ON CONVEX GENERALIZED SYSTEMS [1]

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Abstract. In the paper, the set-convexity and mapping-convexity properties of the extended images of generalized systems are considered. By using these image properties and the tools of topological linear spaces, separation schemes ensuring the impossibility of generalized systems are developed; then, special problem classes are investigated.

Key words. Generalized systems, image space, generalized convexity.

1 Introduction

One of the main ideas behind the image approach is to transform the analysis of general equilibrium problems and the given space into separation problems in the image space. If the image space is finite-dimensional, then one of the basic tools is the weak separation theorem of nonempty disjoint convex sets in the n-dimensional Euclidean space (Minkowski, 1911). This approach allows to give some geometric interpretations of certain properties of equilibrium systems and to provide new ideas. The aim of the paper is twofold:to develop the study of the set convexity properties of the image of a vector function and to show that the image analysis leads to generalize the classical results stated in the original space in which the problem is defined.

By using these properties of images and the tools of topological linear spaces, separation schemes ensuring the impossibility of generalized systems are developed, then special problem classes are investigated.

In Section 2, generalized systems and their images are defined, then the impossibility of such systems is reduced to an empty intersection of a convex set, and certain conic extensions of images with respect to sets different from the closure of the given convex set. In Sections 3 and 4, the set-convexity and mapping-convexity properties of the extended images of generalized systems are considered, respectively. Separation schemes ensuring the impossibility of generalized systems are developed in Section 5. Special problem classes are investigated in Section 6.

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2 Generalized Systems

Generalized systems seem to be a useful tool in practice to describe equilibrium systems and a general framework to analyse constrained and vector extremum problems, complementarity systems, variational and quasi-variational inequalities (see, e.g., Ref. 1). In order to get further information on the mathematical structure of a real-life equilibrium model, the first step can be a transformation in the given space or in the image space providing a more convenient form. In the given space, several such transformations are known (e.g., transformation of an optimization problem into a convex one if it is possible). In the image spaces, extensive research has recently started in this direction. First, some notations are introduced, then the generalized systems and their image and image transformations are defined.

Let V be a topological linear space on the real numbers $\mathbb{R}$, and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$. A set $\mathcal{H} \subseteq V$ is said to be a cone iff $\lambda\mathcal{H} \subseteq \mathcal{H}$, with $\lambda \in \mathbb{R}_+ \setminus 0$, and a convex cone iff, in addition, $\mathcal{H} + \mathcal{H} \subseteq \mathcal{H}$, where $\mathcal{H} + \mathcal{H} = \{h_1 + h_2 \in V : \ h_1 \in \mathcal{H}, \ h_2 \in \mathcal{H}\}$. *cone* $\mathcal{H} = \{y \in V : y = \lambda x, \quad \lambda \geq 0, \quad x \in \mathcal{H}.\}$ A closed and convex cone C is called pointed iff $C \cap -C = \{0\}$. In a topological space, the closure, the interior,the relative interior, the boundary and the convex hull of a set $\mathcal{H}$ is denoted, respectively, by cl $\mathcal{H}$, int $\mathcal{H}$, ri $\mathcal{H}$, bd $\mathcal{H}$, and conv $\mathcal{H}$. Let $S \in \mathbb{R}^n$, $S^* = \{y \in \mathbb{R}^n : \ \langle x, y\rangle \geq 0, \quad \forall x \in S\}$ is the positive polar of the set S, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

Definition 2.1 Let V be a topological linear space over the real numbers $\mathbb{R}$, $\mathcal{H} \subseteq V$ a convex set, X a Banach space, $K \subseteq X$ a nonempty set, Y a parameter set, and $F : K \times Y \to V$ a mapping.

$$F(x; y) \in \mathcal{H} \subseteq V, \qquad x \in K \subseteq X, \quad y \in Y, \tag{1}$$

is a parametric generalized system in the variable x.

The definition of a generalized system is more general than usual, because a convex set $\mathcal{H}$ is considered instead of a cone, although in practice, there are examples with convex cones. A major question related to generalized systems consists in finding values of y such that (1) be impossible, and in finding methods which show the impossibility of such systems. Separation schemes seem to be one of the most important tools for studying the impossibility of the parametric system (1); to this end, the concept of the image of a set is used.

Example 2.1 Let $X \in \mathbb{R}^n$. Given the vector extremum problem

$$\min{}_C f(x), \quad \text{s.t.} \quad x \in K, \tag{P}$$

where, $f : X \to \mathbb{R}^m$, cl C is a pointed convex cone in $\mathbb{R}^m$, $K \subseteq X$, putting $F(x; y) := f(y) - f(x)$, $\mathcal{H} = C$, $Y = K$, we have that y is a solution of (P) iff the parametric system (1) is impossible. We observe that K can be a continuous or a descrete set, so that continuous as integer programming problems can be considered in the image space analysis.

Example 2.2 Consider the variational inequality

$$\text{find } y \in K \quad \text{s.t.} \quad \langle A(y), x-y\rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$. Putting $F(x;y) := \langle A(y), y-x\rangle, \quad Y := K$, then y is a solution of (VI) iff (1) is impossible.

Definition 2.2 Let V be a topological linear space over the real numbers $\mathbb{R}$, X a Banach space, $K \subseteq X$ a nonempty set, Y a parameter set and $F : K \times Y \to V$ a mapping; then, $\mathcal{K}_y$ $=F(K;y)$, $y \in Y$, is the image of the set K through the map F at a given value of parameter y.

The impossibility of a generalized system means that the intersection of the image $\mathcal{K}_y$ and the given set $\mathcal{H}$ is empty, namely, $\mathcal{K}_y \cap \mathcal{H} = \emptyset$. To prove directly whether $\mathcal{K}_y \cap \mathcal{H} = \emptyset$ or not, is generally too difficult; therefore, in order to show such a disjunction, it will be proved that the two sets, or the set $\mathcal{H}$ and an extension of the image depending on $\mathcal{H}$, lie in two disjoint level sets of a functional; when the functional can be found linear $\mathcal{K}_y$ and $\mathcal{H}$ will be said "linearly separable". The image of a generalized system is not convex, in general, and it may not be connected. In the case of a convex cone $\mathcal{H}$, a regularization of the image set, its conic extension with respect to the cone cl $\mathcal{H}$, denoted by $\mathcal{E}_y$, has been introduced in the form of

$$\mathcal{E}_y = F(K;y) - \text{ cl } \mathcal{H}, \qquad y \in Y. \tag{2}$$

A consequence of (2) is that $\mathcal{K}_y \subseteq \mathcal{E}_y$. The importance of the conic extension for the image of generalized systems is enforced by the following statement (see Ref. 2): If

$$\mathcal{H} + \text{ cl } \mathcal{H} = \mathcal{H}, \tag{3}$$

then the parametric system (1) is impossible iff

$$\mathcal{E}_y \cap \mathcal{H} = \emptyset. \tag{4}$$

Hence, proving impossibility is equivalent to showing disjunction between $\mathcal{H}$ and $\mathcal{E}_y$. In certain cases, it is easier to prove (4) because the conic extension may have some advantageous properties that $\mathcal{K}_y$ has not. In the case of convex optimization, the conic extension is a convex set so that the weak separation theorem of nonempty disjoint convex sets in the n-dimensional Euclidean space can be used.

However, assumption (3) is not fulfilled in every generalized system, e.g., in some cases of vector variational inequalities under inequality constraints. The next example shows such a cone.

Example 2.3 Consider the problem (P) defined in the Example 2.1 and suppose that C is defined by

$$C := \text{int}\mathbb{R}^4_+ \cup \{(x,y,z,w) \in \mathbb{R}^4_+ : \; y=0; \; z=0\} \setminus \{0\}.$$

Recalling that $\mathcal{H} = C$, then, $(1,0,0,1) \in \mathcal{H}$,$(0,1,0,0) \notin \mathcal{H}$, $(0,1,0,0) \in$ cl $\mathcal{H}$, but

$$(1,0,0,1) + (0,1,0,0) = (1,1,0,1) \notin \mathcal{H}.$$

It follows from Example 2.3 that a modification of Theorem 2.1 is necessary to cover these situations related to generalized systems still preserving the advantages of the extended image.

Theorem 2.1 Let V be a linear topological vector space on the real numbers $\mathbb{R}$; $\mathcal{K}, \mathcal{A}, \mathcal{H} \subseteq V$ be arbitrary sets, and $\mathcal{E}(\mathcal{A}) = \mathcal{K} + \mathcal{A}$. If

$$\mathcal{H} - \mathcal{A} = \mathcal{H}, \tag{5}$$

then

$$\mathcal{K} \cap \mathcal{H} = \emptyset \quad \text{iff} \quad \mathcal{E}(\mathcal{A}) \cap \mathcal{H} = \emptyset. \tag{6}$$

Proof. (6) is a direct consequence of the relation

$$(\mathcal{K} + \mathcal{A}) \cap \mathcal{H} = \mathcal{K} \cap (\mathcal{H} - \mathcal{A}).$$

□

Corollary 2.1 Let us consider the parametric system (1), an arbitrary set $\mathcal{A} \subseteq V$ and let $\mathcal{E}_y(\mathcal{A}) = K_y + A$. If

$$\mathcal{H} - \mathcal{A} = \mathcal{H},$$

then the parametric system (1) is impossible iff

$$\mathcal{E}_y(\mathcal{A}) \cap \mathcal{H} = \emptyset. \tag{7}$$

In image space approaches, two important cases can be distinguished: that where the set $\mathcal{H}$ equals a cone with nonempty interior embracing, e.g., scalar and vector optimization problems under inequality constraints, and that where $\mathcal{H}$ is a cone with empty interior corresponding to, e.g., scalar and vector optimization problems under equality and inequality constraints.

Corollary 2.2 Let us consider the parametric system (1) and a convex cone $\mathcal{H}_1 \subseteq V$ with nonempty interior and a convex cone $\mathcal{H}_2 \subseteq V$ with empty interior. If $\mathcal{A}_1 = -\operatorname{int} \mathcal{H}_1 \cup \{0\}$ and $\mathcal{A}_2 = -\mathcal{H}_2 \cup \{0\}$, then the generalized systems related to $\mathcal{H}_1$ and $\mathcal{H}_2$ are impossible iff

$$\mathcal{E}_y(\mathcal{A}_i) \cap \mathcal{H}_i = \emptyset, \qquad i = 1, 2, \tag{8}$$

respectively.

Proof. Since $\mathcal{H}_1 - \mathcal{A}_1 = \mathcal{H}_1 + (\operatorname{int} \mathcal{H}_1 \cup \{0\}) = \mathcal{H}_1 \cup (\mathcal{H}_1 + \operatorname{int} \mathcal{H}_1) = \mathcal{H}_1 \cup ((\operatorname{cl} \mathcal{H}_1 \cup \operatorname{int} \mathcal{H}_1) + \operatorname{int} \mathcal{H}_1) =$
$\mathcal{H}_1 \cup (\operatorname{cl} \mathcal{H}_1 + \operatorname{int} \mathcal{H}_1) \cup (\operatorname{int} \mathcal{H}_1 + \operatorname{int} \mathcal{H}_1) = \mathcal{H}_1 \cup \operatorname{int} \mathcal{H}_1 \cup \operatorname{int} \mathcal{H}_1 = \mathcal{H}_1$, thus, we obtain

$$\mathcal{H}_1 - \mathcal{A}_1 = \mathcal{H}_1,$$

from which the first statement follows.

Similarly,

$$\mathcal{H}_2 - \mathcal{A}_2 = \mathcal{H}_2 + (\mathcal{H}_2 \cup \{0\}) = \mathcal{H}_2 \cup (\mathcal{H}_2 + \mathcal{H}_2),$$

and because H_2 is a convex cone,

$$\mathcal{H}_2 + \mathcal{A}_2 = \mathcal{H}_2,$$

so the second statement is proved. □

Definition 2.3 A generalized system (1) is said to be image convex iff $\forall y \in Y$ such that (1) is impossible, the sets $\mathcal{K}_y$ and $\mathcal{H}$ are linearly separable.

In the next parts, the set-convexity and mapping-convexity properties of extended images $\mathcal{E}_y(\mathcal{A})$ as well as separation schemes based on these results will be studied.

3 Set-Convexities of Extended Images

Some concepts of set-convexity based on recent results (see Refs. 1–5) are considered here in order to develop the study of the extended images. When there will be no fear of confusion and y will be fixed, then $\mathcal{K}_y$ and $\mathcal{E}_y$ will be indicated merely by $\mathcal{K}$ and $\mathcal{E}$, respectively. First, the definitions which seem to be useful in the image space approach are recalled.

Definition 3.1 If V is a topological linear space on $\mathbb{R}$ and $\mathcal{A} \subseteq V$ is a convex subset, then
(i) $\mathcal{K}$ is said to be a convex set with respect to $\mathcal{A}$ (in short, $\mathcal{A}$-convex) iff

$$(1-\alpha)(\mathcal{K}+\mathcal{A}) + \alpha(\mathcal{K}+\mathcal{A}) \subseteq (\mathcal{K}+\mathcal{A}), \qquad \forall \alpha \in]0,1[; \tag{9}$$

(ii) $\mathcal{K}$ is said to be a nearly convex set with respect to $\mathcal{A}$ (in short, $\mathcal{A}$-convex) iff there exists an $\alpha \in (0,1)$ such that

$$(1-\alpha)(\mathcal{K}+\mathcal{A}) + \alpha(\mathcal{K}+\mathcal{A}) \subseteq (\mathcal{K}+\mathcal{A}); \tag{10}$$

if $\alpha = 1/2$, then the set is said to be midpoint $\mathcal{A}$-convex;
(iii) $\mathcal{K}$ is said to be a closely convex set with respect to $\mathcal{A}$ iff the closure of the set $\mathcal{K}+\mathcal{A}$ is a convex set;
(iv) $\mathcal{K}$ is said to be a locally convex or a locally nearly convex set with respect to $\mathcal{A}$ iff $\mathcal{K}+\mathcal{A}$ is convex or nearly convex in a neighbourhood of every point belonging to $\mathcal{K}+\mathcal{A}$;
(v) $\mathcal{K}$ is said to be a coercive set at $k_0 \in \mathcal{K}$ iff there exists a nonempty, (cl $\mathcal{A}$)-compact and strictly convex set $\mathcal{A}_1 \subseteq V$ such that $\mathcal{K} \subseteq \mathcal{A}_1$ and $k_0 \in \text{bd } \mathcal{A}_1$.

Definition 3.1 can be extended for the image of a mapping defined on an arbitrary set. If $\mathcal{A} = \{0\}$, then the convexity, nearly and closely convexity of sets are obtained from (i),(ii) and (iii) in linear topological spaces, respectively. Under the assumption that $\mathcal{A}$ is a convex cone, the $\mathcal{A}$-convex sets were introduced in Ref. 6. Recently, Aleman (Ref. 7) as well as Gwinner and Jeyakumar (Ref. 8) introduced the notion of nearly convex sets. Some properties of nearly convex sets can be found in Refs. 3,7,8. Recently, nearly convexity was used for generalizing quasiconvex functions in the n-dimensional Euclidean space (Ref. 9). The class of nearly convex sets is wider than the convex sets; e.g., the set of all rational numbers in R is nearly convex, but not convex. The most recent definition is the one of closely convex set (Blaga and Kolumbán, Ref. 10). An example is given by the set of all irrational numbers in R. The notion of coercivity was studied in the image space by Pellegrini (Ref. 11).

Two important properties are that both the interior and the closure of a convex set are convex sets in a topological linear space (see e.g., Ref. 12). The next lemma, which is a direct consequence of the definitions and the above statements, shows the relations among the various types of $\mathcal{A}$-convex sets (Refs. 3–5).

Lemma 3.1 If $\mathcal{A} \subseteq V$ is a convex subset and $\mathcal{A}_1 \subseteq V$ a subset, then the implications indicated by the arrows in the next diagram are true:

$$\begin{array}{ccccc} \mathcal{E}(\mathcal{A}_1) \text{ convex} & \Rightarrow & \mathcal{E}(\mathcal{A}_1) \text{ nearly convex} & \Rightarrow & \mathcal{E}(\mathcal{A}_1) \text{ closely convex} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathcal{E}(\mathcal{A}_1+\mathcal{A}) \text{ convex} & \Rightarrow & \mathcal{E}(\mathcal{A}_1+\mathcal{A}) \text{ nearly convex} & \Rightarrow & \mathcal{E}(\mathcal{A}_1+\mathcal{A}) \text{ closely convex.} \end{array}$$

Remark 3.1 By reversing the arrows in Lemma 3.1,implications are obtained that, in general, are not true (Ref. 3).

Theorem 3.1 Let us consider the parametric system (1) and a convex cone $\mathcal{H}_1 \subseteq V$ with nonempty interior and a convex cone $\mathcal{H}_2 \subseteq V$ with empty interior. If $\mathcal{A}_1 = -\text{ int } \mathcal{H}_1 \cup \{0\}$ and $\mathcal{A}_2 = -\mathcal{H}_2 \cup \{0\}$, then the following implications are true:

$$\begin{array}{lll} \mathcal{E}(\mathcal{A}_i) \;\text{ convex} & \Rightarrow \mathcal{E}(-\text{ cl } \mathcal{H}_i) \;\text{ convex}, & i=1,2, \\ \mathcal{E}(\mathcal{A}_i) \;\text{ nearly convex} & \Rightarrow \mathcal{E}(-\text{ cl } \mathcal{H}_i) \;\text{ nearly convex}, & i=1,2, \\ \mathcal{E}(\mathcal{A}_i) \;\text{ closely convex} & \Rightarrow \mathcal{E}(-\text{ cl } \mathcal{H}_i) \;\text{ closely convex}, & i=1,2. \end{array}$$

Proof. If $\mathcal{A} \subseteq V$ is a subset containing the zero-element and $\mathcal{H} \subseteq V$ a convex cone such that $\mathcal{A} \subseteq \mathcal{H}$, then by Lemma 3.1, the following implications are true:

$$\begin{array}{ll} \mathcal{E}(\mathcal{A}) \;\text{ convex} & \Rightarrow \mathcal{E}(\mathcal{H}) \;\text{ convex}, \\ \mathcal{E}(\mathcal{A}) \;\text{ nearly convex} & \Rightarrow \mathcal{E}(\mathcal{H}) \;\text{ nearly convex}, \\ \mathcal{E}(\mathcal{A}) \;\text{ closely convex} & \Rightarrow \mathcal{E}(\mathcal{H}) \;\text{ closely convex}. \end{array}$$

If the sets $\mathcal{A}$ and $\mathcal{H}$ are chosen $\mathcal{A}_i$ and $-$ cl $\mathcal{H}_i$, $i = 1, 2$, respectively, then the theorem is proved. □

Remark 3.2 Theorem 3.1 relates the set-convexity properties of the extended images to those of the classical conical extension, which are usually made with respect to the set $-\text{ cl }\mathcal{H}$; thus the set-convexity results related to the latter one can be considered for the former ones as well.

Theorem 3.2 Let us consider the parametric system (1) and a convex cone $\mathcal{A} \subseteq V$ with $\text{int } \mathcal{A} \neq 0$. Then, the implications indicated by the arrows in the next diagram are true:

$$\begin{array}{ccc} \text{cl } \mathcal{E}(\mathcal{A}) \text{ convex} & \Rightarrow & \mathcal{E}(\text{ int } \mathcal{A}) \text{ convex} \\ \Uparrow & & \Downarrow \\ \text{cl } \mathcal{E}(\text{ int } \mathcal{A}) \text{ convex} & \Leftarrow & \mathcal{E}(\text{ int } \mathcal{A}) \text{ nearly convex.} \end{array}$$

Proof. By Lemma 3.1,

$$\begin{array}{ccc} \text{cl } \mathcal{E}(\mathcal{A}) \text{ convex} & \Rightarrow & \text{int } \mathcal{E}(\mathcal{A}) \text{ convex} \\ \Uparrow & & \Downarrow \\ \text{cl } (\text{ int } \mathcal{E}(\mathcal{A})) \text{ convex} & \Leftarrow & \text{int } \mathcal{E}(\mathcal{A}) \text{ nearly convex.} \end{array}$$

By Ref. 13, if $\mathcal{A} \subseteq V$ is a convex cone with nonempty interior and $\mathcal{K} \subseteq V$ a subset, then

$$\text{int } \mathcal{E}(\mathcal{A}) = \mathcal{E}(\text{int } \mathcal{A}),$$

thus the theorem is proved. □

By Theorem 3.2, if an extended image is open, then its convexity is equivalent to its nearly convexity which property can be exploited using separation arguments. Aleman (Ref. 7) and Paeck (Ref. 5) proved that, if $\mathcal{A} \subseteq V$ is open, then $\mathcal{A}$ is convex iff $\mathcal{A}$ is nearly convex. A corollary (Ref. 8) is that if $\mathcal{A} \subseteq V$ is closed, then $\mathcal{A}$ is convex iff $\mathcal{A}$ is nearly convex.

Corollary 3.1 Let us consider the parametric system (1) and a convex cone $\mathcal{H} \subseteq V$ with nonempty interior. If $\mathcal{A} = -\text{ int}\mathcal{H} \cup \{0\}$, then $\text{cl } \mathcal{E}(\mathcal{A})$ is convex or $\text{int } \mathcal{E}(\mathcal{A})$ is nearly convex iff $\text{int } \mathcal{E}\ (-\text{int } \mathcal{H})$ is convex.

It is recalled that a convex set is strictly convex if its boundary does not contain any line segment.

Corollary 3.2 Let us consider the parametric system (1) and a convex cone $\mathcal{H} \subseteq V$ with nonempty interior. If $\mathcal{A} = -\text{ int } \mathcal{H} \cup \{0\}$, and the set $\mathcal{K} \cap \text{bd } \mathcal{E}(\mathcal{A})$ is convex or the set $(\mathcal{K} \cap \text{bd } \mathcal{E}(\mathcal{A})) \cup \text{ int } \mathcal{E}(\mathcal{A})$ is strictly convex, then $\mathcal{E}(\mathcal{A})$ is convex iff $\text{int } \mathcal{E}(-\text{ int } \mathcal{H})$ is convex.

Proof. The statements follow directly from the following relations:

$$\mathcal{E}(\mathcal{A}) = \mathcal{K} + (-\text{int}\mathcal{H} \cup \{0\}) = \mathcal{K} \cup (\mathcal{K} - \text{ int } \mathcal{H}) = (\mathcal{K} \cap \text{bd } \mathcal{E}(\mathcal{A})) \cup \text{ int } \mathcal{E}(\mathcal{A}),$$

□

4 Convexlikeness of the Image Mappings

Since the beginning of the 80's, convexlike and generalized convexlike conditions have been of interest for deriving generalized alternative theorems of Gordan, Motzkin, Farkas type and Lagrange multiplier results for constrained optimization problems. A real-valued mapping with convexlike property was first considered by Fan (Ref. 13) for generalizing the von Neumann minimax theorem.

The image mappings inherite certain structural features of the given problem, so that the analysis of the relationships between the image and given spaces can provide some possibilities in the image spaces for convexifying classes of originally nonconvex problems. The purpose of this part is to study the different convexlikeness concepts of the image mappings and to characterize them by the corresponding set-convexity properties.

Definition 4.1 If V is a topological linear space on $\mathbb{R}$, $\mathcal{A} \subseteq V$ a convex subset, X a nonempty set and $F : X \to V$ a mapping, then
(i) F is said to be a convexlike mapping with respect to $\mathcal{A}$ (in short, $\mathcal{A}$-convexlike) iff

$$(1-\alpha)F(X) + \alpha F(X) \subseteq F(X) + \mathcal{A}, \qquad \forall \alpha \in (0,1);$$

(ii) F is said to be a nearly convexlike mapping with respect to $\mathcal{A}$ (in short, nearly $\mathcal{A}$-convexlike) iff there exists an $\alpha \in (0,1)$ such that

$$(1-\alpha)F(X) + \alpha F(X) \subseteq F(X) + \mathcal{A};$$

if $\alpha = 1/2$, then the mapping is said to be König convex;
(iii) F is said to be a subconvexlike mapping with respect to $\mathcal{A}$ (in short, $\mathcal{A}$-subconvexlike) iff there exists an $a_0 \in \mathcal{A}$ such that for all $\varepsilon > 0$ (or equivalently, iff for all $a \in \mathcal{A}$)

$$(1-\alpha)F(X) + \alpha F(X) + \varepsilon a_0 \subseteq F(X) + \mathcal{A}, \qquad \forall \alpha \in (0,1);$$

(iv) F is said to be a nearly subconvexlike mapping with respect to $\mathcal{A}$ (in short, $\mathcal{A}$-subconvexlike) iff there exists an $a_0 \in \mathcal{A}$ and $\alpha \in (0,1)$ such that for all $\varepsilon > 0$

$$(1-\alpha)F(X) + \alpha F(X) + \varepsilon a_0 \subseteq F(X) + \mathcal{A};$$

(v) F is said to be a closely convexlike mapping with respect to $\mathcal{A}$ (in short, closely $\mathcal{A}$-convexlike) iff the set $F(X) + \mathcal{A}$ is closely convex.

If $\mathcal{A} = \{0\}$, then convexlike, nearly convexlike and closely convexlike mappings are obtained from (i),(ii) and (iii) in linear topological spaces, respectively. In the definition of the subconvexlikeness, the equivalence of the two characterizations is proved in (Ref. 14). The nearly $\mathcal{A}$-subconvexlike mappings were introduced by Craven and Jeyakumar (Ref. 15), the $\mathcal{A}$-subconvexlike ones by Jeyakumar (Ref. 16). Nearly $\mathcal{A}$-convexlikeness is due to König (Ref. 17). The following statements are simple consequences of Definitions 3.1 and 4.1.

Theorem 4.1 If $\mathcal{A} \in V$ is a convex cone containing the zero-element and $F : X \to V$ a mapping, then
(i) F is an $\mathcal{A}$-convexlike mapping iff the set $\mathcal{E}(\mathcal{A})$ is convex;
(ii) F is a nearly $\mathcal{A}$-convexlike mapping iff the set $\mathcal{E}(\mathcal{A})$ is nearly convex.

Proof. The statements follow from the relation

$$\alpha(F(X)+\mathcal{A})+(1-\alpha)(F(X)+\mathcal{A}) = \alpha F(X)+(1-\alpha)F(X)+\alpha\mathcal{A}+(1-\alpha)\mathcal{A}.$$

□

The first part of Theorem 4.1 can be found in Ref. 18 and the second one in Refs. 5,7,8. By Definitions 3.1 and 4.1 as well as Lemma 3.1 and Theorem 4.1, there is a one-to-one correspondence between the $\mathcal{A}$-convexlike, nearly $\mathcal{A}$-convexlike and closely $\mathcal{A}$-convexlike mappings and the $\mathcal{A}$-convex, nearly $\mathcal{A}$-convex and closely $\mathcal{H}$-convex sets, respectively, thus the diagram of Lemma 3.1 is true for mappings as well. Refs. 3–5 and 19 deal with these correspondences.

Some subclasses and generalizations of the convexlike mappings can be introduced.

Definition 4.2 Let $\gamma : X \times X \times [0,1] \to X$ be a mapping such that $\gamma(X \times X \times [0,1]) \subseteq X$ and $\Psi : V \times V \times [0,1] \to V$. Then, a mapping $F : X \to V$ is a $(\Psi, \gamma, \mathcal{A})$-convexlike mapping if

$$\Psi(F(X) \times F(X) \times [0,1]) \subseteq F(\gamma(X \times X \times [0,1])) + \mathcal{A}.$$

It is well-known that if $X \subseteq \mathbb{R}^n$ is a convex set, $F = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ and the functions f_i, $i = 1, \ldots, m$, are concave, then the mapping F is $(\alpha F(X)+(1-\alpha)F(X), \alpha X+(1-\alpha)X, \mathbb{R}^m_+)$-convexlike mapping, where $\alpha \in [0,1]$.

If $X = \mathcal{M}$ is a complete Riemannian manifold or a geodesic convex set with the geodesics $\gamma(\mathcal{M} \times \mathcal{M} \times [0,1])$ between every two points of $\mathcal{M}$, the mapping $F = (f_1, \ldots, f_m) : \mathcal{M} \to \mathbb{R}^m$ consists of geodesic concave functions (Ref. 20) and $\Psi(F(X) \times F(X) \times [0,1]) = \alpha F(X) + (1-\alpha)F(X)$, $\alpha \in [0,1]$, then F is a (Ψ, γ, R^m_+)-convexlike mapping.

Theorem 4.2 Let us consider the parametric system (1) and a convex cone $\mathcal{H}_1 \subseteq V$ with nonempty interior and a convex cone $\mathcal{H}_2 \subseteq V$ with empty interior. If $\mathcal{A}_1 = -\operatorname{int} \mathcal{H}_1 \cup \{0\}$ and $\mathcal{A}_2 = -\mathcal{H}_2 \cup \{0\}$, then the following implications are true:

$$\begin{aligned}
F \text{ is an } \mathcal{A}_i\text{-convexlike mapping} \quad &\Rightarrow \mathcal{E}(-\operatorname{cl} \mathcal{H}_i) \text{ convex}, \quad i = 1,2,\\
F \text{ is a nearly } \mathcal{A}_i\text{-convexlike mapping} \quad &\Rightarrow \mathcal{E}(-\operatorname{cl} \mathcal{H}_i) \text{ nearly convex}, \quad i = 1,2,\\
F \text{ is a closely } \mathcal{A}_i\text{-convexlike mapping} \quad &\Rightarrow \mathcal{E}(-\operatorname{cl} \mathcal{H}_i) \text{ closely convex}, \quad i = 1,2.
\end{aligned}$$

Theorem 4.3 Let us consider the parametric system (1) and a convex cone $\mathcal{A} \subseteq V$ with $\operatorname{int} \mathcal{A} \neq 0$. Then,

(i) F is an $\mathcal{A}$-subconvexlike mapping iff F is nearly $\mathcal{A}$-subconvexlike and iff the set $\mathcal{E}(\text{ int } \mathcal{A})$ is nearly convex;
(ii) the classes of nearly $\mathcal{A}$-convexlike, $\mathcal{A}$-subconvexlike and closely $\mathcal{A}$-convexlike mappings coincide.

The second part of Theorem 4.3 is a direct consequence of Theorem 3.2. These statements can be found in Ref. 19 and Ref. 5. Corollary 4.1 follows from Corollaries 3.1 and 3.2.

Corollary 4.1 Let us consider the parametric system (1) and a convex cone $\mathcal{H} \subseteq V$ with nonempty interior.
(i) If $\mathcal{A} = -\text{ int } \mathcal{H} \cup \{0\}$, then F is closely $\mathcal{A}$-convexlike or nearly $\mathcal{A}$-convexlike iff F is $(-\text{ int } \mathcal{H})$-convexlike.
(ii) If $\mathcal{A} = -\text{ int } \mathcal{H} \cup \{0\}$, and the set $\mathcal{K} \cap \text{bd } \mathcal{E}(\mathcal{A})$ is convex or the set $(\mathcal{K} \cap \text{bd } \mathcal{E}(\mathcal{A})) \cup \text{ int } \mathcal{E}(\mathcal{A})$ is strictly convex, then F is $\mathcal{A}$-convexlike iff F is $(-\text{ int } \mathcal{H})$-convexlike.

Remark 4.1 The graph $(X, F(X))$ of any generalized $\mathcal{A}$-convexlike mapping can be considered a generalized $(0 \times \mathcal{A})$-convexlike mapping.

By the Aleman and Paeck theorem, if $\mathcal{E}$ is open or closed in the image problem, then the image mapping F is nearly $\mathcal{H}$-convexlike iff F is $\mathcal{H}$-convexlike. If $K \subseteq X$ is a compact set, and the mapping F is continuous, then the image $\mathcal{K}$ is compact. In this case, the extended images formed by closed sets are closed as well (Ref. 21). But in general, the image $\mathcal{K}$ is not open and not closed (e.g., a continuous image mapping can map closed sets onto neither closed nor open sets, Ref. 1). So, it seems to be an interesting question to characterize the convex image problems falling into various $\mathcal{A}$-convexity classes.

Theorems 3.2, 4.2 and Corollaries 3.1, 3.2 and 4.1 state connections between $\mathcal{A}$-convexlike and generalized $\mathcal{A}$-convexlike mappings if the convex cone $\mathcal{A}$ has nonempty interior. It seems to be an open question how to characterize these classes of mappings in the case of a cone or a convex cone with empty interior. The importance of the condition is shown by the fact that such image problems can be raised in the case of equality constraints.

5 Separation Schemes for Generalized Systems

It is observed that the impossibility of the parametric system (1) is equivalent to the condition

$$\mathcal{K}_y \cap \mathcal{H} = \emptyset. \tag{11}$$

Following the scheme introduced in Ref. 22, relation (11) can be proven by showing that the sets $\mathcal{K}_y$ and $\mathcal{H}$ lie in two disjoint level sets of a suitable functional.

It is recalled that a system (1) is said to be image convex in case the sets $\mathcal{K}_y$ and $\mathcal{H}$, if disjoint, admit a separating hyperplane. By means of the analysis developed in the previous sections, sufficient conditions, that ensure the image convexity of the system (1), are stated.

Theorem 5.1 Let V be a linear topological space, $\mathcal{K}_y + \mathcal{A} = \mathcal{E}_y(\mathcal{A})$ where $\mathcal{A}$ is a convex set in V such that $\mathcal{H} - \mathcal{A} = \mathcal{H}$.
(i) If V is a finite dimensional space and the mapping F is $\mathcal{A}$-convexlike, then system (1) is image convex.
(ii) If V is an infinite dimensional space, $\text{int } \mathcal{A} \neq \emptyset$ and any of the following conditions holds:
a) the mapping F is $\mathcal{A}$-convexlike;
b) the mapping F is closely $\mathcal{A}$-convexlike;
c) the mapping F is $\mathcal{A}$-subconvexlike and $\mathcal{A}$ is a convex cone;
d) the mapping F is nearly $\mathcal{A}$-convexlike and $\mathcal{A}$ is a convex cone;
then system (1) is image convex.

Proof. By Theorem 2.2, the hypothesis $\mathcal{H} - \mathcal{A} = \mathcal{H}$ guarantees that $\mathcal{K}_y \cap \mathcal{H} = \emptyset$ iff

$$\mathcal{E}_y(\mathcal{A}) \cap \mathcal{H} = \emptyset. \tag{12}$$

Consequently, we have to prove that if relation (12) holds, then there exists a hyperplane that separates the sets $\mathcal{E}_y(\mathcal{A})$ and $\mathcal{H}$.
(i) If F is $\mathcal{A}$-convexlike, then the set $\mathcal{E}_y(\mathcal{A})$ is convex, and therefore, the thesis holds, since $\mathcal{E}_y(\mathcal{A})$ and $\mathcal{H}$ are two convex disjoint sets.
(ii) If a) holds, then the set $\mathcal{E}_y(\mathcal{A})$ is convex; moreover, since $\text{int } \mathcal{A} \neq \emptyset$, then the set $\text{int } \mathcal{E}_y(\mathcal{A}) \neq \emptyset$, and therefore, it is possible to apply the infinite dimensional Hahn-Banach theorem to achieve the thesis.

Suppose that b) holds. Since $\text{int } \mathcal{A} \neq \emptyset$ and $\mathcal{H} - \mathcal{A} = \mathcal{H}$, then $\text{int } \mathcal{H} \neq \emptyset$. Thus, (12) implies that

$$\mathcal{E}_y(\mathcal{A}) \cap \text{ int } \mathcal{H} = \emptyset. \tag{13}$$

It is known that

$$\text{cl } \mathcal{E}_y(\mathcal{A}) \cap \text{ int } \mathcal{H} \subseteq \text{ cl } (\mathcal{E}_y(\mathcal{A}) \cap \text{ int } \mathcal{H}),$$

therefore, (13) implies that

$$\text{cl } \mathcal{E}_y(\mathcal{A}) \cap \text{ int } \mathcal{H} = \emptyset.$$

Since $\text{cl } \mathcal{E}_y(\mathcal{A})$ and $\text{int } \mathcal{H}$ are convex sets and $\text{int } \mathcal{H} \neq \emptyset$, we obtain that $\mathcal{E}_y(\mathcal{A})$ and $\mathcal{H}$ are linearly separable. To complete the proof we observe that if $\mathcal{A}$ is a convex cone with nonempty interior, then, from Theorem 4.3 (ii), it follows that (b), (c) and (d) are equivalent. □

It is remarked that the existence of a separating hyperplane for the sets $\mathcal{K}_y$ and $\mathcal{H}$ does not guarantee their disjunction which is obtained by introducing further assumptions, usually called regularity conditions for the separation. In Ref. 2, this topic is studied assuming that the system (1) is defined in the form

$$f(x) > 0, \quad g(x) \in C, \qquad x \in \mathbb{R}^n,$$

where

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad g : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

and $C \subseteq \mathbb{R}^m$ is a nonempty convex cone. It will be shown that the results obtained in Ref.2 can be extended to the parametric system (1) when V is a finite dimensional space.

Some preliminary results are needed.

Lemma 5.1 (Ref. 23) Let C be a closed and convex cone in $\mathbb{R}^m$, then C is pointed iff $\text{int } C^* \neq \emptyset$.

Lemma 5.2 If S is a nonempty set in $\mathbb{R}^n$, then

$$S^{**} = \text{ cl } \text{ cone conv S}.$$

Proof. It is known (see Ref. 24) that

$$\text{S}^* = (\text{ cl S})^* = (\text{cone S})^* = (\text{conv S})^*, \tag{14}$$

and if S is a convex cone, then

$$\text{cl } S = S^{**}.$$

Therefore,

$$(\text{cone conv S})^* = (\text{conv S})^*$$

and

$$\text{cl (cone conv S)} = (\text{cone conv S})^{**} = (\text{conv S})^{**} = \text{S}^{**}.$$

□

The following result characterizes the regular separation in terms of polarity.

Theorem 5.2 If $\text{cl } \mathcal{H}$ is a pointed and convex cone, then

$$\text{cl } \text{ cone conv } \mathcal{E}_y(\mathcal{A}) \cap \text{ cl } \mathcal{H} = 0 \quad \Leftrightarrow \quad \mathcal{E}_y(\mathcal{A})^* \cap \text{ int } (-\mathcal{H})^* \neq \emptyset.$$

Proof. ($\Rightarrow$) Ab absurdo, suppose that

$$\mathcal{E}_y(\mathcal{A})^* \cap \text{ int } (-\mathcal{H})^* = \emptyset.$$

From Lemma 5.1, it follows that $\emptyset \neq \text{ int } (\text{ cl } \mathcal{H})^* = \text{ int } (\mathcal{H})^* = - \text{ int } (-\mathcal{H})^*$. Therefore, the sets $\mathcal{E}_y(\mathcal{A})^*$ and $\text{int } (-\mathcal{H})^*$ are linearly separable, i.e.,

$$(\mathcal{E}_y(\mathcal{A})^*)^* \cap -[\text{ int } (-\mathcal{H})^*]^* \neq (\emptyset \cup \{0\}).$$

From Lemma 5.2, it is obtained that

$$\mathcal{E}_y(\mathcal{A})^{**} = \text{ cl } \text{ cone conv } \mathcal{E}_y(\mathcal{A}),$$

and by (14),

$$-[\text{ int } (-\mathcal{H})^*]^* = -[\text{ cl } (\text{ int } (-\mathcal{H})^*)]^* = -[\text{ cl } (-\mathcal{H})^*]^* = \mathcal{H}^{**} = \text{ cl } \mathcal{H},$$

and the absurdity is achieved.

($\Leftarrow$) Let $\lambda \in \mathcal{E}_y(\mathcal{A})^* \cap \text{ int } (-\mathcal{H})^*$. Due to the fact that cl $\mathcal{H}$ is a pointed closed convex cone,

$$\text{int } (-\mathcal{H})^* = \text{ int } (-\text{ cl } \mathcal{H})^* = \{y \in \mathcal{H}^* : \langle x, y\rangle < 0, \quad 0 \neq x \in \text{ cl } \mathcal{H}\},$$

(see, e.g., Ref. 24). Thus,

$$\langle \lambda, u\rangle \geq 0, \quad \forall u \in \mathcal{E}_y(\mathcal{A}), \quad \text{and} \quad \langle \lambda, u\rangle < 0, \quad \forall u \in \text{ cl } \mathcal{H}\backslash\{0\},$$

and the statement is proved. □

Corollary 5.1 Let cl $\mathcal{H}$ be a pointed and convex cone with $0 \notin \mathcal{H}$. If

$$\text{cl cone conv } \mathcal{E}_y(\mathcal{A}) \cap \text{ cl } \mathcal{H} = 0,$$

then $\mathcal{E}_y(\mathcal{A})$ and $\mathcal{H}$ lie in two disjoint level sets of a linear functional.

Now, important results of convex analysis are recalled which ensure the convexity of system (1).

Corollary 5.2 Let V be a linear topological space. If one of the following conditions holds then the parametric system (1) is convex.
(i) The space V is finite dimensional and

$$\text{ri conv}(\mathcal{K}_y) \cap \text{ri}(\mathcal{H}) = \emptyset.$$

(ii) The space V is infinite dimensional,

$$\text{int conv}(\mathcal{K}_y) \neq \emptyset, \quad \text{and} \quad \text{int conv}(\mathcal{K}_y) \cap \mathcal{H} = \emptyset.$$

(iii) The space V is infinite dimensional,

$$\text{int } (\mathcal{H}) \neq \emptyset, \quad \text{and} \quad \text{conv}(\mathcal{K}_y) \cap \text{ int } (\mathcal{H}) = \emptyset.$$

Remark 5.1 If, furthermore, it is required that the sets $\mathcal{K}_y$ and $\mathcal{H}$ be properly linearly separated, then the conditions stated in the previous corollary are necessary, too.

6 Classes of Convex Problems

In this part, some classes of problems are given in the original space so that they generate convex image problems. It is emphasized that a systematization of the convex image problems should allow a deeper view on the nature and the degree of difficulties of nonconvex optimization problems classified only from the point of view of global optimization.

The first remark is that, to our knowledge, the class of optimization problems with convex image is not very wide. Here, the famous Toeplitz-Hausdorff theorem (Ref. 25) is cited which was recently extended by Barvinok (Ref. 26).

Theorem 6.1 (Toeplitz-Hausdorff) The image $F(B)$ of the unit ball $B \subset \mathbb{R}^n$ under a quadratic map $F : \mathbb{R}^n \to \mathbb{R}^2$ is convex.

Theorem 6.2 (Barvinok) If $F = (F_1, \dots, F_k)$ is a vectorial quadratic form on $\mathbb{R}^{n \times d}$, then the image $F(B)$ of the unit ball $B \subset \mathbb{R}^{n \times d}$ under a quadratic vector map $F : \mathbb{R}^{n \times d} \to \mathbb{R}^k$ is convex provided

$$d > (\sqrt{8k+1} - 1)/2, \qquad \text{or equivalently,} \qquad (d+1)(d+2)/2 > k.$$

By varying the values of d and k in the theorem, convexity results, similar to the Toeplitz-Hausdorff theorem, can be obtained.

While the convexity of $\mathcal{K}$ obviously implies the $\mathcal{A}$-convexity by its definition, Theorem 4.1 leads us to find some examples in which $\mathcal{E}(\mathcal{A})$ is convex while $\mathcal{K}$ is not. Thus, a class of originally nonconvex optimization problems can be included in the $\mathcal{H}$-convex or $\mathcal{A}$-convex classes providing a linear separation in the image space between $\mathcal{H}$ and $\mathcal{E}$ or $\mathcal{A}$ and $\mathcal{E}(\mathcal{A})$. A consequence of the linear separation is that, considering an $\mathcal{H}$-convexlike nonlinear optimization problem with inequality constraints, the duality theorem with zero duality gap can be proved.

In the case of convex optimization, the conic extension is a convex set by Theorem 4.1. This $\mathcal{H}$-convex problem class was considered with convex image character for the first time. Hayashi and Komiya considered convexlike optimization problems and established a theorem of the alternative involving $\mathcal{H}$-convexlike functions and studied Lagrangean duality (Ref. 27). Elster and Nehse obtained a saddlepoint optimality condition in the $\mathcal{H}$-convexlike cases (Ref. 28).

Zero duality gap results are presented in infinite-dimensional infinitely constrained optimization problems where the image mapping is nearly $\mathcal{H}$-convex (König convex) (Ref. 29). Gwinner and Jeyakumar used characterizations in order to derive a solvability theorem for nearly-$\mathcal{H}$-convexlike inequality systems (Ref. 8). Weir and Jeyakumar introduced the notion of $\mathcal{H}$-preinvex mapping for obtaining optimality conditions and duality theorems (Ref. 14). Examples for coercivity in the image space can be found in Ref. 11.

To conclude this section a particular case of a real vector valued function $F : X \to \mathbb{R}^m$, $F(x) = (f_1(x), \dots, f_m(x))$ is considered. The aim is to analyse conditions under which the set $F(x) = \{u \in \mathbb{R}^m : \ u = F(x), \ x \in X\}$ admits a supporting hyperplane at a point $F(x^*)$, i.e., there exists $\lambda \in \mathbb{R}^m_+, \ \lambda \neq 0$, such that

$$\langle \lambda, u - F(x^*) \rangle \geq 0, \qquad \forall u \in F(X). \tag{15}$$

This particular choice of $\lambda \in \mathbb{R}^m_+, \ \lambda \neq 0$, is useful in the analysis of optimization problems.

Example 6.1 Consider the problem (P), defined in the Example 2.1, and let $C :=$ int $\mathbb{R}^m_+$, $F(x) := f(x)$. If (15) holds, then the system

$$f(x^*) \ - \ f(x) \ \in C,$$

is impossible, and x^* is an optimal solution of (P).

Obviously, the existence of a supporting hyperplane will depend on some convexity properties of the mapping F. Now, the definition (Refs. 30–31) and some properties of invex functions are recalled.

Definition 6.1 Let $f : X \to R^n$ be differentiable at $x^* \in X$. Then, the function f is said to be invex with respect to $h : X \times X \to R^n$, at x^*, if

$$f(x) - f(x^*) \geq \langle h(x, x^*), \nabla f(x^*)\rangle, \qquad \forall x \in X.$$

It is well-known that if f_i,, $i = 1, \dots, m$, are invex functions with respect to the same h, then $\sum_{i=1}^{m} \lambda_i f_i$ is invex for $\lambda_i \geq 0,\ i = 1, \dots, m$.

Theorem 6.3 Let X be an open set in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$. Suppose that the functions f_i, $i = 1, \dots, m$, are invex at x^* with respect to the same function h. Then, $F(X)$ admits a supporting hyperplane of equation $\langle \lambda, u - F(x^*)\rangle = 0$, $u \in F(X)$, iff the function $L(x, \lambda) = \langle \lambda, F(x)\rangle$ has a stationary point at x^*, i.e., $\nabla L(\lambda, x^*) = 0$.

Proof. If (15) holds, then x^* is a global minimum point for $L(\lambda, x)$ on X, and therefore, $\nabla L(\lambda, x^*) = 0$. Vice-versa, it follows from the hypotheses that the function $L(\lambda, \cdot)$ is invex at x^*, $\forall \lambda \in \mathbb{R}^m_+$; therefore we have that the equality $\nabla L(\lambda, x^*) = 0$ implies the inequality $L(\lambda, x^*) \leq L(\lambda, x)$, $\forall x \in X$, which is equal to condition (15). □

To deepen the analysis, the existence of a supporting hyperplane to a suitable conical extension of the image of the function F is investigated. We use some preliminary results. Let

$$\text{conv}\ \ F(X) = \{u \in \mathbb{R}^m :\ u = \sum_{i=1}^{p} \mu_i F(x_i),\ x_i \in X,\ \sum_{i=1}^{p} \mu_i = 1,\ \mu_i \geq 0,\ i = 1, \dots, p,\ p \in \mathbb{N}\},$$

be the convex hull of the set $F(X)$.

Lemma 6.1 The following conditions are equivalent:

$$(i) \qquad \langle \lambda, u - F(x^*)\rangle \geq 0\ , \quad \forall u \in F(X); \qquad (16)$$

$$(ii) \qquad \langle \lambda, u - F(x^*)\rangle \geq 0\ , \quad \forall u \in \text{conv}\ F(X). \qquad (17)$$

Proof. (17) ⇒ (16) is immediate, since $F(X) \subseteq \text{conv}\ F(X)$.
(16) ⇒ (17). Let $x_i \in X$, $i = 1, \dots, p$. Since

$$\langle \lambda, F(x_i) - F(x^*)\rangle \geq 0 \quad \text{and} \quad \mu_i \geq 0, \qquad \forall i = 1, \dots, p,$$

we obtain that

$$\langle \lambda, \mu_i(F(x_i) - F(x^*))\rangle \geq 0, \qquad \forall i = 1, \dots, p,$$

and therefore,

$$\langle \lambda, \sum_{i=1}^{p} \mu_i F(x_i) - F(x^*)\rangle \geq 0.$$

□

Lemma 6.2 Let $C \subseteq \mathbb{R}^m$ be a convex cone. Then,

$$\text{conv}\,[F(X) + C] = \text{conv}\,F(X) + C.$$

Proof. $\subseteq$) If $x_i \in X$, $c_i \in C$, $i = 1, \dots, p$, then

$$\sum_{i=1}^{p} \mu_i(F(x_i) + c_i) = \sum_{i=1}^{p} \mu_i F(x_i) + \sum_{i=1}^{p} \mu_i c_i \in \text{conv}\,F(X) + C.$$

$\supseteq$) If $x_i \in X$, $i = 1, \dots, p$ and $c \in C$, then

$$\sum_{i=1}^{p} \mu_i F(x_i) + c = \sum_{i=1}^{p} \mu_i(F(x_i) + c),$$

and the inclusion is proven. □

Theorem 6.4 Let the functions f_i, $i = 1, \dots, m$, be invex with respect to the same function h at the point x^*. If there exists $\lambda \in (R^m_+ \cap C^*)$ such that $\nabla L(\lambda, x^*) = 0$, then $\langle \lambda, u - F(x^*) \rangle = 0$ is the equation of a supporting hyperplane for the set conv $[F(X) + C]$.

Proof. From Theorem 6.3, we have that $\langle \lambda, u - F(x^*) \rangle = 0$ is the equation of a supporting hyperplane for $F(X)$ and, by Lemma 6.1, also for conv $F(X)$.

Let $u \in$ conv $[F(X) + C]$. From Lemma 6.2, there exist $x_i \in X$, $\mu_i \geq 0$, $i = 1, \dots, p$, with $\sum_{i=1}^{p} \mu_i = 1$, and $c \in C$ such that $u = \sum_{i=1}^{p} \mu_i F(x_i) + c$.
Then, $\langle \lambda, u - F(x^*) \rangle = \langle \lambda, \sum_{i=1}^{p} \mu_i F(x_i) - F(x^*) \rangle + \langle \lambda, c \rangle \geq 0$.
The last inequality follows from Lemma 6.1 and the condition $\lambda \in C^*$. □

7 Concluding Remarks

The characterizations of the weakened convexity properties for the image mapping provide a technical tool for ensuring the convexity of image problems by the Hahn-Banach Separation Theorem. It is an open question how to characterize the relationships among the different convex image problems if the set $\mathcal{H}$ is convex or a convex cone, both with empty interior.

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