MINIMAX AND EXTREMUM PROBLEMS ASSOCIATED TO A VARIATIONAL INEQUALITY

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Abstract

The theory of the gap functions is extended to the variational inequality introduced by Minty. Exploiting the minimax formulation of a variational inequality, a new class of gap functions is defined. Descent methods,based on the minimization of the new class of gap functions, are analysed.

1 Introduction

In recent years the interest in variational formulations of equilibrium problems has widely grown. In particular, variational inequalities (in short VI) have shown to be a powerful tool in order to develop classical methods existing in the literature, as descent methods, interior point methods, pivotal methods etc. [7, 10]. In addition to this, VI have received interest since they allow to generalize the classical optimality conditions for constrained extremum problems and to formalize equilibrium conditions for problems of different nature as network flow, economic and mechanical engineering equilibrium problems [9, 10]. In this paper we want to deepen the analysis of the connections between minimax theory and VI in order to develop the gap function approach for solving VI.

Given the variational inequality:

$$\text{find } y^* \in K \ \text{ s.t. } \langle F(y^*), x - y^* \rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $F : K \longrightarrow R^n$, $K \subseteq R^n$ and $\langle \cdot, \cdot \rangle$ is the inner product in R^n, a gap function $p : K \longrightarrow R$ is a non-negative function that fulfils the condition $p(y) = 0$ if and only if y is a solution of VI. Therefore solving a VI is equivalent to the minimization of the gap function on the feasible set in which VI is defined. A first example of gap function was given by Auslender [1] who considered the function $p(y) := \sup_{x \in K} \langle F(y), y - x \rangle$; unfortunately, in general, this function is not differentiable: in the last years the efforts of the research have been directed to the study of differentiable gap functions in order to simplify the computational aspects of the problem. Important results in this sense have been obtained by Fukushima, Zhu and Marcotte [7, 20].

Our aim is to show that the gap function approach, developed in [7, 20] for the problem VI, can be extended to the so called Minty Variational Inequality

$$\text{find } x^* \in K \ \text{ s.t. } \langle F(y), y - x^* \rangle \geq 0, \quad \forall y \in K, \tag{VI^*}$$

which, under the hypotheses of continuity and pseudomonotonicity of the operator F, is equivalent to VI [11]. In Section 2 we will point out that the gap function

formulation given by Zhu and Marcotte [20] can be obtained considering a suitable minimax formulation of the problem VI.
The same argument is used in Section 3 to define a continuously differentiable gap function associated to the problem VI^*.
In Section 4, on the same line followed by Fukushima [7], we will develop a descent method for the variational inequality VI^*.
We recall the main notations and definitions that will be used in the sequel. A function $f : K \longrightarrow R$ is said quasi–convex on K iff:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq max\{f(x_1), f(x_2)\},$$

$\forall x_1, x_2 \in K, \forall \lambda \in [0,1]$.

A function $f : K \longrightarrow R$ is said strictly quasi–convex [17] if strict inequality holds $\forall x_1 \neq x_2 \in K, \forall \lambda \in (0,1)$. The main properties of a strictly quasi–convex function are reported in the Appendix.

A function $f : K \longrightarrow R$ is said strongly convex on K with modulus a $(a \geq 0)$ iff $\forall x_1, x_2 \in K, \forall \lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) - a[\lambda(1-\lambda)/2]\|x_1 - x_2\|^2.$$

We will say that the mapping $F : R^n \longrightarrow R^n$ is monotone on K iff:

$$\langle F(y) - F(x), y - x\rangle \geq 0, \quad \forall x, y \in K;$$

it is strictly monotone if strict inequality holds $\forall x \neq y$.
We will say that the mapping F is pseudomonotone on K iff:

$$\langle F(y), x - y\rangle \geq 0 \text{ implies } \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in K.$$

2 Gap functions and Variational Inequalities

Consider the variational inequality:

$$\text{find } y^* \in K \text{ s.t. } \langle F(y^*), x - y^*\rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $F : K \longrightarrow R^n$, $K \subseteq R^n$. It is well known that the problem VI is a particular case of a general equilibrium problem [4]:

$$\text{find } y^* \in K \text{ s.t. } f(x, y^*) \geq 0, \quad \forall x \in K,$$

where $f : K \times K \longrightarrow R$.
Let us consider a preliminary result that relates a general equilibrium problem to a suitable minimax problem under the hypothesis that f is zero on the diagonal of $K \times K$.

Lemma 2.1 *Suppose that $f(x,x) = 0$ $\forall x \in K$. Then, the following statements are equivalent:*
i) there exists $y^* \in K$ s.t. $f(x, y^*) \geq 0, \quad \forall x \in K$.
ii) $\max_{y \in K} \inf_{x \in K} f(x, y) = 0$.

Proof. $f(x,x) = 0, \ \forall x \in K$, implies that $\inf_{x\in K} f(x,y) \leq 0, \forall y \in K$.
Therefore *i)* implies that $\max_{y\in K} \ inf_{x\in K} f(x,y) = 0$.
Viceversa, if *ii)* holds, then there exists y^* such that $\inf_{x\in K} f(x,y^*) = 0$, which is equivalent to *i)*.

Therefore a variational inequality can be transformed into an equivalent minimax problem: this property is of fundamental importance in the analysis of the gap function theory for variational inequalities.

Definition 2.1 *Let K be a closed subset in R^n. The function $p : R^n \longrightarrow R$ is a gap function for VI iff:*
i) $p(y) \geq 0, \quad \forall y \in K$;
ii) $p(y) = 0$ iff y is a solution for VI.

The function $p(y) := \sup_{x\in K} \langle F(y), y-x\rangle$, which was introduced by Auslender [1], is a gap function for VI; in general, it is not differentiable.

Remark 2.1 VI is equivalent to the minimax problem:

$$\min_{y\in K} \sup_{x\in K} \langle F(y), y-x\rangle.$$

The problem of defining a continuously differentiable gap function was first solved by Fukushima [7] whose approach was generalized by Zhu and Marcotte [20]; subsequently Giannessi [9] embedded the variational inequality problem into a general separation scheme.
We recall the results obtained by Zhu and Marcotte. Let $G(x,y) : K \times K \longrightarrow R$, non–negative, continuously differentiable, strongly convex on the convex set K with respect to x, $\forall y \in K$, and such that
i) $G(y,y) = 0, \quad \forall y \in K$;
ii) $\nabla_x G(y,y) = 0, \quad \forall y \in K$.

Proposition 2.1 *[20] Let $h(x,y) := \langle F(y), y-x\rangle - G(x,y)$ and $g(y) := max_{x\in K} h(x,y)$. Then*
i) $g(y)$ is a gap function for VI;
ii) If $F \in C^1$ then $g \in C^1$ and

$$\nabla g(y) = F(y) - (\nabla F(y))^T (H(y) - y) - \nabla_y G(H(y), y)$$

where $H(y) := argmax_{x\in K} h(x,y)$.

If we consider the particular case $G(x,y) := \frac{1}{2}\langle x-y, M(x-y)\rangle$, where M is a symmetric and positive definite matrix of order n, we obtain the gap function introduced by Fukushima [7].

Remark 2.2 Since $G(x, y)$ is strongly convex with respect to x, then $g(y)$ is well–defined because the maximum with respect to x of $h(x, y)$ exists and it is unique; therefore VI is equivalent to the minimax problem

$$\min_{y\in K} \max_{x\in K} h(x, y).$$

We observe that equivalent minimax formulations of the variational inequality allow to define differentiable gap functions for VI. We will show that the approach proposed by Zhu and Marcotte and, therefore, the derived minimax formulations, arise from considering suitable regularizations of the variational inequality VI. The following result holds:

Proposition 2.2 *Let K be a closed convex subset in R^n. Let $G(x, y) : K \times K \longrightarrow R$ be a non negative, differentiable function on the convex set K with respect to x and such that*
i) $G(y, y) = 0, \quad \forall y \in K$*;*
ii) $\nabla_x G(y, y) = 0, \quad \forall y \in K$*.*
Then the problem VI is equivalent to the following:

$$\textit{find } y^* \in K \quad \textit{s.t. } \langle F(y^*), x - y^*\rangle + G(x, y^*) \geq 0, \quad \forall x \in K. \qquad VI(F, G)$$

Proof. Since G is non negative it is obvious that, if y^* is a solution of VI, then it is also a solution of $VI(F, G)$.
Viceversa suppose that y^* is a solution of $VI(F, G)$. This is equivalent to say that y^* is a global minimum point of the problem

$$min_{x\in K} \langle F(y^*), x - y^*\rangle + G(x, y^*).$$

Since K is a closed convex set then y^* is a solution of the variational inequality

$$\langle F(y^*) + \nabla_x G(y^*, y^*), x - y^*\rangle \geq 0, \quad \forall x \in K.$$

Condition *ii)* implies that y^* is a solution of VI.

We note that no convexity hypotheses on the function G are needed to prove Proposition 2.2.

Remark 2.3 From Lemma 2.1 we have that $VI(F, G)$ is equivalent to the condition

$$\min_{y\in K} \sup_{x\in K} [\langle F(y), y - x\rangle - G(x, y)] = 0$$

and therefore, under the hypotheses of Proposition 2.2 the function

$$h(y) := \sup_{x\in K} [\langle F(y), y - x\rangle - G(x, y)]$$

is a gap function for the problem VI.

3 The gap function associated to Minty Variational Inequality

It is well-known that the variational problem

$$\text{find } x^* \in K \text{ s.t. } \langle F(y), y - x^* \rangle \geq 0, \quad \forall y \in K \tag{VI^*}$$

which is also known [8] as Minty variational inequality, is closely related to the problem VI. When the operator F is continuous and pseudomonotone VI^* is equivalent to VI and, therefore, many existence theorems, based on monotonicity assumptions, prove the existence of a solution for both problems VI and VI^* [5, 14]. Following the scheme described in the previous section we will show that, exploiting the minimax formulation of the problem VI^* [13], it is possible to define a gap function associated to VI^*.

We will introduce a suitable regularization of the problem VI^* from which we will derive a continuously differentiable gap function, using the classical results of the minimax theory.

Let $H(x, y) : K \times K \longrightarrow R$ be a non negative, differentiable function such that

$$H(x, x) = 0, \quad \forall x \in K; \tag{1}$$

$$\nabla_y H(x, x) = 0, \quad \forall x \in K. \tag{2}$$

Consider the following variational inequality:

$$\text{find } x^* \in K \text{ s.t. } \langle F(y), y - x^* \rangle + H(x^*, y) \geq 0, \quad \forall y \in K. \tag{$VI^*(F,H)$}$$

Proposition 3.1 *Let K be a convex set in R^n. Suppose that $H(x, y) : K \times K \longrightarrow R$ is a non negative, differentiable function that fulfils (1) and (2) and that $F : R^n \longrightarrow R^n$ is a differentiable pseudomonotone operator on K. Then $VI^*(F, H)$ is equivalent to VI^*.*

Proof. Since H is non negative it is obvious that, if y^* is a solution of VI^*, then it is also a solution of $VI^*(F, H)$.
Viceversa suppose that x^* is a solution of $VI^*(F, H)$. This is equivalent to say that x^* is a global minimum point of the problem

$$\min_{y \in K} [\langle F(y), y - x^* \rangle + H(x^*, y)].$$

Since K is a convex set x^* is a solution of the variational inequality

$$\langle \nabla_y q(x^*, x^*) + \nabla_y H(x^*, x^*), y - x^* \rangle \geq 0, \quad \forall y \in K,$$

where $q(x, y) := \langle F(y), y - x \rangle$. From (2) we obtain

$$\langle \nabla_y q(x^*, x^*), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

Since $\nabla_y q(x, y) = F(y) + \nabla F(y)(y - x)$ then $\nabla_y q(x^*, x^*) = F(x^*)$, that is x^* is a solution of VI. Since F is pseudomonotone then x^* is also a solution of VI^*.

Corollary 3.1 *In the hypotheses of the previous proposition the function*

$$h(x) := \sup_{y \in K} [\langle F(y), x - y \rangle - H(x, y)]$$

is a gap function for VI^*.

Proof. It is sufficient to observe that, for Lemma 2.1, $VI^*(F, H)$ is equivalent to the condition

$$\min_{x \in K} \sup_{y \in K} [\langle F(y), x - y \rangle - H(x, y)] = 0$$

and that $h(x) \geq 0, \ \forall x \in K$. The thesis follows from Proposition 3.1.

From Propositions 2.2 and 3.1 we immediately obtain the following result:

Corollary 3.2 *Let $F : R^n \longrightarrow R^n$ be a differentiable pseudomonotone operator on K and $G(x, y) : K \times K \longrightarrow R$ be a non negative, differentiable function such that*

1. $G(x, x) = 0 \quad \forall x \in K$*;*
2. $\nabla_x G(x, x) = 0 \quad \forall x \in K$*.*

Then $h(x) := \sup_{y \in K} [\langle F(y), x - y \rangle - G(y, x)]$ is a gap function for VI^*.

Remark 3.1 The previous corollary follows from the simple remark that if $G(x, y)$ fulfils the hypotheses *i)* and *ii)* of Proposition 2.2 then $G(y, x)$ fulfils conditions (1) and (2). Therefore, an analogous result can be stated to derive a gap function for the variational inequality VI, considering a function $H(x, y)$ that fulfils (1) and (2).

In the last part of the section we analyse the differentiability properties of the function $h(x)$.

Proposition 3.2 *Let K be a compact convex set in R^n. Suppose that F is continuous, $H(x, y) : K \times K \longrightarrow R$ is continuously differentiable and the function $\phi(x, y) := \langle F(y), y - x \rangle + H(x, y)$ is strictly quasi convex with respect to y, $\forall x \in K$, then $h(x)$ is continuously differentiable and its gradient is given by*

$$\nabla h(x) = F(y(x)) - \nabla_x H(x, y(x))$$

where $y(x)$ is the solution of the problem $\min_{y \in K} \phi(x, y)$.

Proof. We observe that

$$h(x) = - \inf_{y \in K} \phi(x, y) \tag{3}$$

Since $\phi(x, y)$ is strictly quasi convex with respect to y then there exists a unique minimum point $y(x)$ of the problem (3) (see the Appendix, Proposition 5.1). Since

F is continuous and H is continuously differentiable then $\nabla_x \phi$ is continuous and therefore, from theorem 1.7, chapter 4 of [1], it follows that

$$\nabla h(x) = -\nabla_x \phi(x, y(x)).$$

Moreover it is known that the function $y(x)$ is continuous (see the Appendix, Proposition 5.2) and therefore $\nabla h(x)$ is continuous at x.

Remark 3.2 In the Proposition 3.2, the hypothesis of compactness on the set K can be replaced by any other assumption that guarantees that the infimum in (3) is attained. Moreover, we observe that, if $H(x, y)$ is concave with respect to x, for every fixed $y \in K$, then $h(x)$ is a continuously differentiable convex function.

4 Descent methods for Minty Variational Inequality

In this section we will consider a descent direction method for solving the problem

$$\min_{x \in K} \; h(x) \tag{4}$$

where h is the gap function associated to the variational inequality VI^*:

$$h(x) := \sup_{y \in K} [\langle F(y), x - y \rangle - H(x, y)].$$

We will assume that

1. K is a nonempty compact convex set in R^n;
2. $\phi(x, y) := \langle F(y), y - x \rangle + H(x, y)$ is a strictly quasi convex function with respect to y, $\forall x \in K$;
3. F is a continuously differentiable operator with $\nabla F(y)$ positive definite $\forall y \in K$;
4. $H(x, y) : K \times K \longrightarrow R$ is a non negative continuously differentiable function that fulfils conditions (1) and (2) and such that $\nabla_x H(x, y) + \nabla_y H(x, y) = 0, \forall x, y \in K$.

Remark 4.1 The hypotheses 1 and 3 guarantee that VI^* has a solution x^* and, moreover, that x^* is also a solution of VI (see [11]).
Condition 4 is fulfilled by the function $H(x, y) := \frac{1}{2}\langle M(x - y), x - y \rangle$ where M is a symmetric positive definite matrix of order n. This function was considered by Fukushima [7] in order to define a gap function for the variational inequality VI.

We recall that, from Proposition 3.2, h is a continuously differentiable function and $\nabla h(x) = F(y(x)) - \nabla_x H(x, y(x))$, where $y(x)$ is the solution of the problem

$$\min_{y \in K} \phi(x, y). \tag*{$P(x)$}$$

Lemma 4.1 *Suppose that the hypotheses* 1–4 *hold. Let $y(x)$ the solution of $P(x)$. Then x^* is a solution of VI^* iff $x^* = y(x^*)$.*

Proof. We have shown in the proof of Proposition 3.2 that, in the hypotheses 1,2,3, $P(x)$ admits a unique minimum point. Since $\nabla F(y)$ is a positive definite matrix $\forall y \in K$ and F is continuously differentiable then F is a strictly monotone operator ([16], Theorem 5.4.3). Therefore x^* is a solution of VI^* iff $0 = h(x^*) = -\min_{y\in K} \phi(x^*, y)$ iff $y(x^*) = x^*$.

Remark 4.2 Lemma 4.1 can be proved without the assumption $\nabla_x H(x,y) + \nabla_y H(x,y) = 0, \forall x, y \in K$. This additional hypothesis allows to prove that $d(x) := y(x) - x$ is a descent direction for h at the point $x \in K$, whenever $y(x) \neq x$.

Proposition 4.1 *Suppose that the hypotheses* 1–4 *hold. Let $y(x)$ be the solution of the problem $P(x)$ with $y(x) \neq x$. Then $d(x) := y(x) - x$ is a descent direction for h at the point $x \in K$.*

Proof. We have to show that $\langle \nabla h(x), d(x)\rangle < 0$. Since K is a convex set $y(x)$ fulfils the condition $\langle \nabla_y \phi(x, y(x)), z - y(x)\rangle \geq 0, \forall z \in K$ that is, putting $q(x,y) := \langle F(y), y - x\rangle$,

$$\langle \nabla_y q(x, y(x)), z - y(x)\rangle + \langle \nabla_y H(x, y(x)), z - y(x)\rangle \geq 0, \quad \forall z \in K.$$

In particular for $z := x$ we obtain

$$\langle \nabla_y q(x, y(x)), x - y(x)\rangle \geq -\langle \nabla_y H(x, y(x)), x - y(x)\rangle. \tag{5}$$

Since $\nabla_y q(x,y) = F(y) + \nabla F(y)(y - x)$, taking into account assumption 4 and (5), we have

$\langle \nabla_x h(x), y(x) - x)\rangle = \langle F(y(x)), y(x) - x\rangle - \langle \nabla_x H(x, y(x)), y(x) - x\rangle \leq \langle F(y(x)), y(x) - x\rangle + \langle \nabla_y q(x, y(x)), x - y(x)\rangle = \langle F(y(x)), y(x) - x\rangle + \langle F(y(x)), x - y(x)\rangle + \langle \nabla F(y(x))(y(x) - x), x - y(x)\rangle = \langle \nabla F(y(x))(y(x) - x), x - y(x)\rangle < 0$ if $x \neq y(x)$, and we achieve the thesis.

Using the same arguments exploited by Fukushima for proving theorem 4.1 in [7], it is possible to estabilish the following convergence result:

Theorem 4.1 *Suppose that the hypotheses* 1–4 *hold. Consider the sequence $\{x_k\}$ defined by the relation*

$$x_{k+1} := x_k + t_k d_k, \quad k = 0, 1, ...$$

where $d_k(x_k) := y(x_k) - x_k$ and $t_k \in [0, 1]$ is the solution of the problem

$$min\{h(x_k + t d_k) : \ 0 \leq t \leq 1\}. \tag{6}$$

Then, for any $x_0 \in K$ the sequence $\{x_k\} \subset K$ and converges to the solution of the variational inequality VI^*.

Proof. Since $\nabla F(y)$ is positive definite $\forall y \in K$ and F is continuously differentiable then F is a strictly monotone operator ([16], Theorem 5.4.3) and therefore both problems VI and VI^* have the same unique solution.
The convexity of K implies that the sequence $\{x_k\} \subset K$ since $t_k \in [0,1]$. It is proved in the Appendix that the function $y(x)$ is continuous, which implies the continuity of $d(x)$. It is known (see e.g.[15]) that the map

$$U(x,d) := \{y : y = x + td, 0 \le t \le 1, h(y) = \min_{0 \le t \le 1} h(x+td)\}$$

is closed whenever h is a continuous function. Therefore the algorithmic map $x_{k+1} = U(x_k, d(x_k))$ is closed, (see e.g. [15]). Zangwill's convergence theorem [19] (see the Appendix) implies that any accumulation point of the sequence $\{x_k\}$ is a solution of VI^*. Since VI^* has a unique solution, the sequence $\{x_k\}$ converges to the solution of VI^*.

Remark 4.3 The proposed algorithm can be applied to a linear variational inequality over a compact convex set, where the operator is defined by $F(y) := Ay+b$ with A positive definite matrix of order n.The function $H(x,y)$ may be defined as $H(x,y) := \frac{1}{2}\langle M(x-y), x-y\rangle$ where M is a symmetric positive definite matrix of order n.

The literature on descent methods for constrained extremum problems is very wide. Many of the generalizations concerning classical descent methods are likely to be suitable for the algorithm proposed in Theorem 4.1; for example, it could be considered an inexact line-search rule when solving (6). Moreover, owing to the fact that the gap function approach descends from equivalent minimax formulations of VI and VI^*, descent methods for minimax problems [1] could be analysed.

5 Appendix

In this section we report, for completeness, the main properties of a strictly quasi–convex function, the proof of the continuity of the optimal solution map of a parametric optimization problem and Zangwill convergence theorem. These properties have been used for proving some of the results (Proposition 3.2 and Theorem 4.1) in the previous sections.

Definition 5.1 *[17] A function $f : K \longrightarrow R^n$ is said strictly quasi–convex iff:*
$x_1, x_2 \in K,\ x_1 \neq x_2,\ f(x_1) \le f(x_2),\ 0 < \lambda < 1$ *implies*

$$f(\lambda x_1 + (1-\lambda)x_2) < f(x_2).$$

We recall that some authors have given different definitions of strict quasi–convexity, see e.g. [12]. For a deeper analysis on this topic see [2] and references therein.
In the next result we summarize the main properties of a strictly quasi–convex function that we have considered in the paper.

Proposition 5.1 *[18] Let f be a strictly quasi–convex function on the convex set $K \subseteq R^n$. Then the following statements hold:*

1. *f is quasi–convex on K;*
2. *Every local minimum point of f on K is also a global minimum point on K;*
3. *If f attains a global minimum point x^* on K then x^* is the unique minimum point for f on K.*

The second topic of interest is the continuity of the optimal set mapping of a parametric optimization problem. We recall the main concepts concerning point to set maps that we will use in what follows.

Definition 5.2 *Let X, Y be metric spaces.*

1. *A point to set map $A : X \longrightarrow 2^Y$ is upper semicontinuous (u.s.c.) according to Berge at a point $\lambda^* \in X$ if, for each open set B containing $A\lambda^*$, there exists a neighborhood V of λ^* such that*
$$A\lambda \subset B, \quad \forall \lambda \in V.$$
2. *A point to set map $A : X \longrightarrow 2^Y$ is called closed at $x \in X$ iff*
$$x^k \longrightarrow x \in X,\ y^k \longrightarrow y \in Y, \textit{with } y^k \in A(x^k)\ \forall k, \textit{implies that } y \in A(x^k).$$
A point to set map is called closed on $S \subset X$ if it is closed at every point of S.

Consider the problem
$$\inf f(x, \lambda), \quad x \in K \qquad P(\lambda)$$
where $f(x, \lambda) : R^n \times R^p \longrightarrow R$, K is a closed convex set in R^n.
Suppose that there exists the minimum for $P(\lambda^*)$ and the set of the solutions of $P(\lambda^*)$ is bounded. Let $y(\lambda)$ be the optimal solution mapping of the problem $P(\lambda)$, The following result holds:

Proposition 5.2 *Let $f(x, \lambda) : R^n \times R^p \longrightarrow R$ be lower semicontinuous and strictly quasi–convex with respect to x, for every fixed $\lambda \in R^p$ and upper semicontinuous at the point $(y(\lambda^*), \lambda^*)$. Then $y(\lambda)$ is a single valued map which is continuous at the point λ^*.*

Proof. Since $f(x, \lambda)$ is lower semicontinuous and strictly quasi–convex with respect to x, for every fixed $\lambda \in R^p$ it follows that $y(\lambda)$ is single-valued.
Applying theorem 4.3.3 of [3], we obtain that $y(\lambda)$ is u.s.c. according to Berge at the point λ^*. Since $y(\lambda)$ is single-valued it follows that $y(\lambda)$ is continuous at λ^*.

We recall the statement of Zangwill Convergence Theorem [19] as reported in [15]. Given an optimization problem P on X, let M be the set of the points of X that fulfil a suitable necessary optimality condition. Suppose that, in order to solve P, it is used an algorithm represented by a point to set map $A : X \longrightarrow 2^X$.

Definition 5.3 *We say that $z : X \longrightarrow R$ is a descent function (relative to the algorithm A) if it is continuous and has the following properties:*

1. *$x \notin M$ implies $z(y) < z(x) \quad \forall y \in A(x)$,*

2. *$x \in M$ implies $z(y) \leq z(x) \quad \forall y \in A(x)$.*

Theorem 5.1 *[19] Let P be an optimization problem on X and M be the set of the points of X that fulfil a certain necessary optimality condition.*
Let $A : X \longrightarrow 2^X$ be the algorithmic point to set mapping and consider a sequence $\{x^k\}$ generated by the algorithm, i.e. satisfying $x^{k+1} \in A(x^k)$.
If the following three conditions hold:

1. *Every point x^k is contained in a compact set $K \subset X$;*

2. *There exists a function of descent z;*

3. *The point to set map A is closed on $X \setminus M$ and $\forall x \in X \setminus M,\ A(x) \neq \emptyset$.*

Then, for every x which is the limit of a convergent subsequence of $\{x^k\}$, we have that $x \in M$.

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