A361030 is an integer sequence

Peter Bala, March 10 2023

Let $\mathrm{A}(n) = $ A361030$(n) = 20160\dfrac{(3n)!}{n!(n+3)!^2}$. We show that $\mathrm{A}(n)$ is an integer.

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Let $\mathrm{B}(n) = $ A007272$(n) = 60\dfrac{(2n)!}{n!(n+3)!}$. Let $\mathrm{C}(n) = $ A361038$(n) =$ $1680\dfrac{(3n)!}{(2n)!(n+3)!}$.

That $\mathrm{B}(n)$ and $\mathrm{C}(n)$ are integers follows from the easily verified identities

$$\mathrm{B}(n) = 10\binom{2n}{n} - 15\binom{2n}{n+1} + 6\binom{2n}{n+2} - \binom{2n}{n+3} \tag{1}$$

and

$$\mathrm{C}(n) = 280\binom{3n}{n} - 228\binom{3n}{n+1} + 54\binom{3n}{n+2} - 5\binom{3n}{n+3}. \tag{2}$$

Clearly, $\mathrm{A}(n) = \dfrac{1}{5}\mathrm{B}(n)\mathrm{C}(n)$. Thus to prove that $\mathrm{A}(n)$ is integral it is enough to prove that 5 divides $\mathrm{B}(n)\mathrm{C}(n)$. This is a consequence of the following result.

Proposition. *5 divides the integers* $\mathrm{C}(5n)$, $\mathrm{C}(5n+1)$, $\mathrm{C}(5n+2)$, $\mathrm{C}(5n+4)$ and $\mathrm{B}(5n+3)$.

Proof. From (2) we see that

$$\begin{aligned}\mathrm{C}(n) &\equiv 4\binom{3n}{n+2} - 3\binom{3n}{n+1} \pmod 5\\ &= \frac{5(n-2)(3n)!}{(n+2)!(2n-1)!} \pmod 5. \end{aligned} \tag{3}$$

Thus the integer

$$\mathrm{C}(5n) \equiv 5\frac{(5n-2)}{(10n-1)}\binom{15n}{5n+2} \pmod 5.$$

Since $10n-1$ is coprime to 5 for all n it must be the case that $10n-1$ divides the integer $(5n-2)\binom{15n}{5n+2}$ for all n. Hence $\mathrm{C}(5n)$ is divisible by 5. The remaining cases of the proposition are done exactly similarly.

From (3), the integer

$$C(5n+1) \equiv 5\frac{(5n-1)}{(10n+1)}\binom{15n+3}{5n+3} \pmod 5.$$

Since $10n+1$ is coprime to 5 for all n we conclude that $C(5n+1)$ is divisible by 5.

From (3), the integer

$$C(5n+2) \equiv 25\frac{n}{(10n+3)}\binom{15n+6}{5n+4} \pmod 5.$$

Since $10n+3$ is coprime to 5 for all n we conclude that $C(5n+2)$ is divisible by 5.

From (3), the integer

$$C(5n+4) \equiv 5\frac{(5n+2)}{(10n+7)}\binom{15n+12}{5n+6} \pmod 5.$$

Since $10n+7$ is coprime to 5 for all n we conclude that $C(5n+4)$ is divisible by 5.

Finally, from (1), the integer

$$\begin{aligned} B(n) &\equiv \binom{2n}{n+2} - \binom{2n}{n+3} \pmod 5 \\ &= \frac{5}{5n+3}\binom{2n}{n+2} \pmod 5. \end{aligned}$$

Hence the integer

$$B(5n+3) \equiv 5\frac{1}{(5n+6)}\binom{10n+6}{5n+1} \pmod 5.$$

Since $5n+6$ is coprime to 5 for all n we conclude that $B(5n+3)$ is divisible by 5. $\square$