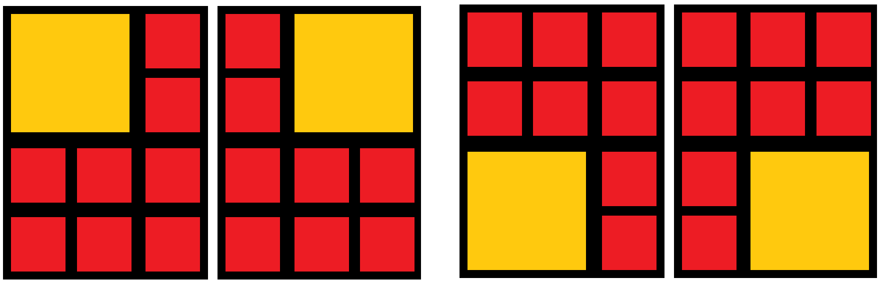
Milan, December 2022

This document discusses the counting of distinct tilings of m*n rectangles using squares of integer length sides. In particular, it focuses on "free" tilings, as opposed to "fixed" tilings.

In the context of free tilings, two or more distinct fixed tilings that are mappable on to each other by rotation or reflection are considered identical.

The following are 4 distinct fixed tilings that correspond to just 1 free tiling:



The following OEIS sequences give the numbers of fixed and free tilings of rectangles of width m for m from 2 through 10.

	2	3	4	5	6	7	8	9	10
Fixed	A000045	A002478	A054856	A054857	A219925	A219926	A219927	A219928	A219929
Free	A001224	A359019	A359020	A359021	A359022	A359023	A359024	A359025	A359026

The correspondence between fixed and free tilings depends on their symmetry, according to the following table. It should be noted however that these rules apply to m*n rectangles where m is not equal to n. Squares have specific rules that will be discussed later.

Symmetry of tilings by integer-sided squares of non-square rectangles, along with the contribution of each free tiling to the number of fixed tilings:

Symmetry	Contribution
Only reflective symmetry in the vertical axis	2
Only reflective symmetry in the horizontal axis	2
Only 180 degree rotational symmetry	2
All of the above symmetries	1
No symmetry	4

As a consequence, for any specific width of rectangle, we can say that (Formula 1):

free(n) = excl_v(n)/2 + excl_h(n)/2 + excl_rot(n)/2 + all_sym(n) + asym(n)/4

That is, the number of free tilings of rectangles of height n can be calculated, using the appropriate divisors, according to the number of fixed tilings that have respectively, exclusively vertical reflective symmetry, exclusively horizontal symmetry, etc.

One easy way, to count tilings of a rectangle of size m*n (for m != n), is to build a recursive function that works as follows:

Find the first free square in the rectangle (starting at the lowest row, and the leftmost position of that row).

If no free square is found, add 1 to the appropriate counter (if the tiling has all symmetries, add 1 to counter1; if no symmetries, add 1 to counter4; if just 1 symmetry, add 1 to counter2).

Otherwise, for each possible square size, try to add, at the free position found, a square of the that size. If successful, restart the procedure recursively.

When all tilings have been found, the sum of the counters will give fixed(n) for the specific width m.

Further, we can divide each counter by the appropriate factor, and their sum will then be free(n).

When examining tilings for an m*n rectangle where m=n, it is necessary to use a different symmetry table, with different ratios between free and fixed tilings:

Symmetry	Contribution
Only one reflective symmetry (vertical, horizontal, diagonal)	4
Precisely two reflective symmetries (vertical and horizontal, or both diagonals), and consequent 180 degree rotational symmetry	2
Only 180 degree rotational symmetry	4
Only 90 degree rotational symmetry	2
All of the above symmetries	1
No symmetry	8

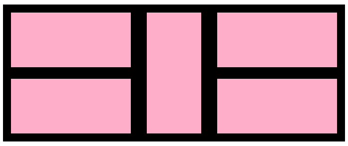
Note that throughout this document, we refer to rectangles that are "vertical". That is, they have a base of width m, and a height of n. The following paragraph, on width 2 rectangles, is the only exception.

Furthermore, by "vertical symmetry", we intend symmetry in the vertical axis, etc.

Counting free tilings of rectangles of width 2.

The number of free tilings in this case is given by the OEIS sequence A001224. This had already been shown to correspond to “the number of tilings of a 2 x n rectangle with dominoes when left-to-right mirror images are not regarded as distinct”. It is obvious that the tiling consists of vertical dominoes and/or pairs of horizontal dominoes[1]. It is clear then that it is possible to map any such domino tiling to a square tiling and vice versa by considering that any vertical domino corresponds to 2 unit squares, and any pair of horizontal dominoes corresponds to a 2*2 square.

For example, the following domino tiling of a 5*2 rectangle



corresponds to this square tiling of the same rectangle.

As there is a one-to-one correspondence between the tilings, their number for any given n must be the same.

[1] In the context of the sequence A001224, the rectangle that the dominoes are tiling is horizontal rather than vertical.

Counting free tilings of rectangles of other widths.

Recall Formula 1 (for m!= n):

free(n) = excl_v(n)/2 + excl_h(n)/2 + excl_rot(n)/2 + all_sym(n) + asym(n)/4

We can also say that (obviously):

fixed(n) = excl_v(n) + excl_h(n) + excl_rot(n) + all_sym(n) + asym(n)

In the examination of polyominoes, we already know that the asymmetrical polyominoes are the most, and are more difficult to find. For this reason, we eliminate the asymmetrical tilings from the equations:

4*(free(n) - excl_v(n)/2 - excl_h(n)/2 - excl_rot(n)/2 - all_sym(n)) = fixed(n) - excl_v(n) - excl_h(n) - excl_rot(n) - all_sym(n)

And so:

free(n) = (fixed(n) + excl_v(n) + excl_h(n) + excl_rot(n) + 3*all_sym(n)) / 4

However, this may be further simplified by considering tilings that have, for example, "at least" vertical symmetry, instead of the tilings that have exclusively vertical symmetry. The formula then becomes (Formula 2):

free(n) = (fixed(n) + at_least_v(n) + at_least_h(n) + at_least_rot(n)) / 4

It is in fact easier to find tilings with "at least" vertical reflective symmetry than those with exclusively vertical symmetry.

Therefore, for any specific width m, if we had a formula for the number of fixed tilings, and formulae for tilings with the 3 symmetries, then we could find a formula for the free tilings of rectangles of that width.

Counting free tilings of rectangles of width 3.

In this case, fixed(n) is given by sequence A002478, which has various formulae, one of which reads:

a(n) = a(n-1) + 2*a(n-2) + a(n-3)

We will now try to find formulae for tilings possessing the 3 symmetries.

Tilings with vertical reflective symmetry.

It is clear that such tilings must consist of combinations of (i) rows of 3 unit squares, and (ii) squares of side 3.

For height n, therefore, the number of tilings with this symmetry is equal to the number of ways that n may be expressed as a sum of 1's and 3's, which is enumerated by sequence A000930[2].

In other words:

at_least_v(n) = A000930(n)

Tilings with horizontal reflective symmetry.

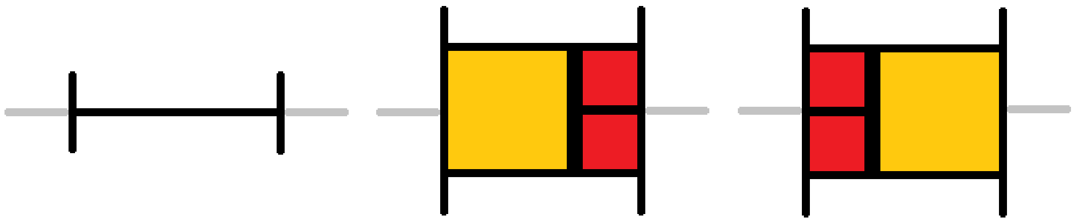
It is necessary to distinguish between rectangles of even and odd height.

1. Even height

It is clear that the axis of reflection must pass through one of the following combinations of tiles:

a. Through a clean separation of tiles above and below the axis;
b. Midway through a 2*2 square on the left, and on the right 2 unit squares;
c. As (b), the other way round.

These possibilities are shown here:



In the first case, above the line there may be any fixed tiling of size n/2, and below the line the same tiling reflected in the axis.

In the second and third cases, above the 2*2 square there may be any fixed tiling of size (n-2)/2, and below the line the same tiling reflected in the axis.

Therefore, for even n:

at_least_h(n) = A002478(n/2) + 2* A002478((n – 2)/2)

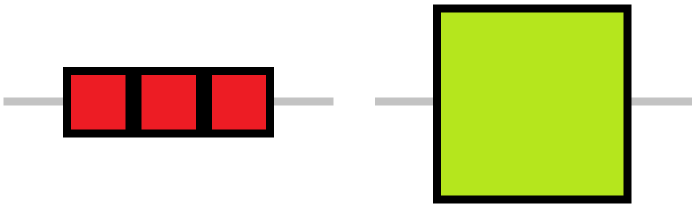
[2] For example, A000930(4) = 3, since 4=1+1+1+1=3+1=1+3. The sequence derives from the "Narayana Cows Sequence" which describes the growth of a herd of cows according to their reproductive cycle.

2. Odd height

The axis of reflection must pass through one of the following combinations of tiles:

a. A row of 3 unit squares
b. A 3*3 square

These possibilities are shown here:



Therefore, for odd n, following the same logic as used above:

at_least_v(n) = A002478((n-1)/2) + A002478((n – 3)/2)

Tilings with 180 degree rotational symmetry.

1. Even height

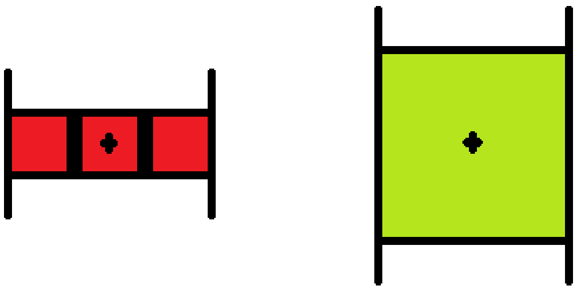
The centre of rotation must be on a line that separates cleanly the tiles above and below it. Above the line there may be any fixed tiling of height n/2, and below the line, the same tiling rotated 180 degrees.

Therefore, for even n:

at_least_rot(n) = A002748(n/2)

2. Odd height

The centre of rotation must be at the centre of a unit square or a 3*3 square, as illustrated in this diagram (equivalent to the diagram for horizontal reflective symmetry):



Therefore, for odd n:

at_least_rot(n) = A002478((n-1)/2) + A002478((n – 3)/2)

Conclusion.

For even n, recalling Formula 2:

free(n) = (A002478(n) + A000930(n) + A002478(n/2) + 2* A002478((n – 2)/2) + A002748(n/2)) / 4

And therefore, for even n:

free(n) = (A002478(n) + A000930(n) + 2 * A002478(n/2) + 2* A002478((n – 2)/2)) / 4

For odd n:

free(n) = (A002478(n) + A000930(n) + A002478((n-1)/2) + A002478((n – 3)/2) + A002478((n-1)/2) + A002478((n – 3)/2))/4

And therefore, for odd n:

free(n) = (A002478(n) + A000930(n) + 2*A002478((n-1)/2) + 2*A002478((n – 3)/2))/4

Some footnotes:

- The formulae are valid only for n > 1
- They may be further simplified by taking into account that A002478(n/2)=A000930(n)
- The sequences have been defined with offset 0; so free(0)=1 in that there is precisely one way to square tile a 3*0 rectangle, i.e., by using no tiles at all.

Counting free tilings of rectangles of width 4.

In this case, fixed(n) is given by sequence A054856, which has various formulae, one of which reads:

a(n) = 2*a(n-1)+3*a(n-2)-a(n-4)-a(n-5)

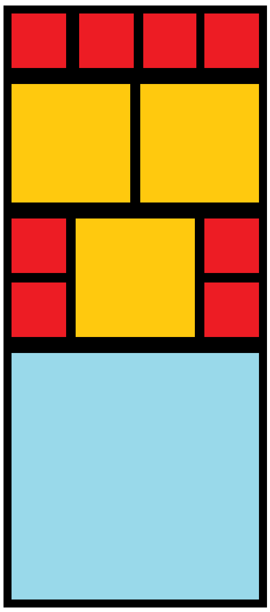
We will now try to find formulae for tilings possessing the 3 symmetries.

Tilings with vertical reflective symmetry.

It is clear that such tilings must consist of combinations of the following formations of tiles:

a. A row of 4 unit squares
b. A row of two 2*2 squares
c. In the middle a 2*2 square, flanked on the left and on the right by a vertical dominoes each of 2 unit squares
d. A 4*4 square

as shown in the following diagram:



The number of tilings of height n with this symmetry can then be deduced by counting the possible compositions of n as a sum of 1's, 2's, 2's and 4's. But there are two 2's in the list – one corresponding to the two 2*2 squares, the other to the 2*2 square in the middle. This can be resolved by expressing "2" in two different ways:

at_least_v(n) = compo(n, 1, 2, (1+1), 4)

That is, the number of tilings is given by the number of possibile combinations of 1, 2, (1+1) and 4 that have sum n.

For example, at_least_v(3)=5 because 3= 1+1+1 = 2+1 = 1+2 = 1+(1+1) = (1+1)+1.

There are several OEIS sequences related to compositions:

A000045(n+1) = compo(n, 1, 2)
A000073(n) = compo(n, 1, 2, 3)

A000078(n) = compo(n, 1, 2, 3, 4)
A000930(n) = compo(n, 1, 3)

However, there does not appear to be a sequence for compo(n, 1, 2, (1+1), 4). Initially, I had no formula for this function, and I generated its values by a recursive function that gets significantly slow around n = 40.

The algorithm for implementing the function compo can be made more efficient by counting just the compositions of n as a sum of 1's, 2's and 4's, but for each composition, count not just 1, but 2^k, where k is the number of 2's in the composition. At n=32, this gives a tenfold performance improvement.

Then Walter Trump gave me this formula for n > 4:

For v(n) = compo(n, 1, 2, (1+1), 4), v(n) = v(n-4) + 2*v(n-2) + v(n-1)

It was then possible to calculate the first 1000 values in less than 1 second.

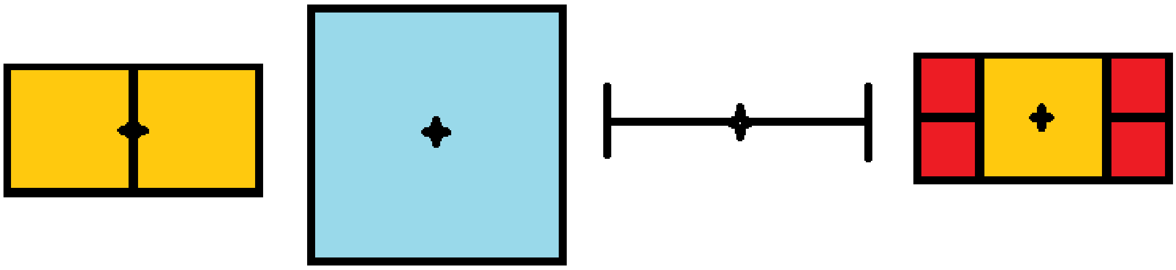
Tilings with 180 degree rotational symmetry.

1. Even height.

An even height 4*n tiling with 180 degree rotational symmetry must have its centre of rotation in of the following combinations of tiles:

a. Two 2*2 squares, side by side
b. A 4*4 square
c. A line that cleanly separates the tiles above and below the centre of rotation
d. a 2*2 square, flanked on the left and on the right by a vertical dominoes each of 2 unit squares

as shown in the following diagrams.



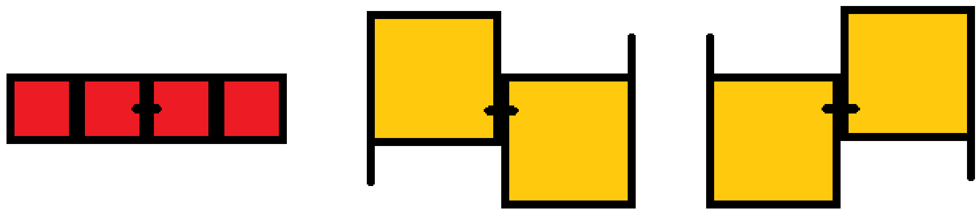
In this case, following the same logic used for the 3*n rectangles, we can see that for even n:

at_least_rot(n) = 2* A054856((n – 2)/2) + A054856((n – 4)/2) + A054856(n/2)

2. Odd height:

An odd height 4*n tiling with 180 degree rotational symmetry must have at its centre of rotation in of the following combinations of tiles:

a. a row of 4 unit squares
b. on the left, a 2*2 square, slightly above the centre, and on the right, the same, slightly below
c. as (b) but vice versa



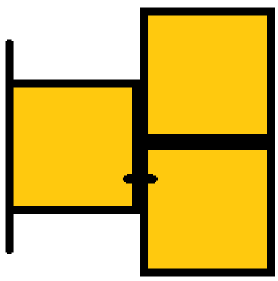
Obviously, the first combination causes a contribution of A054856((n – 1)/2).

In the second and third cases, the formula is more complex. We will examine the second case.

In the space on the right, just above the X, there can be either (i) 2 unit squares, or (ii) another 2*2 square.

In case (i), there is a line all the way across the tiling, so the contribution is A054856((n – 3)/2).

In case (ii), we find something like:



Again, in the space at the top left, there can be either (i) 2 unit squares, or (ii) another 2*2 square.

In case (i), the contribution is A054856((n – 5)/2). And so on until we reach the top of the rectangle. So the contribution of case (b) is:

$$\sum \text{A054856}(i) \text{ for } i = 0 \text{ thru } (n-3)/2$$

Case (c) is just a repeat of case (b).

In summary, we can therefore say that for odd n:

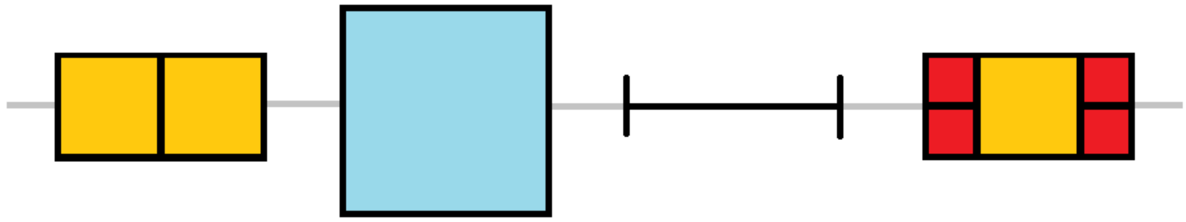
$$\text{at\_least\_rot}(n) = \text{A054856}((n-1)/2) + 2*\sum \text{A054856}(i) \text{ for } i = 0 \text{ thru } (n-3)/2$$

Tilings with horizontal reflective symmetry.

1. Even height

The possible midway formations are the same as those in the case of 180 degree rotational symmetry, plus two more:

e. A 2*2 square on the left, and a line dividing tiles on the right
f. As (e) but vice versa



By the same logic as before, we can say that for even n:

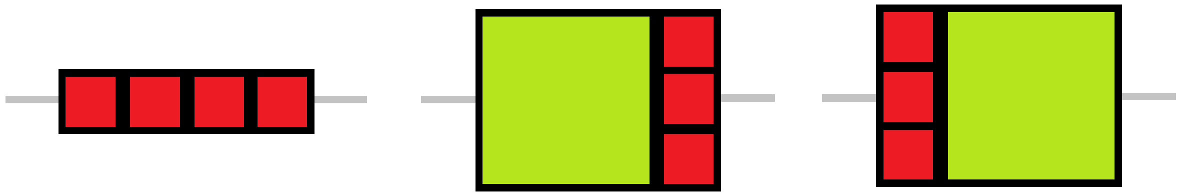
$$\text{at\_least\_h}(n) = 2 * \text{A054856}((n-2)/2) + \text{A054856}((n-4)/2) + \text{A054856}(n/2) +$$

$$2 * \sum \text{A054856}(i) \text{ for } i = 0 \text{ thru } (n-2)/2$$

2. Odd height

An odd height 4*n tiling with up-down reflective symmetry must have at its midway one of the following combinations of tiles:

a. a row of 4 unit squares
b. on the left, a 3*3 square, and on the right a strip of 3 unit squares
c. as (b) but vice versa



Therefore, for odd n:

at_least_h(n) = A054856((n – 1)/2) + 2* A054856((n – 3)/2)

Conclusion.

For even n, recalling Formula 2:

free(n) = (A054856(n) + compo(n, 1, 2, (1+1), 4) + 2* A054856((n – 2)/2) + A054856((n – 4)/2) + A054856(n/2) + 2* A054856((n – 2)/2) + A054856((n – 4)/2) + A054856(n/2) + 2*∑ A054856(i) for i = 0 thru (n-2)/2)/4

And therefore, for even n:

free(n) = (A054856(n) + compo(n, 1, 2, (1+1), 4) + 4* A054856((n – 2)/2) + 2*A054856((n – 4)/2) + 2*A054856(n/2) + 2*∑ A054856(i) for i = 0 thru (n-2)/2)/4

For odd n:

free(n) = (A054856(n) + compo(n, 1, 2, (1+1), 4) + A054856((n – 1)/2) +
2*∑ A054856(i) for i = 0 thru (n-3)/2 +
A054856((n – 1)/2) + 2* A054856((n – 3)/2))/4

And therefore, for odd n:

free(n) = (A054856(n) + compo(n, 1, 2, (1+1), 4) + 2*A054856((n – 1)/2) + 2*∑ A054856(i) for i = 0 thru (n-3)/2 + 2* A054856((n – 3)/2))/4

One footnote:

- The formulae are valid only for n > 4

Acknowledgements

I thank Craig Knecht for introducing me to the counting of tilings, and Walter Trump for one or two hints that simplified the formulae, and a formula for compo(1, 2, (1+1), 4).

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