Logical Semantics of Modularisation

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Abstract. An algebra of theories, signatures, renamings and the operations import and export is investigated. A normal form theorem for terms of this algebra is proved. Another algebraic approach and the relation with a fragment of second order logic are also considered.

1 Introduction

Modularisation is (together with parametrisation) a key feature in order to describe and design complex objects in a manageable and comprehensible way. In this paper we study the logical aspects of modularisation in formal (programming or specification) languages. This is done by investigating some natural and useful operations on *theories*, the objects that express the logical semantics of such languages. The usual names for these operations in the jargon of computer science are import (in logical terms: combination of theories), export (restricting the signature of a theory) and renaming. The results are presented in an algebraic fashion.

1.1 Relation with Other Work

Operators on modules and their semantics have been studied in e.g. [1] (in the context of CLEAR), [5] (in the context of PLUSS), [3] and [4] (using category theory), [16] (using model class semantics). Our main source of inspiration has been [2], where the approach is similar to Wirsing's in [16], extended to theory semantics and countable model semantics. The role of the interpolation theorem for the theory semantics of import and export has been pointed out in [7]. Besides giving a survey of logical aspects of modularisation, this paper contains, to the best of our knowledge, the following new points:

- investigation of the behaviour of import and export in combination with non-bijective renamings on theories;
- a normal form theorem for theory terms constructed with these operators;
- a (trivial) counterexample for interpolation in conditional equational logic;
- definition of import and export on theories using two orthogonal closure properties;
- relation with the $[\&, \exists]$-fragment of second order logic.

1.2 Survey of the Rest of the Paper

In Sect. 2 we introduce signatures, theories, renamings and operations defined on them. Axioms for these operations are given in Sect. 3, where also the relation of one of these properties with the interpolation theorem is considered, as well as some results on interpolation in (conditional) equational logic. Section 4 is about normal forms of so-called theory terms. In the last two sections we sketch some related ideas: reducing the theory operations of Sect. 2 to two orthogonal closure operators, and a theory semantics for the [&, ∃]-fragment of second order logic.

1.3 Acknowledgements

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2 Signatures, Renamings and Theories

We assume some logical language L with a derivability relation $\vdash$. L contains signature elements, e.g. sorts, functions, predicates. Signature elements can have a type: an arity (required number of arguments) or a sort type (a list of input and output sorts). We assume that, for every type, there are infinitely many signature elements having that type: this will allow us to apply the *fresh signature element principle* (see the end of this section). A signature is a finite set of signature elements; it is called *closed* if it contains all sorts occurring in the types of its elements (observe that closedness is preserved under union and intersection). The closure $c(\Sigma)$ of a signature is the least closed signature containing Σ. If X is a (collection of) expression(s) in the language of L then $S(X)$ is the closure of the collection of all signature elements occurring in (elements of) X.

From now on, we adopt the default convention that Σ and Π range over closed signatures.

Let Γ be a collection of sentences of L, Σ a signature, then the *closure of* Γ *in* Σ is defined by

$$Cl(\Sigma, \Gamma) =_{\text{def}} \{A \mid \Gamma \vdash A \text{ and } S(A) \subseteq \Sigma\}\ .$$

These closures are called *theories*. The *union* of two theories is the smallest theory containing them, defined by

$$T + U =_{\text{def}} Cl(S(T \cup U), T \cup U)\ .$$

It is obvious that $+$ is commutative, associative and idempotent. The *restriction* of a theory to a signature is defined as

$$\Sigma \,\square\, T =_{\text{def}} Cl(\Sigma \cap S(T), T)\ (= \{A \mid A \in T \text{ and } S(A) \subseteq \Sigma\})\ .$$