Polyhedral and Algorithmic Properties of Quantified Linear Programs*

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Abstract. Quantified linear programs (QLPs) are linear programs with variables being either existentially or universally quantified. The integer variant is PSPACE-complete, and the problem is similar to games like chess, where an existential and a universal player have to play a two-person-zero-sum game. At the same time, a QLP with n variables is a variant of a linear program living in $\mathbb{R}^n$, and it has strong similarities with multistage-stochastic programs with variable right-hand side. We show for the continuous case that the union of all winning policies of the existential player forms a polytope in $\mathbb{R}^n$, that its vertices are games of so called extremal strategies, and that these vertices can be encoded with polynomially many bits. The latter allows the conclusion that solving a QLP is in PSPACE. The hardness of the problem stays unknown.

1 Introduction

In the 1940s, linear programming arose as a mathematical planning model and rapidly found its daily use in many industries. However, integer programming, which was introduced in 1951, became dominant far later at the beginning the 1990s. Certainly, one reason for the delay of the integer programming success story stems from the fact that linear programming resides in the complexity class P, while integer programming is NP complete. Nowadays, we are able to solve very large mixed integer programs of practical size, but companies observe an increasing danger of disruptions, i.e., events occur which prevent companies from acting as planned. Therefore, there is a need for planning and deciding under uncertainty. Uncertainty, however, often pushes the complexity of traditional optimizations problems, which are in P or NP, to PSPACE. The quantified versions of linear integer programs cover the complexity class PSPACE. The relaxed versions, which we examine in this paper, additionally have remarkable polyhedral properties. The idea of our research is to explore the abilities of linear programming when applied to PSPACE-complete problems, similar as it was applied to NP-complete problems in the 1990s.

1.1 State-of-the-Art

For traditional deterministic optimization one assumes data for a given problem to be fixed and exactly known when the decisions have to be taken. However, data

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are often afflicted with some kinds of uncertainties, and only estimations, maybe in form of probability distributions, are known. Examples are flight or travel times. Throughput-time, arrival times of externally produced goods, and scrap rate are subject to variations in production planning processes. One possibility to deal with these uncertainties is to aggregate a given probability distribution to a single estimated number. Then, the optimum concerning these estimated input data can be computed with the help of traditional optimization tools. In some fields of application, as e.g. the fleet assignment problem of airlines, this procedure was successfully established. In other fields, like production planning and control, this technique could not be successfully applied, although mathematical models do exist [17]. Because of the intuitive complexity of even some deterministic models, its stochastic counterparts are often not considered. Instead, safety stock is introduced, demanded quantities are overestimated, and buffers are oversized. However, this resource intensive behaviour is contrasted by a couple of publications within the last decade, indicating that stochastic problems are not necessarily out of scope [7,12,15,14,19].

1.2 Complexity and Algorithmic Issues

From complexity theory, we know that many interesting optimization problems under uncertainty are PSPACE complete [16]. The negative results from complexity theory seem discouraging to find efficient (i.e. polynomial time) algorithms to solve optimization problems that are PSPACE complete. As a consequence, algorithmics mostly deal with restricted problems which allow finding a polynomial-time algorithm, or at least approximations [10,6]. The self-restriction to *efficient algorithms* (in a formal sense), however, is related to worst-case instances. This does not reflect the fact that the most astonishing and admired successes of computing intelligence are modeled as NP-complete problems (mixed integer linear programs) and PSPACE-complete problems (computer games like Chess [4]).

Contrasting the efficiency debate, we can interpret a set of expressions, which can encode a PSPACE-complete problem, as a very powerful modeling language: more powerful than necessary to encode any NP-complete problem. The fact that it is not possible to find polynomial time algorithms for all problems that are encoded with the help of such a powerful modeling language, leads to the consequence that research for new solutions must be driven from the application-side or even from the instances-side, as e.g. presented in [12]. Relatively unexplored are the abilities of linear programming extensions in the PSPACE-complete world. In this context, Subramani introduced the notion of quantified linear programs [20,21].

1.3 Solution Issues

Prominent solution paradigms for optimization under uncertainty are Dynamic Programming [2], Sampling [11], the exploration of Markov-Chains [22], Robust Optimization [13], and Stochastic Programming [3,7,18,5]. Markov-Chains and