Some remarks on the role of Generalized Convexity in the theory of Variational Inequalities

Giandomenico Mastroeni

Department of Mathematics, University of Pisa, Via Buonarroti 2, 56100 Pisa

Abstract

The paper aims to emphasise the parallelism between the development of the analysis of Generalized Convexity and the theory of Variational Inequalities. More in detail, the relationships between generalized invexity and Prevariational Inequalities are analised.

1 Introduction

It is well-known that Generalized Convexity has led to a great development in many fields of the optimization theory, in particular in the study of minimax and constrained extremum problems. Owing to the close relations existing between these topics and the theory of Variational Inequalities (in short VI), this paper aims to deepen the analysis of the role of Generalized Convexity in the field of VI, giving particular attention to the concepts of generalized invexity, introduced more recently in the literature [6].

In Section 2 we will point out the connections between constrained extremum problems and VI, recalling, in particular, the Kuhn-Tucker conditions for a Variational Inequality.

In Section 3 we will show how generalized invexity assumptions on the function g allow to consider a generalization of the Variational Inequality problem, the so called Prevariational Inequalities [15, 17].

In Section 4 some extensions to Vector Variational Inequalities, introduced in [4], will be outlined.

We recall the main notations and definitions that will be used in the sequel. Let $M \subseteq R^p$. $intM, clM$, will denote the interior and the closure of M, respectively. Let $y := (y_1, ..., y_p)$; $y_{(1^-)} := (y_2, ..., y_p)$,
$y_{(i^-)} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_p),\ i = 2, ..., p-1$, $y_{(p-)} := (y_1, ..., y_{p-1})$. $\langle \cdot, \cdot \rangle$ is the scalar product in R^n.

A function $f : K \longrightarrow R$ is said quasi–convex on the convex set $K \subseteq R^n$ iff:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq max\{f(x_1), f(x_2)\},$$

$\forall x_1, x_2 \in K, \forall \lambda \in [0, 1]$.
f is pseudoconvex on K iff:

$$\langle \nabla f(y), x - y \rangle \geq 0 \text{ implies } f(y) \leq f(x), \quad \forall x, y \in K;$$

f is invex on K iff there exists a vector function $\mu : K \times K \longrightarrow R^n$ such that, $\forall x, y \in K$,

$$f(x) - f(y) \ \geq \ \langle \nabla f(y), \mu(x, y) \rangle.$$

We will say that the mapping $F : R^n \longrightarrow R^n$ is monotone on K iff:

$$\langle F(y) - F(x), y - x \rangle \geq 0 \quad \forall x, y \in K;$$

it is strictly monotone if strict inequality holds, $\forall x \neq y$.
We will say that the mapping F is quasimonotone on K iff:

$$\langle F(y), x - y \rangle > 0 \text{ implies } \langle F(x), x - y \rangle \geq 0, \quad \forall x, y \in K,\ x \neq y;$$

F is pseudomonotone on K iff:

$$\langle F(y), x - y \rangle \geq 0 \text{ implies } \langle F(x), x - y \rangle \geq 0, \quad \forall x, y \in K;$$

F is strictly pseudomonotone on K iff:

$$\langle F(y), x - y \rangle \geq 0 \text{ implies } \langle F(x), x - y \rangle > 0, \quad \forall x, y \in K,\ x \neq y.$$

2 Constrained extremum problems and Variational Inequalities

Consider the following Variational Inequality:

$$find\ y \in K \quad s.t. \quad \langle F(y), x - y \rangle \geq 0, \quad \forall x \in K, \qquad (VI)$$

where $F : X \longrightarrow R^n$, $X, K \subseteq R^n$, and the constrained extremum problem

$$min\ f(x) \quad s.t. \quad x \in K, \tag{P}$$

where $f : X \longrightarrow R$ is a differentiable function.
It is known that, if K is a convex set, then a necessary condition for $y \in K$ to be a local minimizer of (P) is

$$\langle \nabla f(y), x - y \rangle \geq 0, \quad \forall x \in K. \tag{1}$$

The previous condition states the classic first order optimality condition for (P), that is, the directional derivative of f, at the point y, is non negative in every feasible direction starting from the point y.
If f is a pseudoconvex function on K, then (1) is a sufficient condition for y to be a global minimizer of the problem (P).
We observe that it is always possible to associate to the constrained extremum problem (P) the Variational Inequality defined by (1), while, given the Variational Inequality VI, not always it is possible to define a constrained extremum problem of which VI represents the first order optimality condition. This can be achieved by proving the existence of a differentiable function f such that $F = \nabla f$, as stated in the next result [14]:

Theorem 2.1 *Let $F : X \longrightarrow R^n$ be a continuously differentiable operator. Then there exists a differentiable function f such that $F = \nabla f$ iff $\nabla F(y)$ is a symmetric matrix for every $y \in X$.*

Remark 2.1 In the hypoyheses of the Theorem 2.1 and if K is a convex set, every stationary point of (P) is a solution of VI.

Moreover, the convexity properties of the function f can be deduced from the monotonicity assumptions on the operator F, taking into account the following result [2, 8, 16]:

Theorem 2.2 *Let $f : X \longrightarrow R$ be a differentiable function and C be a convex subset of the open set $X \in R^n$.*

1. *f is (strictly) convex on C iff ∇f is (strictly) monotone on C;*
2. *f is (strictly) pseudoconvex on C iff ∇f is (strictly) pseudomonotone on C;*
3. *f is quasiconvex on C iff ∇f is quasimonotone on C.*

For further generalizations of the previous theorem see e.g. [1, 2].

In order to deepen the analysis of the connections between optimization problems and Variational Inequalities, in the last part of this section we recall the Kuhn-Tucker optimality conditions for VI, which are a straightforward generalization of the classical ones stated for an extremum problem (see also [7] and references therein).
Suppose that the feasible set is defined by

$$K := \{x \in X : g(x) \geq 0\}, \tag{2}$$

where $g := (g_1, ..., g_m),\ g_i : X \longrightarrow R,\ i = 1, .., m.$
Let $I(y) := \{i \in [1, .., m] : g_i(y) = 0\}$.

Proposition 2.1 *Assume that g is a differentiable function and that $\exists z \in R^n$ such that $\langle \nabla g_i(\bar{y}), z\rangle > 0,\ \forall i \in I(\bar{y})$.*

1. *If $\bar{y}$ is a solution of VI then $\exists \lambda \geq 0$ such that $(\lambda, \bar{y})$ is a solution of the following system (S)*

$$\begin{cases} F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x)\rangle = 0 \\ \lambda \geq 0,\ g(x) \geq 0,\ x \in X. \end{cases}$$

2. *If $-g$ is a quasiconvex differentiable function, then $\bar{y}$ is a solution of VI iff $\exists \lambda \geq 0$ such that $(\lambda, \bar{y})$ is a solution of the system (S) ;*

Proof. It is sufficient to observe that $\bar{y}$ is a solution of VI iff $\bar{y}$ solves the following constrained extremum problem:

$$\min_{x \in K} \langle F(\bar{y}), x - \bar{y}\rangle \tag{3}$$

and consider the Kuhn-Tucker conditions for (3).
If no convexity assumptions are made on the constraint function g, then the Kuhn-Tucker conditions are only necessary for optimality. If $-g$ is a quasiconvex function, taking into account that the objective function in (3) is linear, these conditions are also sufficient (see e.g. [11]). $\square$

Remark 2.2 In the hypotheses of Theorem 2.1, the system (S) collapses into the classical Kuhn-Tucker conditions for the constrained extremum problem (P).

3 Generalized Invexity and Prevariational Inequalities

The introduction of the notion of invexity has led to consider, following the line developed to generalize the concept of a convex function, the classes of pseudoinvex and quasiinvex functions [6, 5, 9].

Definition 3.1 *Let $f : C \longrightarrow R$ be a differentiable function at $x^* \in C \subseteq R^n$. f is said*

1. *quasiinvex at x^* if $\exists \mu(x, x^*) : C \times C \longrightarrow R^n$ such that, $\forall x \in C$,*

$$f(x) \leq f(x^*) \quad \Longrightarrow \quad \langle \nabla f(x^*), \mu(x, x^*) \rangle \leq 0;$$

2. *pseudoinvex at x^* if $\exists \mu(x, x^*) : C \times C \longrightarrow R^n$ such that, $\forall x \in C$,*

$$\langle \nabla f(x^*), \mu(x, x^*) \rangle \geq 0 \quad \Longrightarrow \quad f(x^*) \leq f(x).$$

Remark 3.1 It is immediate that, if we suppose that C is a convex set and we choose $\mu(x, x^*) := x - x^*$, we recover the local definitions of a quasiconvex and a pseudoconvex function.
Similarly to the convex case, it is possible to show that a pseudoinvex function is also quasiinvex, but not conversely.

Following the development of the concept of generalized invexity, a generalization of the definition of a Variational Inequality has been considered:

$$\textit{find } y \in K \quad s.t. \quad \langle F(y), \mu(x, y) \rangle \geq 0, \quad \forall x \in K, \qquad (PVI)$$

where $F : X \longrightarrow R^n$, $\mu : X \times X \longrightarrow R^n$, $K \subseteq R^n$.
The previous problem is called Prevariational Inequality (PVI) and has been introduced in [15, 17].

Remark 3.2 If $\mu(x, y) := x - y$, we recover the classical definition of a Variational Inequality.

Let K be defined by (2).

Proposition 3.1 *Assume that, $-g_i$ is a quasiinvex function (with respect to $\mu(x, y)$) at $\bar{y}$, for $i \in I(\bar{y})$. If $(\bar{\lambda}, \bar{y})$ is a solution of the system (S)*

$$\begin{cases} F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ \lambda \geq 0, \ g(x) \geq 0, \ x \in X. \end{cases}$$

then $\bar{y}$ is a solution of PVI.

Proof.
Since, $-g_i$ is a quasiinvex function at $\bar{y}$, for $i \in I(\bar{y})$, then $\forall x \in X$,

$$-g_i(x) \leq -g_i(\bar{y}) \quad \Longrightarrow \quad -\langle \nabla g_i(\bar{y}), \mu(x, \bar{y}) \rangle \leq 0.$$

Adding the previous inequalities, we obtain

$$g_i(x) \geq 0, \forall i \in I(\bar{y}) \quad \Longrightarrow \quad \langle \sum_{i \in I(\bar{y})} \bar{\lambda}_i \nabla g_i(\bar{y}), \mu(x, \bar{y}) \rangle \geq 0.$$

Therefore

$$\langle \sum_{i \in I(\bar{y})} \bar{\lambda}_i \nabla g_i(\bar{y}), \mu(x, \bar{y}) \rangle \geq 0, \quad \forall x \in K.$$

Taking into account that $\bar{\lambda}_i = 0$, $\forall i \notin I(\bar{y})$, the first relation in the system (S) implies that $F(\bar{y}) = \sum_{i \in I(\bar{y})} \bar{\lambda}_i \nabla g_i(\bar{y})$, and the statement is proved. □

Corollary 3.1 *Suppose that -g_i is quasiinvex at $\bar{y}$ and that $\exists z \in R^n$ such that $\langle \nabla g_i(\bar{y}), z \rangle > 0$, $\forall i \in I(\bar{y})$; if $\bar{y}$ is a solution of VI, then $\bar{y}$ is also a solution of PVI.*

Proof. It is sufficient to observe that, for Proposition 2.1 *1*, if $\bar{y}$ is a solution of VI, then it is also a solution of the system (S). □

Consider the problem (P) defined in Section 2. The Proposition 3.1 allows to recover the following statement that was also proposed by Hanson [6].

Corollary 3.2 *Suppose that*

1. *-g_i is quasiinvex at $\bar{y}$, for $i \in I(\bar{y})$;*
2. *There exists a pseudoinvex function f at $\bar{y}$ (with respect to $\mu(x, y)$) such that $F = \nabla f$;*
3. *$(\bar{\lambda}, \bar{y})$ is a solution of the system (S).*

Then $\bar{y}$ is a global optimal solution of (P).

4 Vector Variational Inequalities

In the last part of the paper, we generalize the results obtained in the previous sections to Vector Variational Inequalities introduced in [3, 4]:

$$find\ y \in K \quad s.t. \quad F(y)(y-x) \notin C \setminus \{0\}, \quad \forall x \in K, \qquad (VVI)$$

where $F : X \longrightarrow R^{p \times n}$, $K, X \subseteq R^n$, C is a convex cone in R^p.
Let $f : X \times X \longrightarrow R^p$, $f(x,y) := F(y)(y-x)$. The following result, stated in [13], is closely related to Proposition 2.1 and provides an equivalent formulation of VVI, by means of the Kuhn-Tucker conditions.
Let K be defined by (2).

Proposition 4.1 *Assume that $C := R^p_+$, X is an open set in R^n and*

1. *$-g$ is a convex function differentiable at $\bar{y} \in K$;*
2. *For every $i := 1, \cdots, p$, the following system is possible*

$$f_{i^-}(x, \bar{y}) > 0, \quad g(x) > 0, \quad x \in X.$$

Then $\bar{y}$ is a solution of VVI iff $\exists(\theta, \lambda) \in (R^p_+ \times R^m_+)$, $(\theta, \lambda) \neq 0$, such that $(\theta, \lambda, \bar{y})$ is a solution of the following system (VS)

$$\begin{cases} \theta F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ \theta \geq 0, \ \lambda \geq 0, \ g(x) \geq 0, \ x \in X. \end{cases}$$

Remark 4.1 Condition *2* is a generalization of the Slater constraint qualification [12] for extremum problems and has been considered in [10], in a slightly different form, in order to obtain the Kuhn-Tucker conditions for vector optimization problems.

If we consider the particular case, where C:= int R^p_+, then the Slater condition for scalar optimization is a sufficient regularity assumption in order to prove an analogous result to the previous proposition.

Proposition 4.2 [13] *Assume that $C := intR^p_+$, and that*
(a) $-g$ is a convex function differentiable at $\bar{y}$;
(b) there exists $z \in R^n$ such that $g(z) > 0$.
Then $\bar{y}$ is a solution of VVI iff $\exists(\theta, \lambda) \in (R^p_+ \times R^m_+)$, $(\theta, \lambda) \neq 0$, such that $(\theta, \lambda, \bar{y})$ is a solution of the system (VS).

Remark 4.2 We observe that, taking into account Proposition 2.1, we obtain that $\bar{y}$ is a solution of the system (VS) if and only if it is a solution of the scalarized Variational Inequality

$$\langle \theta F(y), x - y \rangle \geq 0, \quad \forall x \in K. \tag{4}$$

The regularity assumption *2* of Proposition 4.1 ensures that the vector of multipliers θ is strictly positive, while the Slater condition b) of Proposition 4.2 allows to prove that $\theta \neq 0$. It is possible to prove that, when $C = R^p_+, (C = intR^p_+)$ and $\theta > 0, (\theta \neq 0)$, if $\bar{y}$ fulfils (4), then it is also a solution of VVI.

The definition of Prevariational Inequality, recalled in Section 3, can be generalized to the vector case.
We will call Vector Prevariational Inequality the following problem:

$$find\ y \in K \quad s.t. \quad -F(y)\mu(x,y) \notin C \setminus \{0\}, \quad \forall x \in K, \qquad (VPVI)$$

where $\mu : K \times K \longrightarrow R^n$.

Remark 4.3 If $\mu(x, y) := x - y$, we recover the definition of a Vector Variational Inequality.

The following result is a generalization of Proposition 3.1, proved in the scalar case. Let K be defined by (2).

Proposition 4.3 *Let $-g_i$ be a quasiinvex function at $\bar{y}$ (with respect to $\mu(x, y)$) at $\bar{y}$, $\forall i \in I(\bar{y})$, and suppose that $(\bar{\theta}, \bar{\lambda}, \bar{y})$ is a solution of the system (VS). The following statements hold:*

1. *if $C = R^p_+$ and $\theta > 0$, then $\bar{y}$ is a solution of $VPVI$;*
2. *if $C = intR^p_+$ and $\theta \neq 0$, then $\bar{y}$ is a solution of $VPVI$.*

Proof.
Since, $-g_i$ is a quasiinvex function then, for $i \in I(\bar{y})$ and $\forall x \in X$,

$$-g_i(x) \leq -g_i(\bar{y}) \quad \Longrightarrow \quad -\langle \nabla g_i(\bar{y}), \mu(x, \bar{y}) \rangle \leq 0.$$

Adding the previous inequalities, we have

$$g_i(x) \geq 0, \forall i \in I(\bar{y}) \quad \Longrightarrow \quad \langle \sum_{i \in I(\bar{y})} \bar{\lambda}_i \nabla g_i(\bar{y}), \mu(x, \bar{y}) \rangle \geq 0.$$

Therefore,

$$\langle \sum_{i\in I(\bar{y})} \bar{\lambda}_i \nabla g_i(\bar{y}), \mu(x,\bar{y})\rangle \geq 0, \quad \forall x \in K.$$

Since $\bar{y}$ is a solution of (VS), we have that

$$\langle \theta F(\bar{y}), \mu(x,\bar{y})\rangle \geq 0, \quad \forall x \in K. \tag{5}$$

1. Suppose that $C = R^p_+$ and $\theta > 0$.
Ab absurdo, suppose that $\exists \bar{x} \in K$ such that

$$-F(\bar{y})\mu(\bar{x},\bar{y}) \;=\; z \in C \setminus \{0\}; \tag{6}$$

Since $\theta > 0$, then $\langle \theta, z\rangle > 0$, and therefore

$$-\langle \theta F(\bar{y}), \mu(\bar{x},\bar{y})\rangle > 0,$$

which is against (5), and the statement *1* is proved.
2. Suppose that $C =$int R^p_+ and $\theta \neq 0$.
Ab absurdo, suppose that $\exists \bar{x} \in K$ such that (6) holds. Since $\theta \neq 0$ and $z > 0$, then $\langle \theta, z\rangle > 0$, and therefore

$$-\langle \theta F(\bar{y}), \mu(\bar{x},\bar{y})\rangle > 0,$$

which is against (5), and the proposition is proved. □

References

[1] Avriel M., Diewert W.E., Schaible S. and Ziemba W.T. (1981) Introduction to concave and generalized concave functions, in *"Generalized Concavity in Optimization and Economics"*, S. Schaible and W.T. Ziemba (eds.), *pp. 21-50*;

[2] Avriel M., Diewert W.E., Schaible S. and Zang I. (1988) Generalized Concavity, Plenum Publishing Corporation, New York;

[3] Giannessi F. (1998) On Minty Variational Principle, in *"New Trends in Mathematical Programming"*, F.Giannessi, S.Komlosi, T.Rapcsak (eds.), Kluwer;

[4] Giannessi F. (1980) Theorems of the Alternative, Quadratic Programs and Complementarity Problems, in *"Variational Inequalities and Complementarity Problems"*,R.W. Cottle, F. Giannessi and J.L. Lions (eds.), Wiley, New York, *pp.151-186*;

[5] Giorgi G. and Molho E. (1992) Generalized invexity: relationships with generalized convexity and applications to optimality and duality conditions, in *"Generalized Concavity for Economic Applications '*, P. Mazzoleni (eds.), Tecnoprint, Bologna, *pp. 53-70*;

[6] Hanson M.A. (1981) On Sufficiency of Kuhn–Tucker Conditions, *Journal of Mathematical Analysis and Applications* , **Vol. 30**, *pp. 545-550*;

[7] Harker P.T.,Pang J.S. (1990) Finite–Dimensional Variational Inequalities and Nonlinear Complementarity Problem: a Survey of Theory, Algorithms and Applications, *Mathematical Programming*, **Vol.48**, *pp.161-220*;

[8] Karamardian S. (1976) Complementarity over cones with monotone and pseudomonotone maps, *Journal of Optimization Theory and Applications*, **Vol.18**, *pp. 445-454*;

[9] Kaul R.N. and Kaur S. (1985) Optimality criteria in nonlinear programming involving nonconvex functions, *Journal of Mathematical Analysis and Applications* , **Vol. 105**, *pp. 104-112*;

[10] Maeda T. (1994) Constraint Qualifications in Multiobjective Optimization Problems: Differentiable Case, *Journal of Optimization Theory and Applications*, **Vol.80**, *pp. 483-500*;

[11] Mangasarian O.L. (1969) Nonlinear Programming, New York, Academic Press;

[12] Mangasarian O.L. and Fromovitz S. (1967) The Fritz–John Necessary Optimality Condition in the Presence of Equality and Inequality Constraints, *Journal of Mathematical Analysis and Applications* , **Vol. 7**, *pp. 37-47*;

[13] Mastroeni G. (1999) Separation methods for Vector Variational Inequalities. Saddle point and gap function, in *Nonlinear Optimization and Applications 2*, G. Di Pillo and F. Giannessi (eds.), Kluwer Academic Publishers;

[14] Ortega J.M., Rheinboldt W.C. (1970) Iterative solutions of nonlinear equations in several variables, Academic Press, New York;

[15] Parida J., Sahoo M. and Kumar A. (1989) A Variational–Like Inequality Problem, Bulletin of the Australian Mathematical Society, **Vol.39**, *pp. 223-231*;

[16] Schaible S. (1992) Generalized monotone maps, in *"Nonsmooth Optimization Methods and Applications"*, F. Giannessi (eds.), Gordon and Breach; *pp. 392-408*;

[17] Yang X.Q. and Chen G.Y.(1992) A Class of Nonconvex Functions and Prevariational Inequalities, *Journal of Mathematical Analysis and Applications* , **Vol. 169**, *pp. 359-373.*