Discrete Curvature Estimation Methods for Triangulated Surfaces

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Abstract. We review some recent approaches to estimate discrete Gaussian and mean curvatures for triangulated surfaces, and discuss their characteristics. We focus our attention on concentrated curvature which is generally used to estimate Gaussian curvature. We present a result that shows that concentrated curvature can also be used to estimate mean curvature and hence principal curvatures. This makes concentrated curvature one of the fundamental notions in discrete computational geometry.

Keywords: Curvature, Gaussian curvature, mean curvature, discrete curvature, triangulated surfaces.

1 Introduction

Curvature is one of the most important tools used in mathematics to understand the geometrical and metric structures of a surface. Gauss-Bonnet theorem [5] uses curvature to link the metric structure of a surface to its topology in such a way that the genus of the surface can be deduced from its total curvature.

In combinatorial geometry, the most common discrete representation for a surface is a triangle mesh. Triangle meshes are generated from sets of points on the surface by an implicit representation of the surface or by the discretization of parametric surfaces. Thus, the interest arises in developing discrete techniques for inferring geometrical, metric and topological properties of a surface from its discretization as a triangle mesh. The problem of curvature estimation on a triangulated surface has been extensively studied in mesh data processing, because of its numerous applications in shape modeling and analysis. Efficiency and accuracy are the major factors that led to the development of methods for estimating curvature in the discrete. Almost all methods for curvature estimation are region dependent and present stability issues while refining a mesh. A survey on curvature estimators can be found in [10].

In mathematics, concentrated curvature has been developed by Aleksandrov [3] in the middle of the last century as an intrinsic Gaussian curvature estimator for polyhedral surfaces. Concentrated curvature satisfies a discrete version of Gauss-Bonnet theorem which makes it an important tool for analyzing triangulated surfaces in combinatorial geometry. In the last decades, concentrated

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curvature returned up to date under some different variants, usually area dependent, and became a very relevant tool for curvature estimation.

The aim of this paper is to present a new method to discretely estimate mean curvature through concentrated curvature which was used until now to estimate Gaussian curvature. Consequently, principal curvatures can be deduced through concentrated curvature. Furthermore, concentrated curvature does not suffer from errors computation and has no stability issues when refining a mesh. We also present a review of recent approaches to curvature estimation, and we discuss their advantages and drawbacks. We also discuss experimental results.

The remainder of this paper is organized as follows. In Section 2, we present the theoretical background on the analytic definition of curvature. In Section 3, we present some related approaches to curvature estimation. In Section 4, we present Gaussian curvature and some related discrete approaches. In Section 5, we discuss approaches to mean curvature estimation. In Section 6, we describe how the notion of concentrated curvature applied to polygonal curves can be used to derive a discrete curvature for polygonal lines and we then define new Gaussian and mean curvatures by simulating the analytic case. In Section 7, we describe how concentrated curvature can be generalized to 3-dimensional manifolds and how its restriction to the boundary surfaces defines a new mean curvature estimator. In Section 8, we present our main result that describes how concentrated curvature can be used to compute mean curvature. In Section 9, we present some experimental results to compare the different curvature estimators. Finally, Section 10 draws some concluding remarks.

2 Background Notions

In this section, we briefly review some fundamental notions on curvature (see [5] for details). Let C be a curve having parametric representation $(c(t))_{t \in R}$. The curvature $k(p)$ of C at a point $p = c(t)$ is given by

$$k(p) = \frac{1}{\rho} = \frac{|c'(t) \wedge c"(t)|}{|c'(t)|^3},$$

where ρ, called the *curvature radius*, corresponds to the radius of the osculatory circle tangent to C at p.

Let S be a smooth surface (at least C^2). Let $\overrightarrow{n_p}$ be the normal vector to the surface at a point p. Let Π be the plane which contains the normal vector $\overrightarrow{n_p}$. Plane Π intersects S at a curve C containing p: the curvature k_p of C at point p is called *normal curvature* at p. When plane Π turns around $\overrightarrow{n_p}$, curve C varies. There are two extremal curvature values $k_1(p) \leq k_2(p)$ which bound the curvature values of all curves C. The corresponding curves C_1 and C_2 are orthogonal at point p [5]. These extremal curvatures are called *principal normal curvatures.* Since the surface is smooth, then *Euler formula* (also called *Dupin indicatrix*) indicates that the curvatures at a point p have an elliptic behavior described by $k(p) = k_1(p)\cos^2(\theta) + k_2(p)\sin^2(\theta)$, where parameter $\theta \in [0; 2\pi]$. The *Gaussian curvature* $K(p)$ and the *mean curvature* $H(p)$ at point p are the