An Inductive Construction for Plane Laman Graphs via Vertex Splitting

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Abstract. We prove that all planar Laman graphs (i.e. minimally generically rigid graphs with a non-crossing planar embedding) can be generated from a single edge by a sequence of vertex splits. It has been shown recently [6,12] that a graph has a pointed pseudo-triangular embedding if and only if it is a planar Laman graph. Due to this connection, our result gives a new tool for attacking problems in the area of pseudo-triangulations and related geometric objects. One advantage of vertex splitting over alternate constructions, such as edge-splitting, is that vertex splitting is geometrically more local.

We also give new inductive constructions for duals of planar Laman graphs and for planar generically rigid graphs containing a unique rigidity circuit. Our constructions can be found in $O(n^3)$ time, which matches the best running time bound that has been achieved for other inductive contructions.

1 Introduction

The characterization of graphs for rigidity circuits in the plane and isostatic graphs in the plane has received significant attention in the last few years [2, 3,8,9,14]. The special case of planar graphs, and their non-crossing realizations has been a particular focus, in part because special inductive constructions apply [3], and in part because of special geometric realizations as pseudo-triangulations and related geometric objects [6,11,12,13].

In [3] it was observed that all 3-connected planar rigidity circuits can be generated from K_4 by a sequence of vertex splits, preserving planarity, 3-connectivity and the circuit property in every intermediate step, a result that follows by duality from the construction of 3-connected planar rigidity circuits by edge splits [2]. We extend this result in two ways. We show that all planar Laman graphs (bases for the rigidity matroid) can be generated from K_2 (a single edge) by a

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sequence of vertex splits, and all planar generically rigid graphs (spanning sets of the rigidity matroid) containing a unique rigidity circuit can be generated from K_4 (a complete graph on four vertices) by a sequence of vertex splits.

One advantage of vertex splitting over alternate constructions, such as edge-splitting, is that vertex splitting is geometrically more local. With very local moves, one can better ensure the planarity of a class of realizations of the resulting graph. This feature can be applied to give an alternate proof that each planar Laman graph can be realized as a pointed pseudo-triangulation, and that each planar generically rigid graph with a unique rigidity circuit can be realized as a pseudo-triangulation with a single non-pointed vertex.

A graph $G = (V, E)$ is a *Laman graph* if $|V| \geq 2$, $|E| = 2|V| - 3$, and

$$i(X) \leq 2|X| - 3 \quad (1)$$

holds for all $X \subseteq V$ with $|X| \geq 2$, where $i(X)$ denotes the number of edges induced by X. Laman graphs, also known as isostatic or generically minimally rigid graphs, play a key role in 2-dimensional rigidity, see [5,7,8,10,14,16]. By Laman's Theorem [9] a graph embedded on a generic set of points in the plane is infinitesimally rigid if and only if it is Laman. Laman graphs correspond to bases of the 2-dimensional *rigidity matroid* [16] and occur in a number of geometric problems (e.g. unique realizability [8], straightening polygonal linkages [12], etc.)

Laman graphs have a well-known inductive construction, called *Henneberg construction*. Starting from an edge (a Laman graph on two vertices), construct a graph by adding new vertices one by one, by using one of the following two operations:

(i) add a new vertex and connect it to two distinct old vertices via two new edges (*vertex addition*)

(ii) remove an old edge, add a new vertex, and connect it to the endvertices of the removed edge and to a third old vertex which is not incident with the removed edge (*edge splitting*)

It is easy to check that a graph constructed by these operations is Laman. The more difficult part is to show that every Laman graph can be obtained this way.

Theorem 1. *[14,8]. A graph is Laman if and only if it has a Henneberg construction.*

An embedding $G(P)$ of the graph G on a set of points $P = \{p_1, ..., p_n\} \subset R^2$ is a mapping of vertices $v_i \in V$ to points $p_i \in P$ in the Euclidean plane. The edges are mapped to straight line segments. Vertex v_i of the embedding $G(P)$ is *pointed* if all its adjacent edges lie on one side of some line through p_i. An embedding $G(P)$ is *non-crossing* if no pair of segments, corresponding to independent edges of G, have a point in common, and segments corresponding to adjacent edges have only their endvertices in common. A graph G is *planar* if it has a non-crossing embedding. By a *plane* graph we mean a planar graph together with a non-crossing embedding.