Syntactical analysis of total termination

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Abstract. Termination is an important issue in the theory of term rewriting. In general termination is undecidable. There are nevertheless several methods successful in special cases. In [5] we introduced the notion of *total termination*: basically terms are interpreted compositionally in a total well-founded order, in such a way that rewriting chains map to descending chains. Total termination is thus a semantic notion. It turns out that most of the usual techniques for proving termination fall within the scope of total termination. This paper consists of two parts. In the first part we introduce a generalization of *recursive path order* presenting a new proof of its well-foundedness without using Kruskal's theorem. We also show that the notion of total termination covers this generalization. In the second part we present some syntactical characterizations of total termination that can be used to prove that many term rewriting systems are not totally terminating and hence outside the scope of the usual techniques. One of these characterizations can be considered as a sound and complete description of totality of orderings on terms.

1 Introduction

Most of the usual techniques for proving termination of term rewriting systems (TRS's) make use of total term orders. In [5] this notion of *total termination* was investigated in detail, with the emphasis on the underlying ordinal theory. Here we provide a syntactical analysis of total termination. A typical property of total orders is that if f is a strictly monotone function and $f(a) > f(b)$, then $a > b$. The main topic of this paper is to characterize totality of an order by properties like this. These characterizations are useful to prove that a TRS is *not* totally terminating. For example, the TRS

$$f(g(x)) \to f(f(x))$$
$$g(f(x)) \to g(g(x))$$

is terminating. Assume it is also totally terminating. Then according to the above observation it would still be terminating if the outer f from the first rule and the outer g from the second rule were stripped, yielding

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$$g(x) \to f(x)$$
$$f(x) \to g(x)$$

which is clearly non-terminating. Hence the system is not totally terminating.

One way to define total termination is the following: a TRS is totally terminating if and only if there is a total well-founded order $>$ on ground terms closed under ground contexts such that $l\sigma > r\sigma$ for each rewrite rule $l \to r$ and each ground substitution σ. In practical applications it is very natural to require this totality: for example in Knuth-Bendix completion such a well-founded term ordering is required, and a highly desirable property is that all new critical pairs can be ordered by the ordering. Totality on non-ground terms can not be achieved since commutativity conflicts with well-foundedness; totality on ground terms is the strongest feasible requirement. The totality property is essential for unfailing completion strategies. In the case of ground AC-equational theories finitely presented, the existence of a reduction ordering AC-compatible and total on $\mathcal{T}(\mathcal{F})/ =_{AC}$ ensures that such theories always admit a canonical rewrite system. For more information on AC-compatible total orders see for example [13, 15]. Additionally most of the usual techniques for proving termination of TRS's like *polynomial interpretations* [11, 1], *elementary interpretations* [12], *Knuth-Bendix order* (KBO), prove in fact total termination.

In section 2 we give some basic definitions and properties over term rewriting in general and total termination in particular. The rest of the paper can be divided into two independent parts: section 3 on precedence based orders, and sections 4 and 5 on syntactical characterization of total termination.

In section 3 we present a slightly generalized version of the *recursive path order* (RPO). For this order we give a new proof of well-foundedness which is independent of Kruskal's theorem. We also show that the class of TRS's whose termination can be proved by RPO falls within the class of totally terminating TRS's. The same holds for other precedence based orders like the Knuth-Bendix ordering.

In section 4 we describe a characterization of total termination that is effective in the sense that it provides a powerful technique to prove that TRS's are not totally terminating. However, it is not a complete characterization: we construct a system that is not totally terminating, but can not be dealt with this technique. Such a system is rather tricky, and it is unlikely that it will appear in any application. In section 5 we describe a complete characterization of total termination: a system is totally terminating if and only if its rewrite relation is contained in a strict partial order having some syntactical properties. These properties cover the characterization of section 4. However, this new characterization is not effective any more.

2 Basic definitions and properties

Below we give some basic notions over TRS's. For more information the reader is referred to [3].