The Number of Labelled Edge-3-coloured Cliques without Rainbow Triangles

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Abstract

In this thesis, we examine the characteristics of edge-3-coloured cliques without rainbow triangles, then obtain a recursive formula for the number of labelled graphs in this class based on these characteristics.

1 Introduction

The problem posed in this thesis belongs to the mathematical branch enumerative combinatorics, whose typical task is to find the number of objects with a given set of properties. [1] ”Many combinatorial objects of interest have a rich and interesting algebraic or geometric structure, which often becomes a very powerful tool toward their enumeration.” [1] In our context, the objects are the labelled edge-3-coloured cliques, the property is without rainbow triangles, and the characteristics of our graphs lead us toward their enumeration.

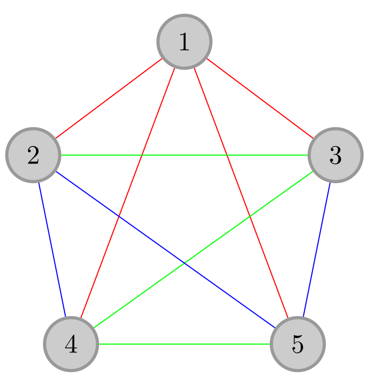
Definition 1 A *clique* is a complete simple undirected graph, i.e., a graph of the form $(V, \binom{V}{2})$.

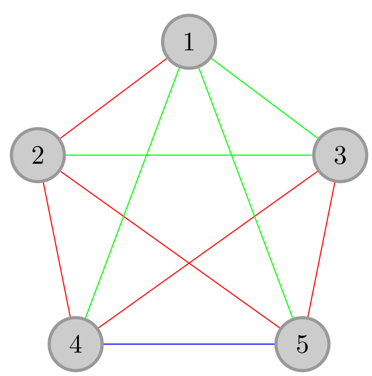
Definition 2 An *edge-k-colouring* of a graph $G = (V, E)$ is a map $\chi : E \to \{1, ..., k\}$.

Definition 3 A *rainbow triangle* of a graph with an edge-3-colouring χ is a set $\{a, b, c\}$ of three vertices, such that $\chi(\{a, b\})$, $\chi(\{b, c\})$, and $\chi(\{a, c\})$ are pairwise distinct.

In order to clarify as well as simplify the presentation of the studied objects, througout this thesis, we use $G = (V, E)$ to denote a labelled clique with edge-3-coloring without rainbow triangles, where $\chi : E \to \{\text{RED, GREEN, BLUE}\}$. And we also present G as the disjoint union of three subgraphs: $G_A = (V, A) := (V, \chi^{-1}(\text{RED}))$, $G_B = (V, B) := (V, \chi^{-1}(\text{GREEN}))$, $G_C = (V, C) := (V, \chi^{-1}(\text{BLUE}))$, namely, $G = G_A \dot\cup G_B \dot\cup G_C := (V, A \dot\cup B \dot\cup C)$, where $E = A \dot\cup B \dot\cup C$.

Below are two examples of G with 5 vertices. On the left, G_A is connected, while G_B and G_C are not. On the right, G_A and G_B are connected, while G_C is not.





[1] Bóna, Miklós. Handbook of Enumerative Combinatorics. CRC Press, 2015.

In Section 2.1, we will examine the connectivities of the subgraphs G_A, G_B, G_C and conclude that, when $|V| \geq 2$, not all three graphs are connected and not all three graphs are disconnected. This allows us to immediately narrow G with $|V| \geq 2$ down to two scenarios: exactly one subgraph is disconnected; exactly two subgraphs are disconnected.

In Section 2.2, we will focus on the scenario where exactly two subgraphs, say G_A, G_B, are disconnected, and conclude that $G_A \dot\cup G_B$ is also disconnected, and explore the relations among the vertex sets of connected components of $G_A, G_B, G_A \dot\cup G_B$.

In Section 3, we will develop the recursive formulas for the number of G upon the connected components of the one disconnected subgraph, when only one subgraphs is disconnected, or upon the connected components of the union of the two disconnected subgraphs, when two subgraphs are disconnected.

2 Characteristics of G

Although we define G as labelled, all the characteristics concluded in this section also apply to non-labelled edge-3-coloured cliques without rainbow triangles.

2.1 Connectivities of the subgraphs G_A, G_B, G_C

Before we start to examine the connectivities of the subgraphs, one heuristic lemma is needed.

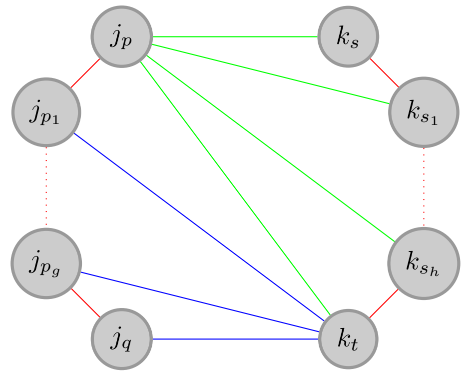
Lemma 1. *Between any two distinct connected components of a disconnected subgraph, all the edges are unicolour.*

Proof. Without loss of generality, we prove the lemma by proving: if $G_A = (V, A)$ is not connected, then between any two distinct connected components of G_A, either all edges are in B, or all of them are in C. Let $J := \{j_1, ..., j_m \mid m \geq 1\} \subseteq V, K := \{k_1, ..., k_n \mid n \geq 1\} \subseteq V$ be the vertex sets of two arbitrary distinct connected components of G_A. There're three possibilities of the numbers of vertices m and n:

1. $m = n = 1$. There is only one edge drawn between J and K, which is either green or blue;

2. $m = 1$, i.e., $J = \{j_1\}$ and $n \geq 2$. Let's first denote the set of red-neighbours of a vertex $k \in K$ as $N_k := \{k_i \mid \{k, k_i\} \in A\} \subseteq K$. If $\{j_1, k_1\} \in B$, then for $\forall k_i \in N_{k_1}$, $\{j_1, k_i\} \in B$. Suppose $\exists k_l \in N_{k_1}$, s.t. $\{j_1, k_l\} \in C$, then $\{j_1, k_1, k_l\}$ is a rainbow triangle. Furthermore, $\{\{j_1, k_s\} \mid k_s \in N_{k_i}, k_i \in N_{k_1}\} \subseteq B$ for the same reason. Since K is red connected, we can iteratively include all the vertices in K because the iteration corresponds exactly to the breadth search of vertices in $(K, \binom{K}{2} \cap A)$, and we conclude that $\{\{j_1, k_i\} \mid k_i \in K\} \subseteq B$;

3. $m, n \geq 2$. Suppose $\exists\, j_p, j_q \in J, \exists\, k_s, k_t \in K$, s.t. $\{j_p, k_s\} \in B, \{j_q, k_t\} \in C$. When $j_p = j_q$, or $k_s = k_t$, the second point above applies. When $j_p \neq j_q$ and $k_s \neq k_t$, since J is red-connected, K is red-connected, we can find a path $(j_p, j_{p_1}, ..., j_{p_g}, j_q)$ in $(J, \binom{J}{2} \cap A)$, and a path $(k_s, k_{s_1}, ...k_{s_h}, k_t)$ in $(K, \binom{K}{2} \cap A)$. $\{j_p, k_s\} \in B \Rightarrow \{j_p, k_{s_1}\} \in B \Rightarrow \cdots \Rightarrow \{j_p, k_{s_h}\} \in B \Rightarrow \{j_p, k_t\} \in B$. Similarly, $\{j_q, k_t\} \in C \Rightarrow \{j_{p_g}, k_t\} \in C \Rightarrow \cdots \Rightarrow \{j_{p_1}, k_t\} \in C$. And we end up with a rainbow triangle $\{j_p, k_t, j_{p_1}, \}$, as the figure below shows.



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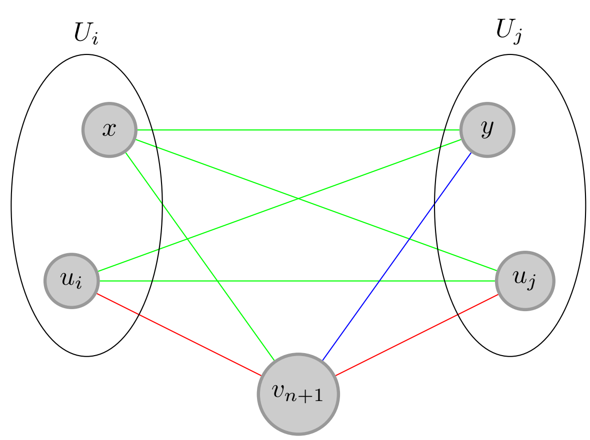
Now we can prove the lemmas about the connectivities of the subgraphs based on Lemma 1.

Lemma 2. *Not all three graphs G_A, G_B, G_C are connected, except when $|V| = 1$.*

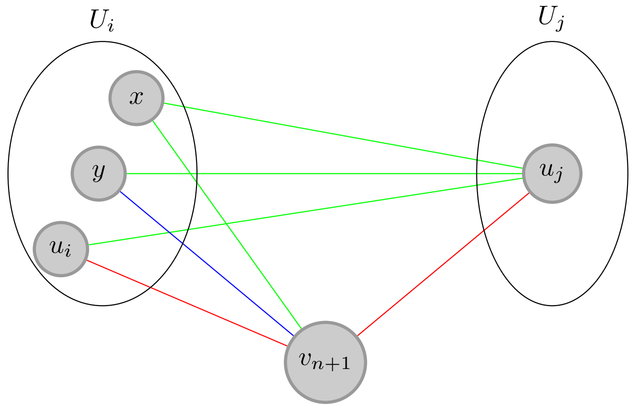
Proof. When $|V| = 0$, all three graphs are disconnected since a graph without vertices is not connected. When $|V| = 1$, all three graphs are connected since a graph with just one vertex is always connected. When $|V| = n \geq 2$, we use induction to prove that not all three graphs are connected.

- (Base case) n = 2: $|E| = 1$, only one of the graph is connected;
- (Induction hypothesis) For a edge-3-coloured clique $G = (V, E)$ without rainbow triangle and $|V| = n$, $V = \{v_1, ..., v_n\}$, we assume w.l.o.g, that G_A is disconnected and has k distinct connected components with vertex sets $U_1, ..., U_k$, where $2 \leq k \leq n$, and G_B, G_C being either connected or disconnected;
- (Induction step) We add a new vertex v_{n+1} and n new edges $\{v_1, v_n + 1\}, ..., \{v_n, v_{n+1}\}$ to G, forming a clique $\hat{G}$ with $n + 1$ vertices. Now we prove that it is impossible, that the colouring of these n new edges will make $\hat{G}$ red-connected, green-connected and blue-connected. Let's assume otherwise, then for the n edges coming out of v_{n+1}, there must be at least k red edges connecting $u_1 \in U_1$,...,$u_k \in U_k$, and 1 green edge connecting $x \in V$, 1 blue edge connecting $y \in V$, so it's necessary that $n \geq k+1+1$, which implies $k \leq n - 2$. For $k \leq n - 2$, either $\exists U_i, U_j \in \{U_1, ..., U_k\}$, s.t. $i \neq j, |U_i| \geq 2, |U_j| \geq 2$ or $\exists U_m \in \{U_1, ..., U_k\}$, s.t. $|U_m| \geq 3$. Therefor there exists two possible scenarios considering the relation between x and y:

  1. $x \in U_i, y \in U_j$ with $i \neq j, |U_i| \geq 2, |U_j| \geq 2$
     If $\{x, y\} \in B$, then after Lemma 1, $\{u_i, y\} \in B$, then $\{v_{n+1}, u_i, y\}$ is a rainbow triangle, as indicated by the figure below. Similarly, if $\{x, y\} \in C$, then $\{x, u_j\} \in C$, and $\{v_{n+1}, x, u_j\}$ is a rainbow triangle.



2. $x, y \in U_i$ with $|U_i| \geq 3$
Since $k \geq 2$, $\exists U_j \neq U_i$. If $\{x, u_j\} \in B$, then $\{y, u_j\} \in B$, and $\{v_{n+1}, y, u_j\}$ is a rainbow triangle, like the figure below shows. If $\{x, u_j\} \in C$, $\{v_{n+1}, x, u_j\}$ is a rainbow triangle.



Thus, it's impossible to colour the n new edges, s.t. $\hat{G}$ is connected with all three colours if G has at least one disconnected subgraph, completing our induction step.

□

Lemma 3. *Not all three graphs G_A, G_B, G_C are disconnected, except when $|V| = 0$.*

Proof. When $|V| = 0$, all three graphs are disconnected. When $|V| = 1$, all three graphs are connected. When $|V| > 1$, from Lemma 2 we conclude that not all three graphs are connected. Suppose w.l.o.g that (V, A) is disconnected with k distinct connected components with vertex sets $U_1, ..., U_k$, $2 \leq k \leq |V|$. Because of Lemma 1, for arbitrary two components $U_i \neq U_j$, $\{\{u_i, u_j\} \mid u_i \in U_i, u_j \in U_j\} \subseteq (B \dot{\vee} C)$. And this allows us to define a clique $H := (U, F)$, whose vertices are the vertex sets of the connected components of (V, A), i.e., $U := \{U_1, ..., U_k\}, F := \binom{U}{2}$, and a edge-2-colouring $\Omega : F \to \{\text{GREEN, BLUE }\}$, $\Omega(\{U_i, U_j\}) := \chi(\{x, y\})$ for any $x \in U_i, y \in U_j, \forall i \neq j$, and this is well defined. Further, let $H_b := (U, F_b := \Omega^{-1}(\text{GREEN})), H_c := (U, F_c := \Omega^{-1}(\text{BLUE}))$, then $F = F_b \dot{\cup} F_c, H = H_b \dot{\cup} H_c$. And the equivalences follow:

$$G_B = (V, B) \text{ is connected} \Leftrightarrow H_b = (U, F_b) \text{ is connected} \tag{3.1}$$
$$G_C = (V, C) \text{ is connected} \Leftrightarrow H_c = (U, F_c) \text{ is connected} \tag{3.2}$$

We prove (3.1), and (3.2) follows analogously.
($\Rightarrow$) For arbitrary $U_i, U_j \in U$, we can find $x, y \in V$, s.t. $x \in U_i, y \in U_j$. Since G_B is connected, there exists a path $\gamma := (x, ..., y)$ in G_B. Then we can construct a walk from U_i to U_j by adding all the $U_l \in U$, which γ visits sequentially (with possible stops, when two adjacent vertices in γ belong to the same connected component). And this walk lies in H_b because of the definition of Ω and H_b, and the existence of such walk in H_b proves that H_b is connected.
($\Leftarrow$) For arbitrary $x, y \in G_B$: when x, y lie in the same $U_i \in U$, there exists $U_{j \neq i} \in U$, s.t. $\Omega(\{U_i, U_j\}) =$ GREEN, since H_b is connected. Then for $\forall\, w \in U_j, (x, w, y)$ is a path in G_B; when $x \in U_i, y \in U_j$ with $i \neq j$, there exists a path $(U_i, U_{i_1}, U_{i_2}, ..., U_j)$ in H_b, then for any $x_{i_1} \in U_{i_1}$, any $x_{i_2} \in U_{i_2}, ..., (x, x_{i_1}, x_{i_2}, ..., y)$ is a path in G_B, so G_B is connected.

Having these equivalences, if we can prove that for any edge-2-colouring of $H = (U, F)$ where $|U| \geq 2$, at least H_b or H_c is connected, then it follows that at least G_B or G_C is connected. This can be done by induction:

1. When $k = 2$, either H_b or H_c is connected since there's only one edge.
2. Suppose H_b is connected for k vertices. We add a new vertex and k new edges to form a clique $\hat{H}$ with $k + 1$ vertices. When at least one of the new edges is coloured green, then $\hat{H}$ is green connected because H_b

is connected. When all the edges are coloured blue, then $\hat{H}$ is blue connected via the new $(k+1)$th vertex. Thus either H_b or H_c is connected.

□

2.2 When two subgraphs are disconnected

In this section, we first examine the relation between vertex sets of connected components of two disconnected subgraphs, then conclude the disconnectivity of the union of two disconnected subgraphs, and at last examine the relations between vertex sets of connected components of the disconnected subgraphs and their union.

Lemma 4. *Suppose G_A, G_B are disconnected. Let $x \in V$, let J be the vertex set of connected component of x in G_A, and K the vertex set of connected component of x in G_B. Then $J \subseteq K$ or $K \subseteq J$. Consequently, either $J = K$, or J is the disjunct union of vertex sets of more than one connected components of G_B including K, or K is the disjunct union of vertex sets of more than one connceted components of G_A including J.*

Proof. Suppose $J \nsubseteq K$ and $K \nsubseteq J$, then $\exists\, j \in J$, s.t. $j \notin K$, and $\exists\, k \in K$, s.t. $k \notin J$. And $x \in J \cap K$, so $x \neq j, x \neq k$. Since $x, j \in J$, there exists a path $\gamma = (x, x_1, ..., x_m, j)$ in G_A. Let $y \in \{x_1, ..., x_m, j\}$ be the vertex nearest to x in γ, s.t. $y \notin K$ (when $m = 0$, $y = j$), then y lies in another connected component of G_B, let $\hat{K}$ be the vertex set of that connected component. Because the vertex before y in γ, say z (when $m = 0$, $z = x$), lies in K and $\{z, y\} \in A$, after Lemma 1, $\{\{k_1, k_2\} \,|\, k_1 \in K, k_2 \in \hat{K}\} \subseteq A$. Since $k \in K, y \in \hat{K}$, so $\{k, y\} \in A$. But $y \in J$ at the same time, so $k \in J$, which contradicts our assumption. So $J \subseteq K$, or $K \subseteq J$. Consequently, either

1. $J = K$; or

2. $J \subsetneq K$, then for an arbitrary $\hat{j} \in K \backslash J$: $\hat{j} \in \hat{J} \neq J$, where $\hat{J}$ is the vertex set of another component of G_A. Then $\hat{J} \subseteq K$. Because if $K \subseteq \hat{J}$, then $J \subseteq \hat{J}$, which contradicts to the disjunction of the vertex sets of the connected components of G_A. So K is the disjoint union of vertex sets of more than one connected components of G_A including J, or

3. $K \subsetneq J$, then J is the disjoint union of vertex sets of more than one connected components of G_B including K. □

Lemma 5. *If $G_A = (V, A), G_B = (V, B)$ are disconnected, then $G_A \dot\cup G_B = (V, A \dot\cup B)$ is disconnected.*

Proof. Let J, J' be the vertex set of any two distinct connected components of G_A, where $J \cap J' = \emptyset$. Then for J, J' to be connected in $G_A \dot\cup G_B$, they must be the subset of the vertex set of the same connected component, say M of G_B, i.e., $J \subseteq M, J' \subseteq M$ (Lemma 4). But since G_B is also disconnected, there exists at least another connected component of G_B, whose vertex set is, say M', where $M' \cap M = \emptyset$, which implies $M' \cap J = M' \cap J' = \emptyset$. Then J and M' are disconnected in G_A as well as in G_B, hence as well as in $G_A \dot\cup G_B$. So $G_A \dot\cup G_B$ is disconnected. □

Lemma 6. *Suppose G_A, G_B are disconnected, then $G_A \dot\cup G_B$ is disconnected. Let $J_1, ..., J_m$; $K_1, ..., K_l$ be the vertex sets of the distinct connected components of G_A; G_B respectively, then there are three possibilities of the vertex sets $Z_1, ..., Z_r$ of the connected components of $G_A \dot\cup G_B$:*

1. *$Z_t = J_s = K_i$;*
2. *$Z_t = J_s$ is the disjoint union of more than one connected components of G_B;*
3. *$Z_t = K_i$ is the disjoint union of more than one connected components of G_A.*

Proof. Lemma 4 + Lemma 5. □

3 Recursive formula for the number of G

We define

- D_1 := scenario, where exactly one of the three subgraphs is disconnected;
- D_2 := scenario, where exactly two of the three subgraphs are disconnected;
- $p(n,k)$:= number of ways to partition n labelled vertices into k connected components;
- $f(n)$:= number of G with n vertices;
- $f_X(n)_{\{X\in\{A,B,C\}\}}$:= number of G with n vertices, where only the subgraph G_X is connected;
- $f_{XY}(n)_{\{X\neq Y\in\{A,B,C\}\}}$:= number of G with n vertices, where only the subgraphs G_X, G_Y are connected;
- $u_X(n_l)_{\{X\in\{A,B,C\}\}}$:= number of ways to form a n_l-vertices connected component of the disconnected subgraph G_X in D_1;
- $u_{XY}(n_l)_{\{X\neq Y\in\{A,B,C\}\}}$:= number of ways to form a n_l-vertices connected component of $G_X \dot\cup G_Y$ in D_2, where G_X, G_Y are disconnected.

Because of symmetry, $u_A(n_l) = u_B(n_l) = u_C(n_l)$, $u_{AB}(n_l) = u_{AC}(n_l) = u_{BC}(n_l)$, $f_A(n) = f_B(n) = f_C(n)$, $f_{AB}(n) = f_{AC}(n) = f_{BC}(n)$. So we have:

$$f(n) = 3f_A(n) + 3f_{AB}(n) \qquad n \geq 2 \tag{1}$$

In order to develop the recursive formula for $f(n)$, we partition the $n \geq 2$ labelled vertices into $k \geq 2$ connected components, assume each connected component is also an edge-3-colored clique without rainbow triangle, and connect the components in ways s.t. no rainbow triangle will arise. Consequently we need to

1. count $p(n,k)$ where $n \geq 2$, $k \geq 2$;
2. count number of ways to connect $k \geq 2$ connected components in D_1, D_2 respectively;
3. count $u_A(n_l), n_l \geq 2$ in D_1, $u_{AB}(n_l), n_l \geq 2$ in D_2 respectively;
4. fit the base case $u_A(1), u_{AB}(1), f_A(1), f_{AB}(1), f(1)$ into the formula.

3.1 Formula for $p(n,k)$ where $n \geq 2, k \geq 2$

Calculating $p(n,k)$ involves both labelled and unlabelled enumerations. To form two connected components for a vertex set $\{1,2,3,4\}$, while $\{1,2\},\{3,4\}$ and $\{1,3\},\{2,4\}$ are two different ways, $\{1,2\},\{3,4\}$ and $\{3,4\},\{1,2\}$ are the same way, i.e., the connected components are not labelled while the vertices are. We will first introduce the equations directly and do some explanations afterwards.

$$\begin{aligned}
p(n,k) = \sum_{\substack{n_1+\cdots+n_k=n \\ 1\leq n_1\leq\cdots\leq n_k<n}} \frac{\binom{n}{n_1,\ldots,n_k}}{\prod\limits_{\substack{i \\ n_{i1}=\cdots=n_{ij}}} j!} &= \sum_{\substack{n_1+\cdots+n_k=n \\ 1\leq n_1\leq\cdots\leq n_k<n}} \frac{n!}{n_1!\cdot\,\cdots\,\cdot n_k!} \frac{1}{\prod\limits_{\substack{i \\ n_{i1}=\cdots=n_{ij}}} j!} \\
= \sum_{\substack{\alpha_1 m_1+\cdots+\alpha_l m_l=n \\ 1\leq m_1<m_2<\cdots<m_l<n \\ \alpha_1,\ldots,\alpha_l\geq 1 \\ \alpha_1+\cdots+\alpha_l=k}} \frac{n!}{\prod\limits_{i=1}^{l}(m_i!)^{\alpha_i}\alpha_i!} &= \sum_{\substack{\alpha_1 m_1+\cdots+\alpha_l m_l=n \\ 1\leq m_1<m_2<\cdots<m_l<n \\ \alpha_1,\ldots,\alpha_l\geq 1 \\ \alpha_1+\cdots+\alpha_l=k}} \frac{k!}{\prod\limits_{i=1}^{l}\alpha_i!}\cdot\frac{n!}{k!\prod\limits_{i=1}^{l}(m_i!)^{\alpha_i}} \\
= \sum_{\substack{\alpha_1 m_1+\cdots+\alpha_l m_l=n \\ 1\leq m_1<m_2<\cdots<m_l<n \\ \alpha_1,\ldots,\alpha_l\geq 1 \\ \alpha_1+\cdots+\alpha_l=k}} \binom{k}{\alpha_1,\ldots,\alpha_l} \frac{n!}{k!\prod\limits_{i=1}^{l}(m_i!)^{\alpha_i}} &= \sum_{\substack{n_1+\cdots+n_k=n \\ 1\leq n_1,\ldots,n_k<n}} \frac{\binom{n}{n_1,\ldots,n_k}}{k!}
\end{aligned} \tag{2}$$

There are seven expressions in eq(2):

- in the second expression, each connected component is assigned a number n_i to denote the number of vertices in that component. Since the connected components are not labelled, we arrange them regarding n_i in an (not strictly) ascending order. In this way, no permutation occurs among connected components with different number of vertices. However, the connected components with same number of vertices do permute in the numerator because the ascending ordering of n_i is not strict, so the number of permutations among them must be included in the denominator;
- in the fourth expression, we group the connected components with same number of vertices. So connected components in the same group permute while among the different groups there is no permutation. And we adjust the expression accordingly;
- the sixth and seventh expressions equate because $\binom{k}{\alpha_1,...,\alpha_l}$ is exactly the number of ways to permutate among $n_1, ..., n_k$ in the seventh expression for a given $\alpha := (\alpha_1, ..., \alpha_l)$ and $n := (n_1, ..., n_l)$ in the sixth expression. After multiplying this number of permutations in the sixth expression, the sum in the seventh expression should include all the possible values of $n_1, ..., n_k$ as long as $1 \leq n_i < n$ with $n_1 + \cdots + n_k = n$;
- it is more suitable to develop a recursive algorithm after the fourth expression because it has less permutations hence is more efficient. However, the seventh expression paves the way of developing generating function from the recursive formula. We will shortly discuss about generating function in Section 3.6.

3.2 Number of ways to connect $k \geq 2$ connected components in D_1, D_2

3.2.1 In D_1

Let w.l.o.g G_B, G_C be connected, G_A disconnected. Then we can again define a labelled edge-2-coloured clique $H = H_b \dot\cup H_c$ exactly like in the proof of Lemma 3. Let's emphasize again, for any given G, H is uniquely defined, and conversely, a given H defines a unique way to connect the distinct connected components of G_A. So the number of ways to connect the components of G_A equals the number of edge-2-colouring of H. Since we require that both G_B, G_C are connected, with equivalence (3.1), (3.2), we conclude that the number of ways to connect the components of G_A s.t. G_B, G_C are connected equals the number of edge-2-colouring of H, s.t. H_b and H_c are connected, and this number can be deducted from the number of connected k-vertices-graphs d_k.
Let $h_b(k), h_c(k), g_k$ denote the number of k-vertices-H, where only H_b, only H_c, both H_b, H_c are connected respectively. Because of symmetry, $h_b(k) = h_c(k)$. Then $2h_b(k) + g_k = 2^{\binom{k}{2}}$, and $d_k = h_b(k) + g_k$. So

$$g_k = 2d_k - 2^{\binom{k}{2}} \tag{3}$$

Furthermore, d_k satisfies the following recurrence[2]:

$$k2^{\binom{k}{2}} = \sum_m \binom{k}{m} m d_m 2^{\binom{k-m}{2}} \quad (k \geq 1) \tag{4}$$

So there is also $g_k = 2d_k - 2^{\binom{k}{2}}$ ways to connect the $k \geq 2$ connected components of G_A s.t. G_B, G_C are connected, where d_k satisfies eq(4).

3.2.2 In D_2

Let w.l.o.g G_A, G_B be disconnected, and G_C connected. We need to count the number of ways to connect the k connected komponents of $G_A \dot\cup G_B$. Because of Lemma 6, there can be only blue edges between the connected components, so there is only one way to connect the k connected komponents of $G_A \dot\cup G_B$.

3.2.3 No rainbow triangle arises

In both cases, the counted ways of connecting will not produce any rainbow triangle. First, by the assumption of recursion, the connected components are without rainbow triangles. Second, for any three vertices not in the same connected component: if they are in two different components, among the three edges they form, there are two lying between the connected components, and we obey Lemma 1 when we connect, which ensures us that these two edges are the same colour, so maximal two colours appear in the three edges; if the vertices are in three different components, since we connect different components with either two colours for D_1, or with one colour for D_2, so no rainbow triangle will arise.

3.3 $u_A(n_l)$ in D_1 and $u_{AB}(n_l)$ in D_2 where $n_l \geq 2$

3.3.1 $u_A(n_l)$ in D_1 where $n_l \geq 2$

We assume G_A is disconnected. For $G_U := (U, \binom{U}{2}) \subseteq G$, where U is the vertex set of any connected component of G_A with $|U| = n_l \geq 2$, exactly one of the three situations appears:

1. $G_U \cap G_A$ connected, while $G_U \cap G_B, G_U \cap G_C$ disconnected, and the number of such $G_U \mathrel{\widehat{=}} f_A(n_l)$;
2. $G_U \cap G_A, G_U \cap G_B$ connected, $G_U \cap G_C$ disconnected, and the number of such $G_U \mathrel{\widehat{=}} f_{AB}(n_l)$;
3. $G_U \cap G_A, G_U \cap G_C$ connected, $G_U \cap G_B$ disconnected, and the number of such $G_U \mathrel{\widehat{=}} f_{AC}(n_l) = f_{AB}(n_l)$.

Consequently,

$$u_A(n_l) = f_A(n_l) + 2f_{AB}(n_l) = u_C(n_l) \qquad n_l \geq 2 \tag{5}$$

3.3.2 $u_{AB}(n_l)$ in D_2 where $n_l \geq 2$

Since $u_{AB}(n_l) = u_{BC}(n_l)$, we assume G_B, G_C are disconnected. For $G_U := (U, \binom{U}{2}) \subseteq G$, where U is the vertex set of any connected component of $G_B \dot\cup G_C$ with $|U| = n_l \geq 2$, exactly one of the three situations appears:

1. $G_U \cap G_B, G_U \cap G_C$ are connected, i.e., the connected component of G_B, G_C coincide, and the number of such $G_U \mathrel{\widehat{=}} f_{BC}(n_l) = f_{AB}(n_l)$;
2. $G_U \cap G_B$ connected, $G_U \cap G_C$ disconnected, with $G_U \cap G_A$ connected or disconnected, i.e., the connected component of G_B include more than one connected components of G_C, and the number of such $G_U \mathrel{\widehat{=}} f_B(n_l) + f_{AB}(n_l) = f_A(n_l) + f_{AB}(n_l)$;
3. $G_U \cap G_C$ connected, $G_U \cap G_B$ disconnected, with $G_U \cap G_A$ connected or disconnected, i.e., the connected component of G_C include more than one connected components of G_B, the number such $G_U \mathrel{\widehat{=}} f_C(n_l) + f_{AC}(n_l) = f_A(n_l) + f_{AB}(n_l)$.

Consequently,

$$u_{BC}(n_l) = 2f_A(n_l) + 3f_{AB}(n_l) \qquad n_l \geq 2. \tag{6}$$

3.4 Base case

Since no matter in D_1 or in D_2, there is only one way to form a one-vertex connected component, so $u_A(1) = u_{AB}(1) = 1$. In order that eq(5), eq(6) also apply for $n_l = 1$, we need $f_A(1) + 2f_{AB}(1) = 2f_A(1) + 3f_{AB}(1) = 1 \Leftrightarrow$

$$f_A(1) = -1, f_{AB}(1) = 1 \tag{7}$$

At the meantime, there is only one way to form a one-vertex-G, so

$$f(1) = 1 \tag{8}$$

3.5 Recursive formulas

3.5.1 Version one

The first version of recursive formula we develop is one more suitable for developing computer algorithm. We combine the fourth expression of eq(2) with eq(6), eq(7):

$$f_A(n) = \sum_{k=2}^{n} \sum_{\substack{\alpha_1 m_1 + \cdots + \alpha_l m_l = n \\ 1 \le m_1 < m_2 < \cdots < m_l < n \\ \alpha_1, \ldots, \alpha_l \ge 1 \\ \alpha_1 + \cdots + \alpha_l = k}} n! \prod_{i=1}^{l} \frac{\Big(2f_A(m_i) + 3f_{AB}(m_i)\Big)^{\alpha_i}}{(m_i!)^{\alpha_i} \alpha_i!}$$

$$\text{where } n \ge 2, f_A(1) = -1, f_{AB}(1) = 1 \quad (9)$$

Combining the fourth expression of eq(2) with eq(3), eq(4), eq(5), eq(7), we obtain:

$$f_{AB}(n) = \sum_{k=2}^{n} \sum_{\substack{\alpha_1 m_1 + \cdots + \alpha_l m_l = n \\ 1 \le m_1 < m_2 < \cdots < m_l < n \\ \alpha_1, \ldots, \alpha_l \ge 1 \\ \alpha_1 + \cdots + \alpha_l = k}} n! \cdot g_k \cdot \prod_{i=1}^{l} \frac{\Big(f_A(m_i) + 2f_{AB}(m_i)\Big)^{\alpha_i}}{(m_i!)^{\alpha_i} \alpha_i!}$$

$$\text{where } n \ge 2, f_A(1) = -1, f_{AB}(1) = 1, g_k = 2d_k - 2^{\binom{k}{2}}, k2^{\binom{k}{2}} = \sum_m \binom{k}{m} m d_m 2^{\binom{k-m}{2}} \quad (10)$$

3.5.2 Version two

The second version of recursive formula is more suitable for developing generating functions. We combine the seventh expression of eq(2) with eq(6), eq(7):

$$f_A(n) = \sum_{k=2}^{n} \sum_{\substack{n_1 + \cdots + n_k = n \\ 1 \le n_1, \ldots, n_k < n}} \frac{\binom{n}{n_1, \ldots, n_k}}{k!} \prod_{l=1}^{k} \Big(2f_A(n_l) + 3f_{AB}(n_l)\Big)$$

$$\text{where } n \ge 2, f_A(1) = -1, f_{AB}(1) = 1 \quad (11)$$

Combining the seventh expression of eq(2) with eq(3), eq(4), eq(5), eq(7), we obtain:

$$f_{AB}(n) = \sum_{k=2}^{n} \sum_{\substack{n_1 + \cdots + n_k = n \\ 1 \le n_1, \ldots, n_k < n}} \frac{\binom{n}{n_1, \ldots, n_k}}{k!} \cdot g_k \cdot \prod_{l=1}^{k} \Big(f_A(n_l) + 2f_{AB}(n_l)\Big)$$

$$\text{where } n \ge 2, f_A(1) = -1, f_{AB}(1) = 1, g_k = 2d_k - 2^{\binom{k}{2}}, k2^{\binom{k}{2}} = \sum_m \binom{k}{m} m d_m 2^{\binom{k-m}{2}} \quad (12)$$

3.6 The outcome and beyond

With help of computer program written based on the combination of eq(1), eq(8), eq(9), eq(10), the first 45 numbers of labelled edge-3-colored cliques without rainbow triangles can be computed within a minute, with $f(45)$ having the magnitude of 10^{298}. After comparing g_k with The On-Line Encyclopedia of Integer Sequences® (OEIS®)[3], we list the first 10 numbers of f_A, f_{AB}, f below.

	$f_A(n)$	$f_{AB}(n)$	$f(n)$
$n = 1$			1
$n = 2$	1	0	3
$n = 3$	7	0	21
$n = 4$	81	12	279
$n = 5$	1491	552	6129
$n = 6$	40989	29340	210987
$n = 7$	1654983	2283000	11813949
$n = 8$	103734729	300436668	1212514191
$n = 9$	11566289259	74748918888	258945624441
$n = 10$	2663874684261	37147677624540	119434656926403

Like mentioned at the end of Section 3.1 and in Section 3.5.2, it is possible to develop generating functions from eq(11), eq(12). Generating function is a powerful tool in both labelled and unlabelled enumerations in the sense that it allows us to obtain an explicit formula for the desired number upon solving, so that we can calculate the number much faster, gain further information concerning the asymptotic growth rate or other possible analytic characteristics.[4] Although not always solvable[5], generating function can be a promising attempt.

References

[1] Miklós Bóna. *Handbook of enumerative combinatorics.* CRC Press, 2015.

[2] Herbert S Wilf. *generatingfunctionology.* Academic Press, 1994.

[3] The on-line encyclopedia of integer sequences® (oeis®). `https://oeis.org/`. Accessed: 2023-12-15.

[4] Manuel Bodirsky. Combinatorics. Course Notes, TU Dresden. Available under https: //wwwpub.zih.tu-dresden.de/ bodirsky/Combinatorics.pdf, 2022.

[5] Herbert S Wilf. *generatingfunctionology.* Academic Press, 1994.