SOME REMARKS ON A MINIMAX FORMULATION OF A VARIATIONAL INEQUALITY

GIANDOMENICO MASTROENI
Department of Mathematics, University of Pisa
Via Buonarroti 2, 56127 Pisa, Italy

Abstract. Saddle point conditions of suitable functions are equivalent to particular classes of variational inequalities. Some existence theorems for variational inequalities, based on monotonicity assumptions on the operator F, allow to prove these saddle point conditions. Applications to the generalized complementarity problem and extensions to vector variational inequalities are considered.

1. Saddle point conditions and variational inequalities

Consider the variational inequality:

$$\text{find } y^* \in K \quad \text{s.t. } \langle F(y^*), x - y^* \rangle \geq f(y^*) - f(x), \quad \forall x \in K, \quad VI(f, K)$$

where $F : K \longrightarrow H$,$f : K \longrightarrow R$, H is a real Hilbert space, K is a nonempty convex subset of H and $\langle \cdot, \cdot \rangle$ is the inner product in H.

Following the approach considered in [8], let us introduce the function $\phi : K \times K \longrightarrow R$,

$$\phi(x, y) := \langle F(y), y - x \rangle + f(y) - f(x).$$

Suppose that ϕ has a saddle point (x^*, y^*) on $K \times K$ that is

$$\phi(x, y^*) \leq \phi(x^*, y^*) \leq \phi(x^*, y), \quad \forall (x, y) \in K \times K.$$

If we compute the previous inequalities for $(x, y) := (y^*, x^*)$ we obtain $\phi(x^*, y^*) = 0$. Therefore the existence of a saddle point of ϕ guarantees that

$$\phi(x, y^*) \leq 0, \quad \forall x \in K, \tag{1}$$

$$\phi(x^*, y) \geq 0, \quad \forall y \in K. \tag{2}$$

The inequality (1) states that y^* is a solution of $VI(f, K)$.

We are interested in studying the relationships between $VI(f, K)$ and the problem which arises from inequality (2) that is

$$\text{find } x^* \in K \ \text{ s.t. } \langle F(y), y - x^* \rangle \geq f(x^*) - f(y), \quad \forall y \in K. \quad VI^*(f, K)$$

More precisely we want to investigate under which conditions the existence of a solution of $VI(f, K)$ guarantees the existence of a solution of $VI^*(f, K)$ and viceversa. We observe that if both problems have a solution, that is (1) and (2) hold, then the point (x^*, y^*) is a saddle point for $\phi(x, y)$ on $K \times K$. The previous cosiderations lead to state the following result:

Proposition 1.1 *Let $\phi(x, y) := \langle F(y), y - x \rangle + f(y) - f(x)$.*

1. *$VI(f, K)$ admits a solution y^* if and only if*
$$\inf_{y \in K} \sup_{x \in K} \phi(x, y) = 0$$
and the infimum is attained at y^*;
2. *$VI^*(f, K)$ admits a solution x^* if and only if*
$$\sup_{x \in K} \inf_{y \in K} \phi(x, y) = 0$$
and the supremum is attained at x^*;
3. *y^* and x^* are the respective solutions of the problems $VI(f, K)$ and $VI^*(f, K)$ if and only if (x^*, y^*) is a saddle point for ϕ on $K \times K$.*

Proof. *1.* Let $h(y) := \sup_{x \in K} \phi(x, y)$. Since $\phi(x, x) = 0, \ \forall x \in K$ then $h(y) \geq 0, \ \forall y \in K$. Moreover $y^* \in K$ is a solution of $VI(f, K)$ if and only if $h(y^*) = 0$. Therefore $0 = h(y^*) = \min_{y \in K} \sup_{x \in K} \phi(x, y)$.

2. Let $q(x) := \inf_{y \in K} \phi(x, y)$. It is immediate that $q(x) \leq 0, \forall x \in K$ and that $x^* \in K$ is a solution for $VI^*(f, K)$ if and only if $q(x^*) = 0$. Therefore $0 = q(x^*) = \max_{x \in K} \inf_{y \in K} \phi(x, y)$.

3. Let y^* and x^* the respective solutions of the problems $VI(f, K)$ and $VI^*(f, K)$; from *1* and *2* it follows that the point (x^*, y^*) is a saddle point for ϕ on $K \times K$.

Viceversa, let $(x^*, y^*) \in K \times K$ be a saddle point for ϕ on $K \times K$ that is:

$\langle F(y^*), y^* - x \rangle + f(y^*) - f(x) \leq \langle F(y^*), y^* - x^* \rangle + f(y^*) - f(x^*) \leq \langle F(y), y - x^* \rangle + f(y) - f(x^*) \quad \forall (x, y) \in K \times K.$

Evaluating the previous inequalities at the point $(x, y) := (y^*, x^*)$ we obtain $\langle F(y^*), y^* - x^* \rangle + f(y^*) - f(x^*) = 0$ from which the thesis follows since the first inequality states that y^* is a solution of $VI(f, K)$,while the second that x^* is a solution of $VI^*(f, K)$.

2. Applications to the classical variational inequality

In this section we will consider the particular case in which f is a constant function on the set K so that $VI(f, K)$ becomes the classical variational inequality

$$\text{find } y^* \in K \text{ s.t. } \langle F(y^*), x - y^* \rangle \geq 0, \quad \forall x \in K. \tag{VI}$$

We will denote by VI^* the problem associated to VI by means of the saddle point condition for the function $\phi(x, y) := \langle F(y), y - x \rangle$,

$$\text{find } x^* \in K \text{ s.t. } \langle F(y), y - x^* \rangle \geq 0, \quad \forall y \in K. \tag{VI^*}$$

Proposition 1.1 states that y^* and x^* are the respective solutions of the problems VI and VI^* if and only if (x^*, y^*) is a saddle point for the function ϕ on $K \times K$.

Let us recall some notations that will be used in what follows:
$\text{int}\,(M), \partial(M)$ will denote the interior and the boundary of the set $M \subseteq H$, respectively. If $K \subseteq H$, $\text{int}_K(M), \partial_K(M)$ denote the relative interior and relative boundary of M in K, $M^* := \{y \in H : \langle y, x \rangle \geq 0, \ \forall x \in M\}$ will denote the positive polar of the set M. Moreover we will say that the mapping F is pseudomonotone on K if:

$$\langle F(y), x - y \rangle \geq 0 \text{ implies } \langle F(x), x - y \rangle \geq 0 \quad \forall x, y \in K.$$

We will see how the main results concerning the existence of a solution of the variational inequality VI, based on monotonicity assumptions on the operator F, also allow to prove the existence of a saddle point for the function ϕ on $K \times K$.

Proposition 2.1 *[1] Let K be a closed convex subset in the real Hilbert space H. Let F be a pseudomonotone mapping from K into H which is continuous on finite-dimensional subspaces. Then $y^* \in K$ solves VI if and only if $x^* := y^*$ solves VI^*.*

Remark 2.1 If we assume that K is a bounded set in H, then the hypotheses of Proposition 2.1 are sufficient to guarantee the existence of a solution of both problems VI and VI^* (see [1]).

The next result states necessary and sufficient conditions for the existence of a solution of the problem VI under the hypothesis of pseudomonotonicity of the mapping F:

Theorem 2.1 *[1] Let K be a closed convex subset in the real Hilbert space H. Let F be a pseudomonotone mapping from K into H which is continuous on finite-dimensional subspaces. Then the following statements are equivalent:*

1. *There exists a solution y^* for VI;*
2. *There exist $u \in K$ and a constant $r > \|u\|$ such that $\langle y - u, F(y)\rangle \geq 0,\ \forall y \in K$ with $\|y\| = r$;*
3. *There exists $r > 0$ such that the set $\{y \in K : \|y\| \leq r\}$ is nonempty and such that, for each $y \in K$ with $\|y\| = r$, there exists $u \in K$ with $\|u\| < r$ and $\langle F(y), y - u\rangle \geq 0$.*
4. *There exists a nonempty closed, bounded, convex subset B of K with $\mathrm{int}\ _K(B)$ that satisfies the following condition:*

$$\forall y \in \partial_K(B), \textit{there exists } u \in \ \mathrm{int}\ _K(B) \textit{ such that } \langle F(y), y - u\rangle \geq 0.$$

5. *There exists a convex set $E \subset H$ with $\mathrm{int}\ E \neq \emptyset$ such that $\emptyset \neq K \cap E$ is bounded and, for each $y \in K \cap \partial(E)$, there exists $u \in K \cap\ \mathrm{int}\ (E)$ such that $\langle F(y), y - u\rangle \geq 0$.*

Remark 2.2 In the previous theorem it is possible to add the further statements, recalling Propositions 2.1 and 1.1:

6. There exists a solution y^* for VI^*;
7. The point (y^*, y^*) is a saddle point for $\phi(x, y) := \langle F(y), y - x\rangle$ on $K \times K$.

Therefore any of the conditions *2,3,4* and *5* of Theorem 2.1 not only guarantees the existence of a solution of VI but also turns out to be necessary and sufficient for the existence of a saddle point of the function ϕ on $K \times K$.

Another interesting result about the existence of solutions of VI, based on monotonicity assumptions on the operator F, is the following:

Proposition 2.2 *[1] Let K be a closed convex subset in the real Hilbert space H. Let F be a pseudomonotone mapping from K into H which is continuous on finite-dimensional subspaces. Suppose that there exists $u \in K$ such that $F(u) \in\ \mathrm{int}\ (K^*)$. Then there exists a solution for VI.*

Remark 2.3 Once again we observe that the hypotheses of the previous proposition are sufficient to guarantee the existence of a saddle point of the function ϕ on $K \times K$.

In the next theorem we summarize the results so far obtained about the existence of a saddle point of the function ϕ in the hypotheses of pseudomonotonicity and continuity on finite-dimensional subspaces of the operator F.

Theorem 2.2 *Let K be a closed convex subset in the real Hilbert space H. Let F be a pseudomonotone mapping from K into H which is continuous on finite-dimensional subspaces. Then*

1. *If K is a bounded set in H then $\phi(x, y)$ admits a saddle point on $K \times K$.*
2. *Any of the conditions* 1,2,3,4,5 *of Theorem 2.1 is necessary and sufficient for $\phi(x, y)$ to admit a saddle point on $K \times K$.*
3. *If there exists $u \in K$ such that $F(u) \in \text{ int } (K^*)$ then $\phi(x, y)$ admits a saddle point on $K \times K$.*

Proof. We preliminarly observe that the hypotheses guarantee that the solutions of both problems VI and VI^* coincide, as stated in Proposition 2.1.

1. It follows from Remark 2.1 and 3 of Proposition 1.1;
2. It follows from Theorem 2.1 recalling Remark 2.2 and 3 of Proposition 1.1;
3. It follows from Proposition 2.2 recalling 3 of Proposition 1.1.

3. Connections with complementarity problems

In this section we will consider the particular case in which the feasible set K is a closed convex cone in H with apex at the origin.

Remark 3.1 In [1, 7] it is proved that if K is a convex cone then y^* solves VI if and only if y^* is a solution of the generalized complementarity problem:

$$\text{find } y \in K \text{ s.t. } F(y) \in K^* \text{ and } \langle F(y), y \rangle = 0. \qquad (GCP)$$

It is possible to express GCP as a constrained extremum problem:

Proposition 3.1 *[2] $y^* \in K$ solves GCP if and only if y^* is a solution of the problem:*

$$\min \langle F(y), y \rangle \quad \text{s.t. } F(y) \in K^*, \; y \in K \tag{3}$$

and $\langle F(y^*), y^* \rangle = 0$.

We will show that the problem VI^* is closely related to the Lagrangean Dual of the problem (3) defined by

$$\sup_{\lambda \in K} \inf_{y \in K} \langle F(y), y \rangle - \langle \lambda, F(y) \rangle. \tag{4}$$

The following result holds:

Proposition 3.2 *Let K be a closed convex cone in the real Hilbert space H. Let F be a pseudomonotone mapping from K into H which is continuous on finite-dimensional subspaces and suppose that*

$$\text{there exists } y' \in K \text{such that } F(y') \in \text{ int } (K^*). \tag{5}$$

Then y^* solves (3) if and only if y^* solves (4).

Proof. Let $y^* \in K$ be a solution of (3). For Proposition 2.2 VI has a solution. Since K is a cone, then GCP has a solution and therefore $\langle F(y^*), y^* \rangle = 0$.

The following equivalences allow to achieve the thesis:

y^* solves (3) and $\langle F(y^*), y^* \rangle = 0 \iff y^*$ solves $GCP \iff y^*$ solves $VI \iff y^*$ solves $VI^* \iff \sup_{\lambda \in K} \inf_{y \in K} \langle F(y), y \rangle - \langle \lambda, F(y) \rangle = 0$ and the maximum is attained at $\lambda := y^*$.

These equivalences follow from Proposition 3.1, Remark 3.1, Proposition 2.1 and *2* of Proposition 1.1, respectively.

Viceversa, let λ^* a solution of (4). Propositions 2.2 and 2.1 imply that the problem VI^* has a solution. By *2* of proposition 1.1 it follows that

$$\sup_{\lambda \in K} \inf_{y \in K} \langle F(y), y \rangle - \langle \lambda, F(y) \rangle = 0.$$

From the previous equivalences the thesis follows.

Remark 3.2 In the Proposition 3.2 it is possible to replace condition (5) with any of the conditions *2,3,4,5,* stated in Theorem 2.1.

4. Vector variational inequalities

The statements obtained in the previous part of the paper for scalar variational inequalities can be generalized, adding suitable assumptions, to vector variational inequalities introduced in [5, 4].

We recall the main definitions of vector optimization that we will use in the sequel.

From now on, K will denote a subset of R^n, C a pointed convex cone in R^p with apex at the origin $\{0\}$ of R^p and nonempty interior. We recall that a cone C is said pointed iff $C \cap (-C) = \emptyset$.

Definition 4.1 *Let $f : R^n \longrightarrow R^p$.*

1. *$y \in K$ is said a C-minimum point of f on the set K iff the following system is impossible:*

$$f(y) - f(x) \ \in \ C \setminus \{0\}, \quad x \in K.$$

2. *$y \in K$ is said a C-maximum point of f on the set K iff y is a C-minimum point for $-f$.*

Definition 4.2 *[10] Let $f : X \times Y \longrightarrow R^p$. A point $(x^*, y^*) \in X \times Y$ is said a C-saddle point of f with respect to $X \times Y$ iff*

1. *x^* is a C-minimum point of $f(\cdot, y^*)$ on X;*
2. *y^* is a C-maximum point of $f(x^*, \cdot)$ on Y.*

Consider the following vector variational inequalities introduced in [4].

$$\text{Find } y \in K \text{ s. t. } \langle F(y), y - x\rangle_p \notin C \setminus \{0\}, \quad \forall x \in K. \tag{6}$$

$$\text{Find } x \in K \text{ s. t. } \langle F(y), x - y\rangle_p \notin C \setminus \{0\}, \quad \forall y \in K. \tag{7}$$

where $F : R^n \longrightarrow R^{p\times n}$ and $\langle \cdot, \cdot\rangle_p$ denotes a vector of p scalar products in R^n.

Remark 4.1 If $p = 1$ and $C = R_+$, then (6) and (7) collapse to VI and VI^* respectively.

Following the line adopted in the scalar case it is possible to prove, under a suitable additional assumption, the equivalence between the existence of solutions of the problems (6) and (7) and the existence of a C-saddle point of the function $f(x, y) := \langle F(y), x - y\rangle_p$.

Proposition 4.1 *Let $f : K \times K \longrightarrow R^p, f(x, y) := \langle F(y), x - y\rangle_p$.*

y^* and x^* are the respective solutions of the problems (6) and (7) and $f(x^*, y^*) = 0$ iff (x^*, y^*) is a C-saddle point of $f(x, y)$ on $K \times K$ with $f(x^*, y^*) = 0$.

Proof. y^* and x^* are the respective solutions of the problems (6) and (7) iff

$$-f(x, y^*) \notin C \setminus \{0\}, \quad \forall x \in K$$

$$f(x^*, y) \notin C \setminus \{0\}, \quad \forall y \in K.$$

Since $f(x^*, y^*) = 0$, the previous relations are equivalent to

$$f(x^*, y^*) - f(x, y^*) \notin C \setminus \{0\}, \quad \forall x \in K \tag{8}$$

$$f(x^*, y) - f(x^*, y^*) \notin C \setminus \{0\}, \quad \forall y \in K. \tag{9}$$

Condition (8) states that x^* is a C-minimum point of $f(x, y)$ on K, while condition (9) that y^* is a C-maximum point of $f(x^*, \cdot)$ on K, that is (x^*, y^*) is a C-saddle point of $f(x, y)$ on $K \times K$.

Let us consider now an application of Proposition 4.1 in the particular case in which $C =$ int (R^p_+) and the operator F is continuous and monotone. The following result has been proved in [4]:

Proposition 4.2 *Let $C :=$ int (R^p_+), K a convex set in R^n and F continuous and monotone on K.Then y is a solution of (6) iff it is a solution of (7).*

The next result is a direct consequence of the previous propositions:

Corollary 4.1 *Let $f : K \times K \longrightarrow R^p$, K a convex set in R^n, $f(x, y) := \langle F(y), x - y\rangle_p$, F continuous and monotone on K, $C :=$ int R^p_+. Then the following statements are equivalent:*

1. *y^* is a solution of (6);*
2. *y^* is a solution of (7);*
3. *(y^*, y^*) is a C-saddle point of $f(x, y)$ on $K \times K$.*

The existence of C-saddle points for a vector valued function has been analysed in [10] where a generalization of the classical results of K. Fan [3] for scalar functions, has been proven.

5. Further developments

As we have observed, the minimax setting provides a natural field in which embedding variational inequalities in their more general formulations.

Recently F. Giannessi has proposed [6] the following generalized variational inequality:

$$\text{find } y^* \in A(K),\ F \in \mathcal{F}(y^*) \ \text{ s.t. } \ \langle F, x - y^*\rangle \geq 0, \quad \forall x \in K,$$

where $A : 2^{R^n} \longrightarrow 2^Y$, $\mathcal{F} : Y \longrightarrow 2^{R^n}$, $K \subseteq R^n$, $Y \subseteq R^n$.

A first existence theorem for the previous problem, in the particular case in which $\mathcal{F}$ is a single valued operator, has been proposed by B. Ricceri.

Theorem 5.1 *Let X and D two subsets in R^n. Put*

$$I := \{x \in X \ : \ D \subseteq \bigcup_{\lambda > 0} \lambda(x - X)\}.$$

Suppose that X is convex and compact and $F : R^n \longrightarrow R^n$ is a continuous function such that $\langle F(x), x\rangle = 0$, $\forall x \in X$ and

$$\sup_{y \in X} \langle F(x), y\rangle > 0, \quad \forall x \in X \setminus I.$$

Then there exists $x^* \in X$ such that

$$\langle F(x^*), y - x^*\rangle \geq 0 \quad \forall y \in D.$$

Remark 5.1 The previous theorem descends from a general result (Theorem 2.1 of [9]) stated for D-regular operators in a real vector space.

Taking into account the results of the previous sections it is possible to prove the following proposition in which X is supposed to be a closed convex cone.

Proposition 5.1 *Let X be a closed convex cone and D a subset in R^n. Put*

$$I := \{x \in X \ : \ D \subseteq \bigcup_{\lambda>0} \lambda(X - x)\}$$

Suppose that

1. $F : R^n \longrightarrow R^n$ is a continuous pseudomonotone function and

$$\textit{there exists } y' \in X \textit{ such that } F(y') \in \text{ int } (X^*); \tag{10}$$

2. $\sup_{y \in X} \langle F(x), x - y \rangle > 0, \quad \forall x \in X \setminus I$;

then there exists $x^* \in X$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in D.$$

Proof. From *1.* (see Proposition 2.2) it follows that there exists $x^* \in X$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in X \tag{11}$$

(11) is equivalent to $\sup_{y \in X} \langle F(x^*), x^* - y \rangle \leq 0$ and, therefore the hypothesis *2.* implies that $x^* \in I$. Then $D \subseteq \bigcup_{\lambda>0} \lambda(X - x^*)$ that is

$$D \subseteq \{z \in R^n \ : \ z = \lambda(x - x^*) \text{ for suitable } \lambda > 0, \ x \in X\}.$$

We recall that, since X is a closed convex cone, $\langle F(x^*), x^* \rangle = 0$ (see Remark 3.1). We obtain:

$$\begin{aligned} \inf_{y \in D} \langle F(x^*), y - x^* \rangle &\geq \inf_{\lambda>0, \ x \in X} \langle F(x^*), \lambda(x - x^*) - x^* \rangle = \\ &= \inf_{\lambda>0, \ x \in X} \lambda \langle F(x^*), x - x^* \rangle \geq 0. \end{aligned}$$

Remark 5.2 We observe that, in the previous proposition, it is possible to replace the condition (10) with any of the conditions 2,3,4,5 of Theorem 2.1.

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