OPTIMIZATION AND CONTROL with APPLICATIONS
Qi, Teo and Yang (eds). Published by KLUWER.

1 GAP FUNCTIONS AND DESCENT METHODS FOR MINTY VARIATIONAL INEQUALITY

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Abstract: A new class of gap functions associated to the variational inequality introduced by Minty is defined. Descent methods for the minimization of the gap functions are analysed in order to develop exact and inexact line-search algorithms for solving strictly and strongly monotone variational inequalities, respectively.

Key words: variational inequality, gap function, descent methods.

1 INTRODUCTION

The gap function approach for Variational Inequalities (for short, VI) has allowed to develop a wide class of descent methods for solving the classic VI defined by the following problem:

$$\text{find } y^* \in K \text{ s.t. } \langle F(y^*), x - y^* \rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

We recall that a gap function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a non-negative function on K, such that $p(y) = 0$ with $y \in K$ if and only if y is a solution of VI. Therefore solving a VI is equivalent to the (global) minimization of the gap function on K.

In the last years the efforts of the scholars have been directed to the study of differentiable gap functions in order to simplify the computational aspects of the problem. See Harker et al (1990), for a survey on the theory and algorithms developed for VI.

The problem of defining a continuously differentiable gap function was first solved by Fukushima (1992) whose approach was generalized by Zhu et al (1994); they proved that

$$g(y) := \max_{x \in K} [\langle F(y), y - x \rangle - G(x, y)]$$

is a continuously differentiable gap function for VI under the following conditions:

$G(x, y) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, is a non–negative, continuously differentiable, strongly convex function on the convex set K with respect to x, such that

$$G(y, y) = 0 \quad \text{and} \quad \nabla_x G(y, y) = 0, \quad \forall y \in K.$$

In the particular case where $G(x, y) := \frac{1}{2}\langle x - y, M(x - y) \rangle$, where M is a symmetric and positive definite matrix of order n, it is recovered the gap function introduced by Fukushima (1992).

Mastroeni (1999) showed that the gap function approach for VI developed by Fukushima (1992), Zhu et al (1994), can be extended to the variational inequality introduced by Minty (1962):

$$\text{find } x^* \in K \text{ s.t. } \langle F(y), x^* - y \rangle \leq 0, \quad \forall y \in K. \tag{VI^*}$$

The interest in the study of Minty Variational Inequality had, at first, theoretical reasons, mainly in the analysis of existence results concerning the classic VI. In fact, under the hypotheses of continuity and pseudomonotonicity of the operator F, VI^* is equivalent to VI (Karamardian (1976)). Recently, John (1998) has shown that VI^* provides a sufficient condition for the stability of equilibrium solutions of autonomous dynamical systems:

$$\frac{dx}{dt} + F(x) = 0, \qquad x \in K,$$

where $x = x(t)$, $t \geq 0$.

Moreover some algorithmic applications have been developed in the field of bundle methods for solving VI (see e.g. Lemarechal et al (1995)).

In this paper, we will deepen the analysis of descent methods for VI^* initiated by Mastroeni (1999). In particular, we will define an inexact line-search algorithm for the minimization of a gap function associated to the problem VI^*.

In Section 2 we will recall the main properties of the gap functions related to VI^*.
In Section 3 we will develop an inexact descent method for VI^*, in the hypothesis of strong monotonicity of the operator F. Section 4 will be devoted to a brief outline of the applications of Minty Variational Inequality and to the, recently introduced, extension to the vector case (Giannessi (1998)).
We recall the main notations and definitions that will be used in the sequel. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said quasi–convex on the convex set K iff:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq max\{f(x_1), f(x_2)\}, \tag{1.1}$$

$\forall x_1, x_2 \in K, \forall \lambda \in [0,1]$.
If f is differentiable on K, then f is quasi–convex on K iff:

$$f(x_1) \leq f(x_2) \quad \Longrightarrow \quad \langle \nabla f(x_2), x_1 - x_2 \rangle \leq 0, \quad \forall x_1, x_2 \in K. \tag{1.2}$$

A function $f : K \longrightarrow \mathbb{R}$ is said strictly quasi–convex iff strict inequality holds in (1.1), for every $x_1 \neq x_2$ and every $\lambda \in (0,1)$. This last definition has been given by Ponstein (1967). Different definitions of strict quasi–convexity can be found in the literature (see e.g. Karamardian (1967)): for a deeper analysis on this topic see Avriel et al (1981) and references therein. A strictly quasi–convex function has the following properties (Thomson et al (1973)):

(i)f is quasi–convex on K,

(ii) every local minimum point of f on K is also a global minimum point on K,

(iii) if f attains a global minimum point x^* on K then x^* is the unique minimum point for f on K.

Let X,Y be metric spaces. A point to set map $A : X \longrightarrow 2^Y$ is upper semicontinuous (for short, u.s.c.) according to Berge at a point $\lambda^* \in X$ if, for each open set $B \supset A\lambda^*$, there exists a neighborhood V of λ^* such that

$$A\lambda \subset B, \quad \forall \lambda \in V.$$

A is lower semicontinuous (for short, l.s.c.) according to Berge at a point $\lambda^* \in X$ if, for each open set B satisfying $B \cap A\lambda^* \neq \emptyset$, there exists a neighborhood V of λ^* such that

$$A\lambda \cap B, \quad \forall \lambda \in V.$$

A is called closed at $\lambda^* \in X$ iff

$$\lambda^k \longrightarrow \lambda^* \in X,\ y^k \longrightarrow y \in Y, \text{with } y^k \in A\lambda^k\ \forall k, \text{implies that } y \in A\lambda^*.$$

A point to set map is called closed on $S \subset X$ if it is closed at every point of S.

We will say that the mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is monotone on K iff:

$$\langle F(y) - F(x), y - x \rangle \geq 0, \quad \forall x, y \in K;$$

it is strictly monotone if strict inequality holds $\forall x \neq y$.
We will say that the mapping F is pseudomonotone on K iff:

$$\langle F(y), x - y \rangle \geq 0 \quad \text{implies} \quad \langle F(x), x - y \rangle \geq 0, \quad \forall x, y \in K.$$

We will say that F is strongly monotone on K (with modulus $\mu > 0$) iff:

$$\langle F(y) - F(x), y - x \rangle \geq \mu \|y - x\|^2, \quad \forall x, y \in K.$$

It is known (Ortega et al (1970)) that, if F is continuously differentiable on K, then F is strongly monotone on K iff

$$\langle \nabla F(y) d, d \rangle \geq \mu \|d\|^2, \quad \forall d \in \mathbb{R}^n, \ \forall y \in K,$$

where ∇F denotes the Jacobian matrix associated to F.

2 A GAP FUNCTION ASSOCIATED TO MINTY VARIATIONAL INEQUALITY

In this section, we will briefly recall the main results concerning the gap function theory for VI^* (Mastroeni (1999)). Following the analysis developed for the classic VI, we introduce the gap function associated to VI^*.

Definition 2.1 *Let $K \subseteq \mathbb{R}^n$. The function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a gap function for VI^* iff:*
i) $p(y) \geq 0, \quad \forall y \in K$;
ii) $p(y) = 0$ and $y \in K$ iff y is a solution for VI^*.

By means of a suitable regularization of the variational inequality, it is possible to define a continuously differentiable gap function for VI^* (Mastroeni (1999)).

Let $H(x, y) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a non-negative, differentiable function, such that

$$H(x, x) = 0, \quad \forall x \in K; \tag{2.1}$$

$$\nabla_y H(x, x) = 0, \quad \forall x \in K. \tag{2.2}$$

Proposition 2.1 *Let K be a convex set in $\mathbb{R}^n$. Suppose that $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, is a non negative, differentiable function on K that fulfils* (2.1) *and* (2.2) *and* $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a differentiable and pseudomonotone operator on* K*. Then*

$$h(x) := \sup_{y \in K} [\langle F(y), x - y \rangle - H(x, y)]$$

is a gap function for VI^**.*

Proof: We observe that $h(x) \geq 0$, $\forall x \in K$. Suppose that $h(x^*) = 0$ with $x^* \in K$. This is equivalent to say that x^* is a global minimum point of the problem

$$\min_{y \in K} [\langle F(y), y - x^* \rangle + H(x^*, y)].$$

The convexity of K implies that x^* is a solution of the variational inequality

$$\langle \nabla_y [q(x^*, x^*) + H(x^*, x^*)], y - x^* \rangle \geq 0, \quad \forall y \in K,$$

where $q(x, y) := \langle F(y), y - x \rangle$. From (2.2) we obtain

$$\langle \nabla_y q(x^*, x^*), y - x^* \rangle \geq 0.$$

Since $\nabla_y q(x, y) = F(y) + \nabla F(y)(y - x)$ then $\nabla_y q(x^*, x^*) = F(x^*)$, which implies that x^* is a solution of VI. By the pseudomonotonicity of F, we obtain that x^* is also a solution of VI^*.
Now suppose that x^* is a solution of VI^*. Since $H(x, y)$ is non negative, we have that

$$\langle F(y), y - x^* \rangle + H(x^*, y) \geq 0, \quad \forall y \in K,$$

which is equivalent to the condition

$$\max_{y \in K} [\langle F(y), x^* - y \rangle - H(x^*, y)] = 0.$$

Since $h(x) \geq 0$, $\forall x \in K$, we obtain

$$h(x^*) = \min_{x \in K} \max_{y \in K} [\langle F(y), x - y \rangle - H(x, y)] = 0.$$

□

Let us consider the differentiability properties of the function $h(x)$.

Proposition 2.2 *Let* K *be a nonempty compact convex set in* $\mathbb{R}^n$*. Suppose that* F *is continuous on an open set* $A \supset K$*,* $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is continuously differentiable on* $A \times A$ *and the function* $\phi(x, y) := \langle F(y), y - x \rangle + H(x, y)$ *is strictly quasi convex with respect to* y*,* $\forall x \in K$*, then* $h(x)$ *is continuously differentiable on* K *and its gradient is given by*

$$\nabla h(x) = F(y(x)) - \nabla_x H(x, y(x))$$

where $y(x)$ *is the solution of the problem* $\min_{y \in K} \phi(x, y)$*.*

Proof: We observe that

$$h(x) = -\inf_{y \in K} \phi(x, y) \tag{2.3}$$

Since $\phi(x, y)$ is strictly quasi convex with respect to y then there exists a unique minimum point $y(x)$ of the problem (2.3) . Applying Theorem 4.3.3 of Bank et al (1983) (see the Appendix), we obtain that $y(x)$ is u.s.c. according to Berge at x and, being $y(x)$ single-valued, it follows that $y(x)$ is continuous at x.

Since F is continuous and H is continuously differentiable then $\nabla_x \phi$ is continuous. Therefore, from theorem 1.7 Chapter 4 of Auslender (1976) (see the Appendix), taking into account that (2.3) has a unique minimum point, it follows that h is differentiable in the sense of Gateaux at x and

$$h'(x) = -\nabla_x \phi(x, y(x)).$$

From the continuity of F, $y(x)$ and $\nabla_x H$, it follows that $h'(x)$ is continuous at x so that h is continuously differentiable and

$$\nabla h(x) = h'(x) = F(y(x) - \nabla_x H(x, y(x)).$$

□

3 EXACT AND INEXACT DESCENT METHODS

In the previous section we have shown, that under suitable assumptions on the operator F and the function H, the gap function associated to the variational inequality VI^*:

$$h(x) := \sup_{y \in K} [\langle F(y), x - y \rangle - H(x, y)]$$

is continuously differentiable on K. This considerable property allows us to define descent direction methods for solving the problem

$$\min_{x \in K} \; h(x). \tag{3.1}$$

After recalling an exact descent method proposed by Mastroeni (1999), we will analyse an inexact line search method. We will assume that

1. K is a nonempty compact and convex set in $\mathbb{R}^n$;
2. $\phi(x, y) := \langle F(y), y - x \rangle + H(x, y)$ is strictly quasi convex with respect to y, $\forall x \in K$;
3. F is a continuously differentiable operator on an open set $A \supset K$;
4. $H(x, y) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a non negative function on K, which is continuously differentiable on $A \times A$. Moreover, we suppose that it fulfils conditions (2.1) and (2.2) and the further assumption:

$$\nabla_x H(x, y) + \nabla_y H(x, y) = 0, \quad \forall x, y \in K. \tag{3.2}$$

Remark 3.1 The hypothesis 4 is fulfilled by the function $H(x,y) := \frac{1}{2}\langle M(x-y), x-y\rangle$ where M is a symmetric matrix of order n. With this choice of the function H, the hypothesis 2 is fulfilled when $\langle F(y), y-x\rangle$ is convex with respect to y, $\forall x \in K$, and M is positive definite; for example when $F(y) = Cy+b$ where C is a positive semidefinite matrix of order n and $b \in \mathbb{R}^n$. A characterization of strict quasi convexity, in the differentiable case, is given in Theorem 3.26 of Avriel et al (1981).
In order to obtain a function H which fulfils (2.1),(2.2) and the condition (3.2), as noted by Yamashita et al (1997), it must necessarily be

$$H(x,y) = \psi(x-y),$$

where $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is nonnegative, continuously differentiable and such that $\psi(0) = 0$.

We recall that, from Proposition 2.2, h is a continuously differentiable function and $\nabla h(x) = F(y(x)) - \nabla_x H(x, y(x))$, where $y(x)$ is the solution of the problem

$$\min_{y\in K} \ \phi(x,y). \qquad P(x)$$

Lemma 3.1 *Suppose that the hypotheses* 1–4 *hold and, furthermore,* $\nabla F(y)$ *is a positive definite matrix,* $\forall y \in K$*. Let* $y(x)$ *be the solution of* $P(x)$*. Then* x^* *is a solution of* VI^* *iff* $x^* = y(x^*)$*.*

Proof: Since $\nabla F(y)$ is a positive definite matrix, $\forall y \in K$, and F is continuously differentiable, then F is a strictly monotone operator (Ortega et al (1970), Theorem 5.4.3). Therefore x^* is a solution of VI^* iff

$$0 = h(x^*) = -\min_{y\in K} \phi(x^*, y)$$

and, by the uniqueness of the solution, iff $y(x^*) = x^*$. □

Next result proves that $y(x) - x$ provides a descent direction for h at the point x, when $x \neq x^*$.

Proposition 3.1 *Suppose that the hypotheses* 1–4 *hold and* F *is strongly monotone on* K *(with modulus* $\mu > 0$*). Let* $y(x)$ *be the solution of the problem* $P(x)$ *and* $d(x) := y(x) - x$*. Then*

$$\langle \nabla h(x), d(x)\rangle \le -\mu \|d(x)\|^2.$$

Proof: Since K is a convex set $y(x)$ fulfils the condition

$$\langle \nabla_y \phi(x, y(x)), z - y(x)\rangle \ge 0, \forall z \in K,$$

that is, putting $q(x,y) := \langle F(y), y-x\rangle$,

$$\langle \nabla_y q(x, y(x)), z - y(x)\rangle + \langle \nabla_y H(x, y(x)), z - y(x)\rangle \ge 0, \quad \forall z \in K.$$

In particular for $z := x$ we obtain

$$\langle \nabla_y q(x, y(x)), x - y(x) \rangle \geq -\langle \nabla_y H(x, y(x)), x - y(x) \rangle. \tag{3.3}$$

Since $\nabla_y q(x,y) = F(y) + \nabla F(y)(y - x)$, taking into account assumption 4 and (3.3), we have
$\langle \nabla_x h(x), y(x) - x \rangle = \langle F(y(x)), y(x) - x \rangle - \langle \nabla_x H(x, y(x)), y(x) - x \rangle \leq \langle F(y(x)), y(x) - x \rangle + \langle \nabla_y q(x, y(x)), x - y(x) \rangle = \langle F(y(x)), y(x) - x \rangle + \langle F(y(x)), x - y(x) \rangle + \langle \nabla F(y(x))(y(x) - x), x - y(x) \rangle =$

$$\langle \nabla F(y(x))(y(x) - x), x - y(x) \rangle \leq -\mu \|d(x)\|^2,$$

and the proposition is proved.

Remark 3.2 If we replace the hypothesis of strong monotonicity of the operator F, with the one of strict monotonicity, we obtain the weaker descent condition:

$$\langle \nabla h(x), d(x) \rangle < 0,$$

provided that $y(x) \neq x$.

The following exact line search algorithm has been proposed by Mastroeni (1999):

Algorithm 1.

Step 1. Let $x_0 \in K$, ϵ be a tolerance factor and $k = 0$. If $h(x_0) = 0$, then STOP, otherwise go to step 2.

Step 2. Let $d_k := y(x_k) - x_k$.

Step 3. Let $t_k \in [0, 1]$ be the solution of the problem

$$min\{h(x_k + td_k) : \ 0 \leq t \leq 1\}; \tag{3.4}$$

put $x_{k+1} = x_k + t_k d_k$.
If $\|x_{k+1} - x_k\| < \epsilon$, then STOP, otherwise let $k = k + 1$ and go to step 2.

The following convergence result holds (Mastroeni (1999)):

Theorem 3.1 *Suppose that the hypotheses* 1–4 *hold and* $\nabla F(y)$ *is positive definite,* $\forall y \in K$*. Then, for any* $x_0 \in K$ *the sequence* $\{x_k\}$ *defined by Algorithm 1 belongs to the set* K *and converges to the solution of the variational inequality* VI^**.*

Proof: Since $\nabla F(y)$ is positive definite $\forall y \in K$, and F is continuously differentiable then F is a strictly monotone operator (Ortega et al (1970), Theorem 5.4.3) and therefore both problems VI and VI^* have the same unique solution. The convexity of K implies that the sequence $\{x_k\} \subset K$ since $t_k \in [0, 1]$. It is proved in the Proposition 2.2 that the function $y(x)$ is continuous, which implies the continuity of $d(x)$. It is known (see e.g. Minoux (1986), Theorem 3.1) that the map

$$U(x,d) := \{y : y = x + td, 0 \leq t \leq 1, h(y) = \min_{0 \leq t \leq 1} h(x + td)\}$$

is closed whenever h is a continuous function. Therefore the algorithmic map $x_{k+1} = U(x_k, d(x_k))$ is closed, (see e.g. Minoux (1986), Proposition 1.3). Zangwill's convergence theorem (Zangwill (1969)) (see the Appendix) implies that any accumulation point of the sequence $\{x_k\}$ is a solution of VI^*. Since VI^* has a unique solution, the sequence $\{x_k\}$ converges to the solution of VI^*. □

Algorithm 1 is based on an exact line search rule: it is possible to consider the inexact version of the previous method.

Algorithm 2.

Step 1. Let x_0 be a feasible point, ϵ be a tolerance factor and β, σ parameters in the open interval $(0, 1)$. Let $k = 0$.

Step 2. If $h(x_k) = 0$, then STOP, otherwise go to step 3.

Step 3. Let $d_k := y(x_k) - x_k$. Select the smallest nonnegative integer m such that

$$h(x_k) - h(x_k + \beta^m d_k) \geq \sigma \beta^m \|d_k\|^2,$$

set $\alpha_k = \beta^m$ and $x_{k+1} = x_k + \alpha_k d_k$.
If $\|x_{k+1} - x_k\| < \epsilon$, then STOP, otherwise let $k = k + 1$ and go to step 2.

Theorem 3.2 *Suppose that the hypotheses* 1–4 *hold, F is a strongly monotone operator on K with modulus μ, $\sigma < \mu/2$, and $\{x_k\}$ is the sequence defined in the Algorithm 2.*

Then, for any $x_0 \in K$, the sequence $\{x_k\}$ belongs to the set K and converges to the solution of the variational inequality VI^*.

Proof: The convexity of K implies that the sequence $\{x_k\} \subset K$, since $\alpha_k \in [0, 1]$. The compactness of K ensures that $\{x_k\}$ has at least one accumulation point. Let $\{\tilde{x}_k\}$ be any convergent subsequence of $\{x_k\}$ and x^* be its limit point.
We will prove that $y(x^*) = x^*$ so that, by Lemma 3.1, x^* is the solution of VI^*.

Since $y(x)$ is continuous (see the proof of Proposition 2.2) it follows that $d(x)$ is continuous; therefore we obtain that $d(\tilde{x}_k) \longrightarrow d(x^*) =: d^*$ and $h(\tilde{x}_k) \longrightarrow h(x^*) =: h^*$. By the line search rule we have

$$h(\tilde{x}_k) - h(\tilde{x}_{k+1}) \geq \sigma \tilde{\alpha}_k \|d(\tilde{x}_k)\|^2, \quad \forall k \in N, \tag{3.5}$$

for a suitable subsequence $\{\tilde{\alpha}_k\} \subseteq \{\alpha_k\}$.

Let us prove the relation (3.5). We observe that, by the line search rule, the sequence $\{h(x_k)\}$ is strictly decreasing. Let $k \in N$ and $x_{\bar{k}} := \tilde{x}_k$, for some $\bar{k} \in N$; we have

$$h(\tilde{x}_k) - h(\tilde{x}_{k+1}) \geq h(x_{\bar{k}}) - h(x_{\bar{k}+1}) \geq \sigma \alpha_{\bar{k}} \|d(x_{\bar{k}})\|^2.$$

Putting $\tilde{\alpha}_k := \alpha_{\bar{k}}$, we obtain (3.5).

Therefore,

$$\tilde{\alpha}_k \|d(\tilde{x}_k)\|^2 \longrightarrow 0.$$

If $\tilde{\alpha}_k > \beta^{m_0} > 0$, for some m_0, $\forall k > \bar{k} \in N$, then $\|d(\tilde{x}_k)\| \longrightarrow 0$ so that $y(x^*) = x^*$.
Otherwise suppose that there exists a subsequence $\{\alpha_{k'}\} \subseteq \{\tilde{\alpha}_k\}$ such that $\alpha_{k'} \longrightarrow 0$. By the line search rule we have that

$$\frac{h(x_{k'}) - h(x_{k'} + \bar{\alpha}_{k'} d(x_{k'}))}{\bar{\alpha}_{k'}} < \sigma \|d(x_{k'})\|^2, \tag{3.6}$$

where $\bar{\alpha}_{k'} = \frac{\alpha_{k'}}{\beta}$.
Taking the limit in (3.6) for $k \longrightarrow \infty$, since $\bar{\alpha}_{k'} \longrightarrow 0$ and h is continuously differentiable, we obtain

$$-\langle \nabla h(x^*), d^* \rangle \leq \sigma \|d^*\|^2. \tag{3.7}$$

Recalling Proposition 3.1, we have also

$$-\langle \nabla h(x^*), d^* \rangle \geq \mu \|d^*\|^2.$$

Since $\sigma < \frac{\mu}{2}$, it must be $\|d^*\| = 0$, which implies $y(x^*) = x^*$. □

4 SOME APPLICATIONS AND EXTENSIONS OF MINTY VARIATIONAL INEQUALITY

Besides the already mentioned equivalence with the classic VI, Minty variational inequality enjoys some peculiar properties that justify the interest in the development of the analysis. We will briefly recall some applications in the field of optimization problems and in the theory of dynamical systems. Finally we will outline the recently introduced extension to the vector case (Giannessi (1998)).

Consider the problem

$$\min\ f(x), \quad \text{s.t.} \quad x \in K, \tag{4.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuously differentiable function on the convex set K.

The following statement has been proved by Komlosi (1999).

Theorem 4.1 *Let $F := \nabla f$. If x^* is a solution of VI^* then x^* is a global minimum point for (4.1).*

In some particular cases, the previous result leads to an alternative characterization of a global minimum point of (4.1).

Corollary 4.1 *Let $F := \nabla f$ and suppose that f is a quasi–convex function on K. Then x^* is a solution of VI^* if and only if it is a global minimum point for (4.1).*

Proof: Suppose that x^* is a global minimum point of (4.1). By the equivalent characterization (1.2) of the quasi-convexity, in the differentiable case, it follows that x^* is a solution of VI^*. The converse implication follows from Theorem 4.1. □

A further interesting application can be found in the field of autonomous dynamical systems:

$$\frac{dx}{dt} + F(x) = 0, \quad x \in K, \tag{DS}$$

where $x = x(t)$, $t \geq 0$.

Suppose that ∇F is continuous on the set

$$\Omega := \{x \in K : \|x\| < A\},$$

where $A > 0$, so that there exists a unique solution $x(t)$ of DS with $x(t_0) = x_0$. Consider an equilibrium point $x^* \in \Omega$, which fulfils the relation $F(x^*) = 0$. It is obvious that

$$x(t) = x^*, \ \forall t \geq 0, \quad x(t_0) = x^*,$$

is a solution for DS. The following definition clarifies the concept of stability of the previous solution.

Definition 4.1 *The equilibrium point x^* is said stable for DS if, for every $0 \leq \epsilon < A$, there exists $0 \leq \delta \leq \epsilon$ such that if $\|x_0 - x^*\| \leq \delta$, then $\|x(t) - x^*\| \leq \epsilon, \forall t \geq 0$, where $x(t)$ is the solution of DS with the initial condition $x(t_0) = x_0$.*

Minty Variational Inequality provides a sufficient condition for the equilibrium point x^* to be stable.

Theorem 4.2 (John (1998)) *Let x^* be an equilibrium point for DS. If*

$$\langle F(y), x^* - y \rangle \leq 0, \quad \forall y \in \Omega,$$

then x^* is stable.

Giannessi (1998) has extended the analysis of VI^* to the vector case and has obtained a first order optimality condition for a Pareto solution of the vector optimization problem:

$$\min_{C \setminus \{0\}} f(x) \quad \text{s.t.} \quad x \in K, \tag{4.2}$$

where C is a convex cone in $\mathbb{R}^\ell$, $f : K \longrightarrow \mathbb{R}^\ell$ and $K \subseteq \mathbb{R}^n$.
The Minty vector variational inequality is defined by the following problem:
find $x \in K$ such that

$$F(y)(x-y) \not\geq_{C\setminus\{0\}} 0, \quad \forall y \in K, \qquad (VVI^*)$$

where, $a \geq_{C\setminus\{0\}} b$ iff $a - b \in C \setminus \{0\}$, $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{\ell\times n}$.
We observe that, if $C := \mathbb{R}_+$, then Minty vector variational inequality collapses into VI^*.

In the hypotheses that $C = \text{int}\mathbb{R}^\ell_+$, $F = \nabla f$ and f is a (componentwise) convex function, Giannessi (1998) proved that x is an optimal solution for (4.2) if and only if it is a solution of VVI^*.
Further developments in the analysis of VVI^* can be found in Giannessi (1998), Komlosi (1999), Mastroeni (2000).

5 CONCLUDING REMARKS

We have shown that the gap function theory developed for the classic VI, introduced by Stampacchia, can be extended, under further suitable assumptions, to the Minty Variational Inequality. These extensions are concerned not only with the theoretical point of view, but also with the algorithmic one: under strict or strong monotonicity assumptions on the operator F, exact or inexact descent methods, respectively, can be defined for VI^* following the line developed for VI.

It would be of interest to analyse the relationships between the class of gap functions associated to VI and the one associated to VI^* in the hypothesis of pseudomonotonicity of the operator F, which guarantees the equivalence of the two problems. This might allow to define a resolution method, based on the simultaneous use of both gap functions related to VI and VI^*.

6 APPENDIX

In this appendix we recall the main theorems that have been employed in the proofs of the results stated in the present paper. Theorem 6.1 (Bank et al (1983)) is concerned with the continuity of the optimal solution map of a parametric optimization problem. Theorem 6.2 (Auslender (1976)) is a generalization of well-known results on directional differentiability of extremal-value functions. Theorem 6.3 is the Zangwill convergence theorem for a general algorithm formalized under the form of a multifunction.

Consider the following parametric optimization problem:

$$v(x) := inf\{f(x,y) \quad s.t. \quad y \in M(x)\},$$

where $f : \Lambda \times Y \longrightarrow \mathbb{R}$, $M : \Lambda \longrightarrow 2^Y$, $Y \subseteq \mathbb{R}^n$ and Λ is a metric space.

Let $\psi : \Lambda \longrightarrow 2^Y$ be the optimal set mapping

$$\psi(x) = \Big\{ y \in M(x) : f(x,y) = v(x) \Big\}.$$

Theorem 6.1 (Bank et al (1983), Theorem 4.3.3) *Let $Y := \mathbb{R}^n$ and $x^0 \in \Lambda$. Suppose that the following condition are fulfilled:*

1. *$\psi(x^0)$ is non-empty and bounded;*
2. *f is lower semicontinuous on $\{x^0\} \times Y$ and a point $y^0 \in \psi(x^0)$ exists such that f is upper semicontinuous at (x^0, y^0);*
3. *$f(x, \cdot)$ is quasiconvex on Y for each fixed $x \in \Lambda$;*
4. *$M(x)$ is a convex set, $\forall x \in \Lambda$;*
5. *$M(x^0)$ is closed and the mapping M is closed and lower semicontinuous, according to Berge, at x^0.*

Then ψ is upper semicontinuous according to Berge at x^0.

We observe that, if $M(x) = K$, $\forall x \in \Lambda$, where K is a nonempty convex and compact set in $\mathbb{R}^n$, then the assumptions *1,4* and *5*, of Theorem 6.1, are clearly fulfilled and it is possible to replace the assumption $Y := \mathbb{R}^n$ with $Y := K$.

Next result is well-known and can be found in many generalized versions: we report the statement of Auslender (1976). We recall that a function $h : \mathbb{R}^p \longrightarrow \mathbb{R}$ is said to be "directionally differentiable" at the point $x^* \in \mathbb{R}^p$ in the direction d, iff there exists finite:

$$\lim_{t \to 0^+} \frac{h(x^* + td) - h(x^*)}{t} =: h'(x^*; d).$$

If there exists $z^* \in \mathbb{R}^p$ suct that $h'(x^*, d) = \langle z^*, d \rangle$ then h is said to be differentiable in the sense of Gateaux at x^*, and z^* is denoted by $h'(x^*)$.

Theorem 6.2 *(*Auslender (1976), Theorem 1.7, Chapter 4) *Let*

$$v(x) := \inf_{y \in Y} f(x,y),$$

where $f : \mathbb{R}^p \times Y \longrightarrow \mathbb{R}$. Suppose that

1. *f is continuous on $\mathbb{R}^p \times Y$;*
2. *$\nabla_x f$ exists and is continuous on $\Omega \times Y$, where Ω is an open set in $\mathbb{R}^p$;*
3. *Y is a closed set in $\mathbb{R}^n$;*
4. *For every $x \in \mathbb{R}^p$, $\psi(x) := \{y \in Y : f(x,y) = v(x)\}$ is nonempty and there exists a neighbourhood $V(x)$ of x, such that $\cup_{z \in V(x)} \psi(z)$ is bounded.*

Then, for every $x \in \Omega$, we have:

$$v'(x; d) = \inf_{y \in \psi(x)} \langle \nabla_x f(x, y), d \rangle.$$

Moreover, if for a point $x^* \in \Omega$, $\psi(x^*)$ contains exactly one element $y(x^*)$, then v is differentiable, in the sense of Gateaux, at x^* and

$$v'(x^*) = \nabla_x f(x^*, y(x^*)).$$

We observe that, when Y is a nonempty compact set, then the assumptions *3* and *4*, of Theorem 6.2, are obviously fulfilled.

The reader can also refer to Hogan (1973) and references therein for similar versions of the previous theorem.

Finally, we recall the statement of Zangwill Convergence Theorem as reported in Minoux (1986). Given an optimization problem P defined on $X \subseteq \mathbb{R}^n$, let $\mathcal{M}$ be the set of the points of X that fulfil a suitable necessary optimality condition. Suppose that, in order to solve P, it is used an algorithm represented by a point to set map $A : X \longrightarrow 2^X$.

Definition 6.1 *We say that $z : X \longrightarrow \mathbb{R}$ is a descent function (related to the algorithm A) if it is continuous and has the following properties:*

1. *$x \notin \mathcal{M}$ implies $z(y) < z(x) \quad \forall y \in A(x)$,*
2. *$x \in \mathcal{M}$ implies $z(y) \leq z(x) \quad \forall y \in A(x)$.*

Theorem 6.3 (Zangwill (1969)) *Let P be an optimization problem on X and $\mathcal{M}$ be the set of the points of X that fulfil a certain necessary optimality condition.*
Let $A : X \longrightarrow 2^X$ be the algorithmic point to set mapping and consider a sequence $\{x^k\}$ generated by the algorithm, i.e. satisfying $x^{k+1} \in A(x^k)$.
Suppose that the following three conditions hold:

1. *Every point x^k is contained in a compact set $K \subset X$;*
2. *There exists a descent function z;*
3. *The point to set map A is closed on $X \setminus \mathcal{M}$ and $\forall x \in X \setminus \mathcal{M},\ A(x) \neq \emptyset$.*

Then, for every x which is the limit of a convergent subsequence of $\{x^k\}$, we have that $x \in \mathcal{M}$.

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