, 2, 7, 32, 177, 1122, ...\} \quad \text{(Hoggatt, A005362)}$$

***i*-paths**

Recurrences for *p* and *a* each have 2^i terms.

Consider the rotated $p(\{0,1,2,...,i-1\},n,k)$ triangles with terms $\binom{n}{k}_i$, $n,k \ge 0$:

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

$$\binom{n}{k}_1$$

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

$$\binom{n}{k}_2$$

1					
1	1				
1	4	1			
1	10	10	1		
1	20	50	20	1	
1	35	175	175	35	1

$$\binom{n}{k}_3$$

1					
1	1				
1	5	1			
1	15	15	1		
1	35	105	35	1	
1	70	490	490	70	1

$$\binom{n}{k}_4$$

$$\text{NB:}\quad p(\{0,1,2,...,i-1\},n,k) = \binom{n+k}{k}_i$$

These appear in Fielder & Alford as triangles derived from successive columns of Pascal's triangles, connection with *i*-paths is not given.

Gessel and Viennot (see also Benjamin & Cameron) give

$$\binom{n}{k}_i = \left| \begin{pmatrix} \binom{n}{k} & \binom{n}{k+1} & \cdots \\ \binom{n+1}{k} & & \\ \vdots & & \end{pmatrix} \right|_{i\times i}$$

By inspection (and see Fielder & Alford),

$$\binom{n}{k}_i = \prod_{j=1}^{k} \frac{\binom{n+i-j}{i}}{\binom{j+i-1}{i}}$$

which generalizes the binomial formula as follows

$$\binom{n}{k} = \frac{n\times(n-1)\times\ldots\times(n-k+1)}{k\times(k-1)\times\ldots\times 1} = \frac{\binom{n}{1}\binom{n-1}{1}\ldots\binom{n-k-1}{1}}{\binom{k}{1}\binom{k-1}{1}\ldots\binom{1}{1}}$$

$$\rightarrow \binom{n}{k}_i = \frac{\binom{n}{i}\binom{n-1}{i}\ldots\binom{n-k-1}{i}}{\binom{k}{i}\binom{k-1}{i}\ldots\binom{1}{i}}.$$

(New?) As a short hand, $\binom{n}{k}_i = \frac{{}_k(n)^i}{{}_k(k)^i}$ where ${}_k(n)^i$ is the product of terms "*n*: *k* left, *i* up" and ${}_0(n)^i \overset{\Delta}{=} 1$. For example,

$$\binom{5}{4}_3 = \frac{{}_4(5)^3}{{}_4(4)^3} = \frac{\begin{pmatrix} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{pmatrix}}{\begin{pmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}} = 35 \quad \text{and} \quad \binom{5}{2}_4 = \frac{{}_2(5)^4}{{}_2(2)^4} = \frac{\begin{pmatrix} 7 & 8 \\ 6 & 7 \\ 5 & 6 \\ 4 & 5 \end{pmatrix}}{\begin{pmatrix} 4 & 5 \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}} = 490.$$

Row sums:

1	2	3	4?	...				n (0-paths)
1	2	4	8	16	32	64	128...	2^n
1	2	5	14	42	132	429	1430...	Catalan A000108
1	2	6	22	92	422	2074	10754...	Baxter A001181
1	2	7	32	177	1122	7898	60398...	Hoggatt A005362
1	2	8	44	310	2606	25202	272582...	Hoggatt A005363
1	2	9	58	506	5462	70226	1038578...	Hoggatt A005364

$$a(\{0,1,2,...,i-1\},n)\,,\ 0\le i\le 6\,,\ 0\le n\le 7$$

(New?) By inspection, n^{th} column successive differences become constant, including with 1st row n and even with previous row ($i=-1$) of $\{1,2,2,2,\ldots\}$.

Convolution formulae:

i	**Row sum** $a_n = a(\{0,1,2,...,i-1\},n)$		**Diagonal sum**	
1	$a_n = a_{n-1} + a_{n-1}$ $\{1,2,4,8,16,32,64,...\}$	2^n	$a_n = a_{n-1} + a_{n-2}$ $\{1,1,2,3,5,8,13,...\}$	Fibonacci A000045
2	$a_n = a_{n-1} + \sum_{k=1}^{n-1} a_{k-1}a_{n-1-k} + a_{n-1}$ $\{1,2,5,14,42,132,429,...\}$	Catalan A000108	$a_n = a_{n-1} + \sum_{k=1}^{n-2} a_{k-1}a_{n-2-k} + a_{n-2}$ {1, 1, 2, 4, 8, 17, 37,...}	Generalized Catalan A004148
3	Convolution type formula? {1, 2, 6, 22, 92, 422, 2074,...}	Baxter A001181	Just decrease subscript? {1, 1, 2, 5, 12, 31, 87,...}	Generalized Baxter?
4	? {1, 2, 7, 32, 177, 1122, 7898,...}	Hoggatt A005362	? {1, 1, 2, 6, 17, 51, 177,...}	Generalized Hoggatt?

For the $i=2$ row sum above, comparison with the earlier recurrence system gives

$$a(\{0,2\},n-1) = \sum_{k=1}^{n-1} a(\{0,1\},k-1)a(\{0,1\},n-1-k)\,.$$

Can this be interpreted from the path lattice? If so, can the interpretation be extended to $i=3$:

$$a(\{0,1,2\},n) = 2a(\{0,1,2\},n-1) + a(\{0,1,3\},n-1) + a(\{0,2,3\},n-1)$$

so

$$a(\{0,1,3\},n-1) \text{ and } a(\{0,2,3\},n-1) = \text{ convolution form in terms of } a(\{0,1,2\},\cdot)\,?$$

(New?) Note the easy way of obtaining the diagonal sum formulae for $i=1,2$ from the row sum formulae.

Related references

Benjamin AT, Cameron NT. Counting on Determinants. Am. Math. Monthly, Vol 112, No. 6, 481-492, 2005.

Chung FRK, Graham RL, Hoggatt VE, Kleiman M. The number of Baxter permutations. J. Combinatorial Theory Ser. A, 24:382-394, 1978.

Dulucq S, Guibert O. Baxter Permutations. Discrete Mathematics, 180:143-156, 1998.

Fielder DC, Alford CO. An investigation of sequences derived from Hoggatt sums and Hoggatt triangles, in G. E. Bergum et al., editors, Applications of Fibonacci Numbers: Proc. Third Internat. Conf. on Fibonacci Numbers and Their Applications, Pisa, Jul 25-29, 1988. Kluwer, Dordrecht, Vol. 3, 1990, pp. 77-88.

Fielder DC, Alford CO. On a conjecture by Hoggatt with extensions to Hoggatt sums and Hoggatt triangles, Fib. Quart., 27 (1989), 160-168.

Gessel I, Viennot G. Binomial determinants, paths, and hook length formulae. Adv. Math. 58 (1985), 300-321.

Hoggatt Jr VE, Bicknell M. Catalan and related sequences arising from inverses of Pascal's triangle matrices, Fib. Quart., 14 (1976), 395-405.

Shapiro LW. A Catalan triangle, Discrete Math. 14 (1976), no. 1, 83-90.