A Fully Implicit Compressible Euler Solver for Atmospheric Flows *

Chao Yang[1,2] and Xiao-Chuan Cai[2]

[1] Institute of Software, Chinese Academy of Sciences, Beijing 100190, P. R. China, yangchao@iscas.ac.cn

[2] Department of Computer Science, University of Colorado at Boulder, Boulder, CO 80309, USA, chao.yang@colorado.edu, cai@cs.colorado.edu

1 Introduction

Numerical methods for global atmospheric modeling have been widely studied in many literatures [5, 7, 9]. It is well-recognized that the global atmospheric flows can be modeled by fully compressible Euler equations with almost no approximations necessary [7]. However, due to the multi-scale nature of the global atmosphere and the high cost of computation, other simplified models have been favorably used in most community codes.

There are two main difficulties in using fully compressible Euler equations in atmospheric flow simulations. One is that the fast waves in the equations lead to very restrictive stability conditions for explicit time-stepping methods; see, e.g., [11]. Another difficulty is that the flow is nearly compressible and the low Mach number results in large numerical dissipation errors in many classical numerical schemes.

To deal with the fast acoustic and inertio-gravity waves in the fully compressible model, we develop a fully implicit method so that the time step size is no longer constrained by the stability condition. And to treat the low-Mach number flow, an improved version of the Advection Upstream Splitting Method (AUSM$^+$-up, [8]) is adapted. This technique has been successfully employed for a shallow water model in [12]. In the fully implicit solver, we use an inexact Newton method to solve the nonlinear system arising at each time step; and the linear Jacobian system for each Newton step is then solved by a Krylov subspace method with an additive Schwarz preconditioner. We show by numerical experiments on a machine with thousands of processors that the parallel Newton-Krylov-Schwarz approach works well for fully compressible atmospheric flows.

* CY was supported in part by NSFC under 61170075 and 91130023, in part by 973 and 863 Programs of China under 2011CB309701 and 2010AA012301. XCC was supported in part by NSF under DMS-0913089 and EAR-0934647.

R. Bank et al. (eds.), *Domain Decomposition Methods in Science and Engineering XX*, Lecture Notes in Computational Science and Engineering 91,
DOI 10.1007/978-3-642-35275-1_81,

2 Governing Equations

Various formulations of the governing equations for mesoscale atmospheric models can be found in, e.g., [6]. In this paper, we focus on the compressible Euler equations by restricting the study on two dimensions (the $x-z$ plane) and omitting the Coriolis terms. The compressible Euler equations for the atmosphere take the following form

$$\frac{\partial Q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial z}+S=0,$$

where

$$Q=\begin{pmatrix}\rho\\ \rho u\\ \rho w\\ \rho\theta\end{pmatrix},F=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uw\\ \rho u\theta\end{pmatrix},G=\begin{pmatrix}\rho w\\ \rho wu\\ \rho w^2+p\\ \rho w\theta\end{pmatrix},S=\begin{pmatrix}0\\ 0\\ \rho g\\ 0\end{pmatrix}, \tag{1}$$

where $g = 9.80665\,\mathrm{m/s}^2$ is the effective gravity on the surface of the Earth. In the equation, the prognostic variables are the density ρ, the velocity (u,w) and the potential temperature θ of the atmosphere. The system is closed with the equation of state

$$p = p_{00}\left(\frac{\rho R\theta}{p_{00}}\right)^{\gamma},$$

where $p_{00} = 1013.25\,\mathrm{hPa}$ is the reference pressure on the surface, $R = 287.04\,\mathrm{J/(kg\cdot K)}$ is the gas constant for dry air and $\gamma = 1.4$. For the sake of brevity, we assume the computational domain Ω is a rectangle and the boundary conditions are given in Sect. 5. In some cases, a physical dissipation is added to the left-hand-side of the momentum and velocity equations. The dissipation term is $-\nabla\cdot(\nu\rho\nabla\phi)$ for $\phi = u$, w, and θ.

To recover the hydrostatic solution from the equation, instead of using (1) directly, the following shifted system is often preferred [6, 11]:

$$Q=\begin{pmatrix}\rho'\\ \rho u\\ \rho w\\ (\rho\theta)'\end{pmatrix},F=\begin{pmatrix}\rho u\\ \rho u^2+p'\\ \rho uw\\ \rho u\theta\end{pmatrix},G=\begin{pmatrix}\rho w\\ \rho wu\\ \rho w^2+p'\\ \rho w\theta\end{pmatrix},S=\begin{pmatrix}0\\ 0\\ \rho' g\\ 0\end{pmatrix} \tag{2}$$

where

$$\rho'=\rho-\bar\rho,\quad p'=p-\bar p,\quad (\rho\theta)'=\rho\theta-\bar\rho\bar\theta$$

and the variables with 'bar' satisfy the hydrostatic condition $\frac{\partial\bar p}{\partial z}=-\bar\rho g$ and $\bar\theta$ is obtained from the equation of state. It is clear that the flux Jacobian of the shifted system (2) in each spatial direction is, respectively,

$$\frac{\partial F}{\partial Q}=\begin{pmatrix}0&1&0&0\\ -u^2&2u&0&c^2/\theta\\ -uw&w&u&0\\ -u\theta&\theta&0&u\end{pmatrix},\quad \frac{\partial G}{\partial Q}=\begin{pmatrix}0&0&1&0\\ -wu&w&u&0\\ -w^2&0&2w&c^2/\theta\\ -w\theta&0&\theta&w\end{pmatrix},$$

where $c=\sqrt{\gamma p/\rho}$ is the sound speed.