Number of tilings of a k X n rectangle
using 1 X 1 tiles, dominoes, right trominoes and/or 2 X 2 tiles

I) Using only right trominoes

1. Introduction

1a) When the rectangle is tiled from left to right, column by column, and one column is completely tiled, the next one is partially tiled. This partial tiling can be described by a binary code, the digit 0 stands for an empty cell and the digit 1 for a covered one. With k rows, the partial tiling of a column is a number p with k digits: $0 \le p \le 2^k - 1$.

$0 = (00..0)_2$ describes an empty column and $2^k - 1 = (11..1)_2$ a complete one.

Example k=2, n=6 with an auxiliary seventh column.

Binary codes

0	2	3	0	1	3	0
0	1	1	0	0	1	0
0	0	1	0	1	1	0

fig. 1a

Two successive partial tilings, encoded by p and q, form a block with code (h, p, q) where h is the height. In fig. 1a, we see 6 blocks: $(2,0,2)$ $(2,2,3)$ $(2,3,0)$ $(2,0,1)$ $(2,1,3)$ $(2,3,0)$. The last block, with the auxiliary column, corresponds to the third one. Generally, a k X n rectangle is tiled in n steps, block by block. Note that only covered cells in the right column of a block are marked with “1”.

The numbers which can follow a number p is written as a list $L(k, p) = [q_1, q_2, .., q_r]$:

$$L(2,0) = [1,2],\ L(2,1) = [3],\ L(2,2) = [3],\ L(2,3) = [0].$$

1b) Example k=3, n=2

0	7
0	1
0	1
0	1

0	7
0	1
0	1
0	1

fig. 1b

Here we have different blocks with the same code (3,0,7): $L(3,0) = [7,7]$. As the multiplicity has to be regarded, $L(k, p)$ is not a set. There is a difference between a block and its code.

1c) Graph k=2 k=3

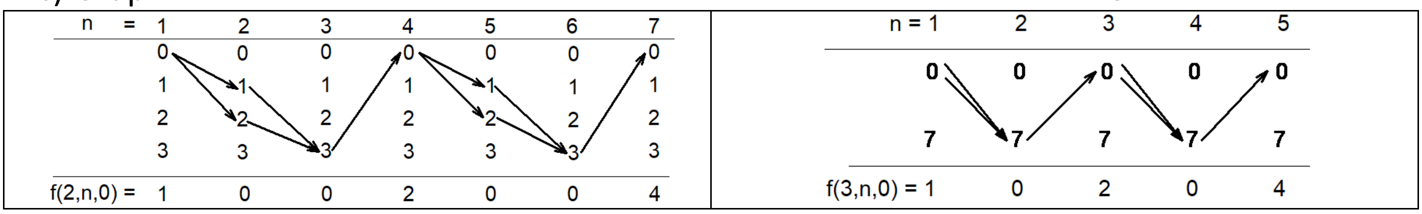


fig. 2

The lists $L(k, p)$, generated by the recurrence I,3), create a directed graph with multiple edges (fig. 2): The frequency $f(k, n, p)$ is the number of paths leading from the root 0 in the first column to p in the n-th column. The number of k X n tilings is $T(n, k) = f(k, n + 1, 0)$. $T(n, k) = f(k, n, 2^k - 1)$ is not true if 1 X 1 tiles ore dominoes are involved.

2. Irreducible blocks

According to 1b), an m X 2 block is a partial tiling of an m X 2 rectangle, $1 \le m \le k$.

Def.: A block is <u>irreducible</u> if any two adjacent rows have a common tile.
Example fig. 1a) The blocks in column 1,2 and 2,3 with codes (2,0,2) and (2,2,3) are irreducible, the block with code (2,3,0) in 3,4 is reducible: It can be split up into two sub-blocks with code (1,1,0). Such a block cannot stand alone because a tiling of a 1 X n rectangle does not exist, but it is needed in the recurrence 3).

0	2	3	0	1	3
0	1	1	0	0	1
0	0	1	0	1	1

fig. 3a

Examples for k=5

Columns	can be split up into blocks with code
1, 2	(3,0,7) and (2,0,2) both irreducible
2, 3	(3,3,0) reducible and (2,2,3) irreducible
3, 4	(4,1,14) reducible and (1,1,0) irreducible
4, 5	(3,3,0) reducible and (2,0,1) irreducible
5, 6	(3,0,7) and (2,1,3) both irreducible

fig. 3b

0	1	0	1	0	1
0	1	0	1	0	1
0	1	0	1	0	1
0	1	1	0	0	1
0	0	1	0	1	1
1	2	3	4	5	6

Fig. 3b shows all irreducible blocks as sub-blocks (an irreducible block with a height >3 does not exist). Codes: (1,1,0), (2,0,1), (2,0,2), (2,1,3), (2,2,3), (3,0,7), (3,0,7) generating the parameter lists $(s, p_2, L(s, p_2))$: $(1,1,[0])$, $(2,0,[1,2])$, $(2,1,[3])$, $(2,2,[3])$, $(3,0,[7,7])$.

<u>3. Recurrence for $L(k,p)$</u>

a) Splitting the code

In fig. 3, each 5 X 2 block is split up into an s X 2 irreducible block (on the bottom) and a remaining (5-s) X 2 block (on top). Generally, for s≤k, any k X 2 block can unambiguously split up into an irreducible s X 2 block$_2$ and a (k-s) X 2 block$_1$. Correspondingly, the code (k,p,q) can be split up into (s,p_2,q_2) and $(k-s,p_1,q_1)$ with $p = p_1 \cdot 2^s + p_2$ and $q = q_1 \cdot 2^s + q_2$. For $s = k$: $p_1 = q_1 = 0$. The following recurrence is based on this split.

b) Recurrence (pseudo code) for the parameter lists $L(k,p)$, $p = 0, \dots, 2^k - 1$:

```
L(0,0)=0
for m = 1 to k
 for p = 0 to 2^m-1
  for each (s,p2,L(s,p2))
   (t=2^s, p1=(p-p2)/t,
    if s<=m and mod(p,t)=p2 then
     for each x ∈ L(m, p1) and each y ∈ L(s,p2)
      append t*x+y to L(m,p)
```

<u>4) Recurrence for generating the frequency $f(k,n,p)$</u>

a) Example k= 4 with the lists $L(4,p)$, generated by 3b):

p	0	1	2	3	4	5	6	7
L(4,p)	10, 6, 9, 5	14, 14, 11, 7	11, 7	8, 4	14, 13	15	15	12
p	8	9	10	11	12	13	14	15
L(4,p)	7, 14, 13, 7	4, 2, 15	15	12, 6	2, 1	6, 3	3	0

$f(4,5,p) = 0$ at the beginning. Then (with $f_p = f(4,4,p)$ and the abbreviation $a{:}+b \; for \; a := a + b$):

$$f(4,5,10){:}+f_0,\, f(4,5,6){:}+f_0,\, f(4,5,9){:}+f_0,\, f(4,5,5){:}+f_0$$
$$f(4,5,14){:}+f_1,\, f(4,5,14){:}+f_1,\, f(4,5,11){:}+f_1,\, f(4,5,7){:}+f_1$$

and so on. The multiplicity of 14 in $L(4,1)$ is no problem: f_1 is added two times.

b) General recurrence (pseudo code)

```
f(k,0,2^k-1)=1,
for m=1 to n+1
 for p=0 to 2^k-1
  for j=1 to length(L(k,p))
   (q(j)= j-th element of L(k,p),
   f(k,m,q(j)) = f(k,m,q(j)) + f(k,m-1,p))
```

Result: The number of tilings of a k X n rectangle, using right trominoes (case 1), is

$$T(n,k) = f(k, n+1, 0) = f(k, n, 2^k - 1)$$

II) Using any combination of tiles

Any combination of 1 X 1 tiles, dominoes, right trominoes and/or 2 X 2 tiles has a special set of irreducible blocks which is the basis of the recurrences above.

1) The universal block

From fig. 3a we extracted an irreducible block with code (1,1,0): A full cell is followed by an empty one if the current row has no common tile with one of the adjacent rows. Therefore, this block belongs to any set of irreducible blocks.

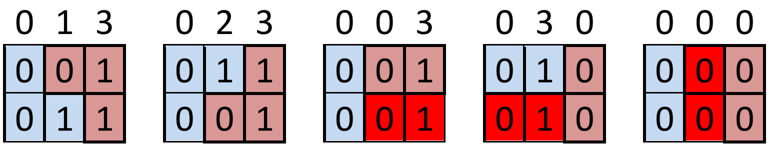
2) Priority rule

1 X 1 tiles and vertical dominoes do not spread over two columns. When such a thin tile occurs in the first column, it must be placed in the left column of the first block. This priority rule must be continued with the next blocks in order to prevent thin tiles from being placed two times. The rule leads to this modified definition:

A block is irreducible if any two adjacent rows have a common tile and no thin tile is placed in the right column.

3) Examples

Case 2: Right trominoes and dominoes



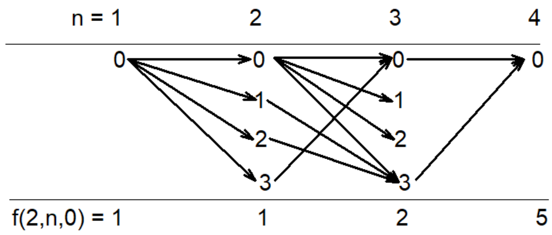


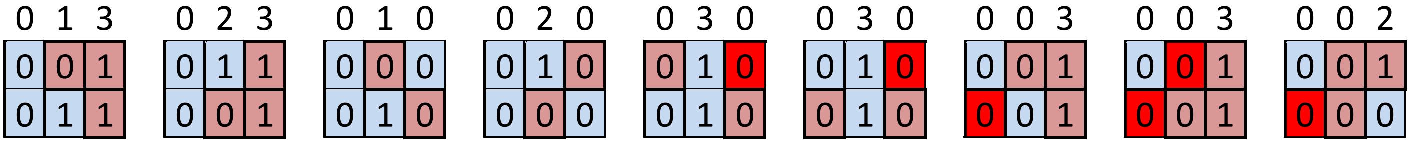
fig. 4a ↑ fig. 4b →

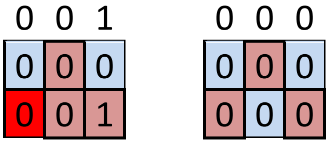
Number of of 2 X n tilings: $T(n,2) = f(2, n+1, 0)$.

$T(2,2) = 2, \quad T(2,3) = 5$

Case 3: Right trominoes and 1 X 1 tiles

Tilings of a 2 X 3 rectangle:





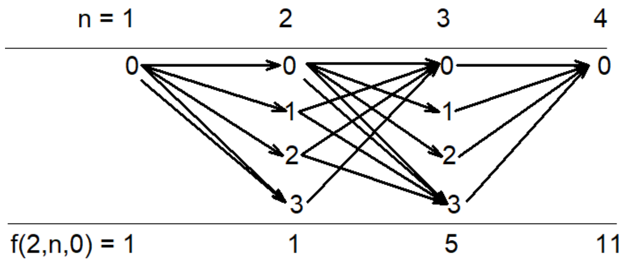


fig. 5a ↑ fig. 5b →

Number of of 2 X n tilings: $T(n,2) = f(2,n+1,0) = \sum_{p=0}^{3} f(2,n,p)$.

$$T(2,2) = 5, \quad T(3,2) = 11$$

Note the application of the priority rule: 03 in the fifth tiling represents an irreducible block, placed as whole, whereas 01 in the third tiling indicates that the 1 X 1 tile is placed in the second step.

Case 4: Right trominoes, dominoes and 1 X 1 tiles

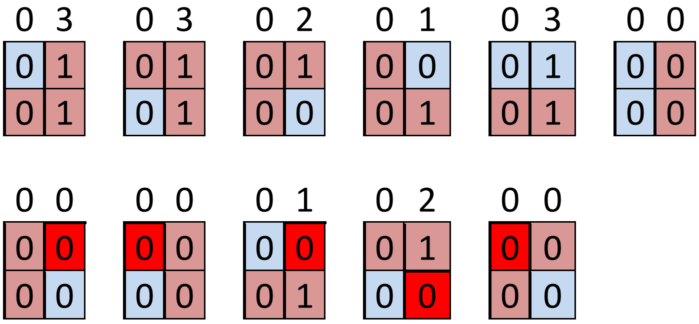


fig. 6a

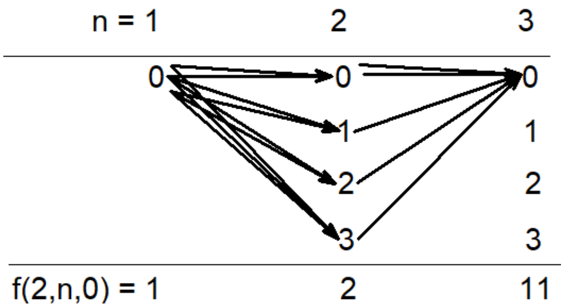


fig. 6b

4) <u>Generalizing the examples</u>

Each arrow from p (n-th column) to 0 ((n+1)-th column) indicates one way of completing the partial tiling p.

Case 1 (only trominoes and/or 2 X 2 tiles): There is just one arrow from $p = 2^k - 1$ to 0.

Therefore, $T(n,k) = f(k,n+1,0) = f(k,n,2^k - 1)$

For the other cases, let $S(n,k) = \sum_{p=0}^{2^k-1} f(k,n,p)$.

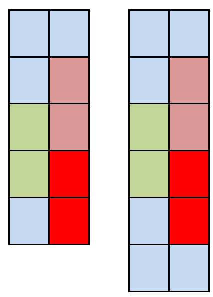
Each partial tiling of the n-th column can be completed by

- dominoes in at most one way: $T(n,k) \le S(n,k)$ (case 2)
- 1 X 1 tiles in exactly one way: $T(n,k) = S(n,k)$ (case 3)
- 1 X 1 tiles and dominoes in at least one way : $T(n,k) \ge S(n,k)$ (case 4)

III Irreducible blocks and corresponding parameters

The recurrence I, 3b) can be used for each combination of 1 X 1 tiles, dominoes, right trominoes and/or 2 X 2 tiles by selecting the block parameters from the table below. Example (right trominoes, 1 X 1 and 2 X 2 tiles):

$$(1, 1, [0])\ \ (1, 0, [0])\ \ (2, 0, [1{,}2{,}3{,}3{,}3])\ \ (2, 1, [3])\ \ (2, 2, [3])\ \ (3, 0, [7{,}7])$$

Tiles	Irreducible blocks	Block parameters	Comments
1) Any tiles	1 0	$(1,1,[0])$	→ Universal block, used for $L(m, 2^m - 1) = [0]$, $2 \le m \le k$
2) Right trominoes	0 1 0 0 0 0 0 1	$(2,0[2,1])$	
	0 1 1 1 1 1 0 1	$(2,1,[3])$ $(2,2,[3])$	
	0 1 0 1 0 1 0 1 0 1 0 1	$(3,0,[7{,}7])$	There is no irreducible block with 5 rows or more. The counter-examples below have dominoes in both columns. This contra-dicts the priority rule. 
3) 2 X 2 tiles	0 1 0 1	$(2,0,[3])$	
4) 1 X 1 tiles	0 0	$(1, 0, [0])$	
5) Dominoes	0 1 0 0 0 0	$(1,0,[1])$ $(2,0,[0])$	
6) Right trominoes and 1 X 1 tiles	0 1 0 1 0 1 0 1	$(2,0,[3{,}3])$	
7) Right trominoes and dominoes	0 1 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 1 0 1	$(3,0,[3{,}6])$ $(4, 0, [15])$	