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Separation methods for Vector Variational Inequalities. Saddle point and gap function

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Abstract

The image space approach is applied to the study of vector variational inequalities. Exploiting separation arguments in the image space, Lagrangian type optimality conditions and gap functions for vector variational inequalities are derived.

Keywords: Vector Variational Inequalities, Image Space, Separation.

1 Introduction

The theory of variational inequalities finds applications in many fields of optimization: from the classical optimality conditions for constrained extremum problems to the equilibrium conditions for network flow, economic and mechanical engineering equilibrium problems [8, 10]. In recent years, variational inequalities, that were first introduced in a scalar form, have been generalized to the vector case [7].

In this paper, by means of the image space analysis, we extend the theory of the gap functions [5, 16] to vector variational inequalities (in short, VVI) defined by the following problem:

$$\text{find} \quad x \in K \quad s.t. \quad F(x)(y-x) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K,$$

where $F : X \longrightarrow R^{p\times n}$, $K \subseteq X \subseteq R^n$, C is a convex cone in R^p; in the definition of a VVI we have used the notation: $x \not\geq_C y$ iff $x - y \notin C$. When $p = 1$ and $C = R_+$, the VVI collapses to the classic variational inequality (VI).
Given the vector optimization problem:

$$\min_{C\setminus\{0\}} h(x) \quad s.t. \quad x \in K, \tag{P}$$

where $h : X \longrightarrow R^p$, in the hypotheses that h is a (componentwise) convex differentiable function on the convex set K, it is known [6] that, if we put $F := \nabla h$, then the VVI is a sufficient optimality condition for (P).

The image space analysis can be applied everytime the problem, we want to deal with, is expressed under the form of the impossibility of a suitable system

$$f(x, y) \in C \setminus \{0\}, \quad g(y) \in D, \quad y \in X, \qquad S(x)$$

where $f : X \times X \longrightarrow R^p$, $X \subseteq R^n$, C is a convex cone in R^p, $g : X \longrightarrow R^m$, D is a closed convex cone in R^m. The space R^{p+n} in which the function (f, g) runs, is called the image space associated to $S(x)$. The impossibility of $S(x)$ is stated by means of separation arguments in the image space, proving that two suitable subsets of the image space lie in disjoint level sets of a separating functional.

We recall that a gap function $p : K \longrightarrow R$ is a non-negative function that fulfils the condition $p(x) = 0$ if and only if x is a solution of VI on K. This definition, which originally has been given for scalar variational inequalities, can be extended to the vector case, so that solving a VVI is equivalent to minimize p on the feasible set K. In Section 2 we will analyse the general features of the image space approach for generalized systems. In Section 3 we will consider the general applications to the VVI while, in Section 4, following the approach introduced in [8], we will show how the separations tecniques in the image space, allow to define a gap function for a VVI.

We recall the main notations and definitions that will be used in the sequel. Let $M \subseteq R^p$. $intM, clM$, will denote the interior and the closure of M, respectively.
Let $y \in R^p$, $y := (y_1, \ldots, y_p)$; $y_{(1^-)} := (y_2, \ldots, y_p)$,
$y_{(i^-)} := (y_1, \ldots, y_{i-1},\ y_{i+1}, \ldots, y_p)$, $i = 2, \ldots, p-1$, $y_{(p^-)} := (y_1, \ldots, y_{p-1})$.
$\langle \cdot, \cdot \rangle$ is the scalar product in R^p, $y \geq 0$ iff $y_i \geq 0$, i=1,. . .,p. $R^p_+ := \{x \in R^p : x \geq 0\}$.
Let $D \subseteq R^m$ be a convex cone, the positive polar of D is the set $D^* := \{x^* \in R^m : \langle x^*, x \rangle \geq 0,\ \forall x \in D\}$. A closed convex cone D is said pointed if $D \cap (-D) = \{0\}$.
Let $g : R^n \longrightarrow R^m$. g is said D–function on the convex set $K \in R^n$ iff:

$$g(\lambda x_1 + (1 - \lambda)x_2) - \lambda g(x_1) - (1 - \lambda)g(x_2) \in D, \quad \forall x_1, x_2 \in K,\ \forall \lambda \in (0, 1).$$

$x^* \in K$ is said a vector minimum point (in short v.m.p.) for (P) iff the following system is impossible:

$$h(x^*) - h(y) \in C \setminus \{0\}, \quad y \in K.$$

2 A separation scheme for generalized systems

In this section we will present the image space analysis for the generalized system $S(x)$, giving particular attention to the linear separation arguments and the regularity conditions that will allow to state the impossibility of $S(x)$. Suppose that the feasible set is defined by $K := \{y \in X :\ g(y) \in D\}$, and consider the following problem:

find $x^* \in K$, s.t. $S(x^*)$ is impossible.

It is easy to see that vector optimization problems and vector variational inequalities can be formulated as the impossibility of the system $S(x)$ choosing a suitable function $f(x, y)$.

The following result is an immediate consequence of the definition of an optimal solution of a vector optimization problem and the statement of a VVI.

Proposition 2.1 *1. Let $f(x, y) := h(x) - h(y)$, then x^* is a v.m.p. for (P) iff $S(x^*)$ is impossible.*

2. Let $f(x, y) := F(x)(x - y)$, then x^* is a solution of VVI iff the system $S(x^*)$ is impossible.

Define the following subsets of the space R^{p+m}, that we will call "the image space" associated to the system $S(x)$:

$$\mathcal{K}(x) := \{(u, v) \in R^{p+m} : \ u = f(x, y), \quad v = g(y), \quad y \in X\},$$

$$\mathcal{H} := \{(u, v) \in R^{p+m} : \ u \in C \setminus \{0\}, \quad v \in D\}.$$

The impossibility of $S(x)$ can be formulated in terms of the sets $\mathcal{K}(x)$ and $\mathcal{H}$.

Proposition 2.2 *$S(x)$ is impossible iff*

$$\mathcal{K}(x) \cap \mathcal{H} = \emptyset. \tag{1}$$

Let $\mathcal{E}(x) := \mathcal{K}(x) - cl\mathcal{H}$;

Proposition 2.3 [4] *If the cone $\mathcal{H}$ fulfils the condition $\mathcal{H} = \mathcal{H} + cl\mathcal{H}$, then (1) is equivalent to the condition*

$$\mathcal{E}(x) \cap \mathcal{H} = \emptyset. \tag{2}$$

Remark 2.1 In [3] it has been proved that if C is an open or closed convex cone, then $\mathcal{H} = \mathcal{H} + cl\mathcal{H}$, provided that D is a closed convex cone.
Moreover, it is known ([9], Lemma 3.1) that $\mathcal{E}(x)$ is a convex set when g is a D–function and $f(x, y)$ is a (clC)–function with respect to y.

Condition (2) can be proved showing that $\mathcal{E}(x)$ and $\mathcal{H}$ lie in two disjoint level sets of a suitable functional; when the functional can be chosen to be linear we say that $\mathcal{E}(x)$ and $\mathcal{H}$ admit a linear separation.

Definition 2.1 *The sets $\mathcal{E}(x)$ and $\mathcal{H}$ admit a linear separation iff $\exists (\mu^*, \lambda^*) \in C^* \times D^*$, $(\mu^*, \lambda^*) \neq 0$, such that*

$$\langle \mu^*, f(x, y) \rangle + \langle \lambda^*, g(y) \rangle \leq 0, \quad \forall y \in X. \tag{3}$$

The existence of a separating hyperplane doesn't guarantee that $\mathcal{E}(x) \cap \mathcal{H} = \emptyset$. In order to ensure the disjunction of the two sets, some restrictions on the choice of the multipliers (μ^*, λ^*) must be imposed.

Proposition 2.4 *Let clC be a pointed cone and assume that the sets $\mathcal{E}(x)$ and $\mathcal{H}$ admit a linear separation.*
i) If $\mu^* \in intC^*$ then $\mathcal{E}(x) \cap \mathcal{H} = \emptyset$.
ii) Suppose that C is an open cone. If $\mu^* \neq 0$ then $\mathcal{E}(x) \cap \mathcal{H} = \emptyset$.

Proof. We recall (see e.g. [2]) that clC is pointed iff $\text{int}C^* \neq \emptyset$ and that

$$intC^* = \{x^* \in C^* : \langle x, x^* \rangle > 0, \quad \forall x \in clC, x \neq 0\}.$$

i) Ab absurdo, suppose that $\mathcal{E}(x) \cap \mathcal{H} \neq \emptyset$. This implies that $\mathcal{K}(x) \cap \mathcal{H} \neq \emptyset$ and, therefore, $\exists z \in K$ such that $f(x, z) \in C \setminus \{0\}$, then, taking into account that $\mu^* \in \text{int}C^*$, we have $0 < \langle \mu^*, f(x, z) \rangle \ \leq \langle \mu^*, f(x, z) \rangle + \langle \lambda^*, g(z) \rangle \leq 0$, and we achieve the absurdity.
ii) Ab absurdo, suppose that $\mathcal{E}(x) \cap \mathcal{H} \neq \emptyset$. Following the proof of part i) $\exists z \in K$ such that $f(x, z) \in C \ = intC$, then, taking into account that $\mu^* \neq 0$, we have $0 < \langle \mu^*, f(x, z) \rangle \ \leq \langle \mu^*, f(x, z) \rangle + \langle \lambda^*, g(z) \rangle \leq 0$, and we achieve the absurdity. □

Remark 2.2 In particular, if we define $f(x, y) := h(x) - h(y), C = R^p_+$ (resp. C open cone), and $\mathcal{E}(x)$ and $\mathcal{H}$ admit a linear separation with $\mu > 0$ (resp. $\mu \neq 0$), then x is a v.m.p. for (P).

The following Theorem gives sufficient conditions that guarantee that the hypotheses of the Proposition 2.4 are fulfilled.

Theorem 2.1 *Suppose that the sets $\mathcal{E}(x)$ and $\mathcal{H}$ admit a linear separation.*

1. *Let $C := R^p_+$. Assume that, for every $i = 1, \ldots, p$, the following system is possible:*
$$f_{i^-}(x, y) > 0, \quad g(y) \in intD, \quad y \in X. \qquad S_i(x)$$
then in (3) we can suppose that $\mu > 0$.

2. *If there exists $\bar{y} \in X$ such that $g(\bar{y}) \in intD$, then in (3) we can suppose that $\mu \neq 0$.*

Proof. *1.* Ab absurdo, suppose that, $\exists i \in \{1, \ldots, p\}$ such that $\mu^*_i = 0$; then $(\mu^*_{i^-}, \lambda^*) \neq 0$ and, since $S_i(x)$ is possible, $\exists \bar{y} \in X$ such that

$$0 < \langle \mu^*_{i^-}, f_{i^-}(x, \bar{y}) \rangle + \langle \lambda^*, g(\bar{y}) \rangle \leq -\langle \mu^*_i, f_i(x, \bar{y}) \rangle = 0,$$

which is absurd.
2. Ab absurdo, suppose that $\mu^* = 0$ in (3); then, $\lambda^* \neq 0$ and, since $g(\bar{y}) \in intD$, it is

$$0 < \langle \lambda^*, g(y) \rangle \leq 0,$$

which is absurd. □

Remark 2.3 The condition given in the statement *1*, which has been also considered in [12], in a slightly different form, is a generalization of the Slater condition for scalar optimization problems [13, 14] taken as the assumption of the statement *2*.

3 Linear separation and saddle point conditions

As observed in [3, 4], linear separation is closely related to the Lagrangian–type optimality conditions. Following the line considered in [3, 4] we will characterize the linear separation in terms of a saddle point condition of the Lagrangian function associated to the system $S(x^*)$, defined by $L : C^* \times D^* \times X \longrightarrow R$,

$$L(x^*; \mu, \lambda, y) := -[\langle \mu, f(x^*, y)\rangle + \langle \lambda, g(y)\rangle].$$

Proposition 3.1 *Suppose that* $f(x^*, x^*) = 0$ *and* $g(x^*) \in D$. *Then* $\mathcal{E}(x^*)$ *and* $\mathcal{H}$ *admit a linear separation iff* $\exists(\mu^*, \lambda^*) \in C^* \times D^*$, $(\mu^*, \lambda^*) \neq 0$, *such that the point* (μ^*, λ^*, x^*) *is a saddle point for* $L(x^*; \mu, \lambda, y)$ *on* $(C^* \times D^*) \times X$.

Proof. Suppose that $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. From (3) we obtain that $\langle \lambda^*, g(x^*)\rangle \leq 0$, which implies that $\langle \lambda^*, g(x^*)\rangle = 0$, since $g(x^*) \in D$ and $\lambda^* \in D^*$. Therefore

$$0 = L(x^*; \mu^*, \lambda^*, x^*) \ \leq L(x^*; \mu^*, \lambda^*, y), \quad \forall y \in X.$$

It remains to show that $L(x^*; \mu, \lambda, x^*) \leq 0, \quad \forall(\mu, \lambda) \in (C^* \times D^*)$. We have that $L(x^*; \mu, \lambda, x^*) = -\langle \lambda, g(x^*)\rangle$ which is negative, $\forall \lambda \in D^*$, and the necessity part of the statement is proved.

Sufficiency. Suppose that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$, that is

$-\langle \lambda, g(x^*)\rangle \leq \ -\langle \lambda^*, g(x^*)\rangle \leq -[\langle \mu^*, f(x^*, y)\rangle + \langle \lambda^*, g(y)\rangle], \quad \forall(\mu, \lambda, y) \in (C^* \times D^*) \times X.$

Computing the first inequality for $\lambda = 0$, we obtain $\langle \lambda^*, g(x^*)\rangle \leq 0$ and, therefore, $\langle \lambda^*, g(x^*)\rangle = 0$. The first inequality coincides with (3) and the proposition is proved. □

Remark 3.1 We observe that the saddle value, $L(x^*; \mu^*, \lambda^*, x^*)$, is equal to zero. This property will be useful in the analysis of the gap function associated to a vector variational inequality defined in Section 5.

Proposition 3.2 *Let* X *be an open convex set in* R^n. *Assume that*

1. $f(x^*, y)$ *is a differentiable* C*–function, with respect to* y*, such that* $f(x^*, x^*) = 0$;

2. g *is a differentiable* D*–function;*

Then (μ^*, λ^*, x^*) *is a saddle point for* $L(x^*; \mu, \lambda, y)$ *on* $(C^* \times D^*) \times X$ *iff it is a solution of the following system (S)*

$$\begin{cases} \nabla_y L(x^*; \mu, \lambda, y) = 0 \\ \langle \lambda, g(y) \rangle = 0 \\ g(y) \in D, \mu \in C^*, \ \lambda \in D^*, \ y \in X. \end{cases}$$

Proof. Suppose that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$, that is

$-\langle \lambda, g(x^*) \rangle \leq -\langle \lambda^*, g(x^*) \rangle \leq -[\langle \mu^*, f(x^*, y) \rangle + \langle \lambda^*, g(y) \rangle], \quad \forall (\mu, \lambda, y) \in (C^* \times D^*) \times X.$

First of all we prove that $g(x^*) \in D$. Ab absurdo suppose that $g(x^*) \notin D = (D^*)^*$; then $\exists \bar{\lambda} \in D^*$ such that $\langle \bar{\lambda}, g(x^*) \rangle < 0$. Since D^* is a cone, then $\alpha \bar{\lambda} \in D^*$, $\forall \alpha \geq 0$ and $-\alpha \langle \bar{\lambda}, g(x^*) \rangle \longrightarrow +\infty, \quad \alpha \longrightarrow +\infty$; this contradicts the first inequality in the saddle point condition.

Computing the first inequality for $\lambda = 0$, we obtain $\langle \lambda^*, g(x^*) \rangle \leq 0$ and, therefore,

$$\langle \lambda^*, g(x^*) \rangle = 0. \tag{4}$$

The second inequality implies that x^* is a global minimum point of $L(x^*; \mu^*, \lambda^*, y)$, since $f(x^*, x^*) = 0$. Then

$$\nabla_y L(x^*; \mu^*, \lambda^*, x^*) = 0. \tag{5}$$

(5), together with (4) and the relation $(\mu^*, \lambda^*) \in (C^* \times D^*)$, allows to achieve the necessity part of the statement.

Sufficiency. Suppose that (μ^*, λ^*, x^*) is a solution of (S). Since $L(x^*; \mu^*, \lambda^*, y)$ is a convex function in the variable y, then $\nabla_y L(x^*; \mu^*, \lambda^*, x^*) = 0$ implies that

$$L(x^*; \mu^*, \lambda^*, x^*) \leq L(x^*; \mu^*, \lambda^*, y), \quad \forall y \in X.$$

Taking into account the complementarity relation $\langle \lambda^*, g(x^*) \rangle = 0$ and the condition $\lambda \in D^*$, we obtain

$$-\langle \lambda, g(x^*) \rangle \leq \ -\langle \lambda^*, g(x^*) \rangle, \quad \forall (\mu, \lambda) \in (C^* \times D^*),$$

and the statement is proved. □

4 Separation methods for Vector Variational Inequalities

The results stated in the previous sections can be applied in the analysis of a VVI, allowing to obtain Kuhn–Tucker type optimality conditions. Consider the vector variational inequality:

$$\text{find} \ \ x \in K \quad s.t. \quad F(x)(y - x) \not\leq_{C \setminus \{0\}} 0, \quad \forall y \in K := \{y \in X : \ g(y) \in D\},$$

where $F : X \longrightarrow R^{p\times n}$, $g : X \longrightarrow R^m$, D is a closed convex cone in R^m, g is a D–function and cl$C$ is a convex pointed cone in R^p.

Following the scheme introduced in Section 2, we define the following subsets of the space R^{p+m}, that we will call the image space associated to VVI:

$$\mathcal{K}(x) := \{(u,v) \in R^{p+m} : \ u = F(x)(x-y), \quad v = g(y), \quad y \in X\},$$

$$\mathcal{H} := \{(u,v) \in R^{p+m} : \ u \in C \setminus \{0\}, \quad v \in D\}.$$

The next result is analogous to Propositions 2.2 and 2.3.

Proposition 4.1 *i) $x^* \in K$ is a solution of VVI iff*

$$\mathcal{K}(x^*) \cap \mathcal{H} = \emptyset. \tag{6}$$

ii) If $\mathcal{H}$ is a convex cone that fulfils the condition $\mathcal{H} := \mathcal{H} + cl\mathcal{H}$, then (6) is equivalent to the condition

$$\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset, \tag{7}$$

where $\mathcal{E}(x^*) := \mathcal{K}(x^*) - cl\mathcal{H}$.

As observed in the Remark 2.1, if g is a D–function, then $\mathcal{E}(x^*)$ is a convex set [9]. Therefore, using separation tecniques in the image space, it is possible to obtain necessary and (or) sufficient Lagrangian–type optimality conditions for VVI.
Let $f : X \times X \longrightarrow R^p, \quad f(x,y) = F(x)(x-y)$.

Theorem 4.1 *Assume that $C := R^p_+$, X is an open convex set in R^n, and*

1. *g is a differentiable D–function;*
2. *for every $i = 1,\dots,p$, the following system is possible*

$$f_{i^-}(x^*,y) > 0, \quad g(y) \in intD, \quad y \in X. \qquad S_i(x^*)$$

Then $x^* \in K$ is a solution of VVI iff $\exists(\mu,\lambda) \in (C^* \times D^*)$, $(\mu,\lambda) \neq 0$, such that (μ,λ,x^*) is a solution of the following system (S)

$$\begin{cases} \mu F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ g(x) \in D, \ \mu \in C^*, \ \lambda \in D^*, \ x \in X. \end{cases}$$

Proof. We observe that $C^* = R^p_+$. Suppose that x^* is a solution of VVI. Then (7) holds. Since $f(x,y)$ is a linear function in the variable y and g is a D-function, the set $\mathcal{E}(x^*)$ is convex (see Remark 2.1). Therefore $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. By Proposition 3.1, we have that $\exists(\mu^*,\lambda^*) \in C^* \times D^*$ such that the point (μ^*,λ^*,x^*) is a saddle point for the Lagrangian function $L(x^*;\mu,\lambda,y) = -[\langle \mu, f(x^*,y)\rangle + \langle \lambda, g(y)\rangle]$.

By Proposition 3.2, we obtain that (μ^*, λ^*, x^*) is a solution of the system (S).
Sufficiency. Let (μ^*, λ^*, x^*) be a solution of the system (S); by Proposition 3.2 we get that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$, and, therefore, $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. Taking into account Theorem 2.1, condition *2* implies that $\mu^* > 0$. Proposition 2.4 ensures that $\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset$ and, therefore, x^* is a solution of VVI. □

In the last part of the section we will consider the, so called, weak case, in which C is an open convex cone. We will see that much less restrictive conditions are required in order to obtain an analogous result to Theorem 4.1 for the weak case. In particular, we will show that the classical Slater condition will be a sufficient regularity assumption on the constraint function g.

Theorem 4.2 *Assume that C is an open convex cone, X is an open convex set in R^n and that*
(a) g is a differentiable D–function;
(b) there exists $y \in X$ such that $g(y) \in intD$.
Then x^* is a solution of VVI iff $\exists(\mu, \lambda) \in C^* \times D^*, (\mu, \lambda) \neq 0$, such that (μ, λ, x^*) is a solution of the system

$$\begin{cases} \mu F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x) \rangle = 0 \\ g(x) \in D, \ \mu \in C^*, \ \lambda \in D^*, x \in X. \end{cases}$$

Proof. Necessity. The proof is analogous to the one of Theorem 4.1.
Sufficiency. The proof is analogous to the one of Theorem 4.1, replacing the hypothesis *2* with the hypothesis *(b)*, and $\mu^* > 0$ with $\mu^* \neq 0$. □

5 Gap functions for Vector Variational Inequalities

Given the variational inequality:

$$\text{find } x^* \in K \ \text{ s.t. } \langle F(x^*), y - x^* \rangle \geq 0, \quad \forall x \in K, \tag{VI}$$

where $F : K \longrightarrow R^n$, $K \subseteq R^n$, a gap function $p : K \longrightarrow R$ is a non-negative function that fulfils the condition $p(x) = 0$ if and only if x is a solution of VI. Therefore, solving a VI is equivalent to the minimization of the gap function on the feasible set K. A first example of gap function was given by Auslender [1] who considered the function $p(x) := \sup_{y \in K} \langle F(x), x - y \rangle$. Similarly to the scalar case, a gap function can be defined for a vector variational inequality.

Definition 5.1 *A function* $p : K \longrightarrow R$ *is a gap function for* VVI *iff*
i) $p(x) \geq 0, \quad \forall x \in K$;
ii) $p(x) = 0$ *if and only if* x *is a solution of* VVI.

Consider the following function $\psi : K \longrightarrow R$:

$$\psi(x) := \min_{(\mu,\lambda)\in S} \sup_{y\in X} [\langle \mu, f(x,y)\rangle \ + \ \langle \lambda, g(y)\rangle],$$

where $S := \{(\mu, \lambda) \in (C^* \times D^*) : \|(\mu, \lambda)\| = 1\}$.
Let $F : X \longrightarrow R^{p\times n}$, $f : X \times X \longrightarrow R^p, f(x, y) = F(x)(x - y)$, and $\Omega := \{x \in K : \psi(x) = 0\}$. The saddle point condition, that characterizes the separation in the image space (see the Proposition 3.1), allows to prove that $\psi(x)$ is a gap function for VVI.

Theorem 5.1 *Let* g *be a D–function on the convex set* $X \subseteq R^n$.

1. *Assume that* $C := R^p_+$ *and that, for every* $i = 1, \ldots, p$ *and* $\forall x^* \in \Omega$, *the following system is possible*

$$f_{i^-}(x^*, y) > 0, \quad g(y) \in intD, \quad y \in X; \qquad S_i(x^*)$$

   then $\psi(x)$ *is a gap function for* VVI.

2. *Assume that* C *is an open convex cone and that*

$$\exists \bar{y} \in X \ \text{such that} \ g(\bar{y}) \in intD; \tag{8}$$

   then $\psi(x)$ *is a gap function for* VVI.

Proof. *1.* It is easy to prove that $\psi(x) \geq 0, \ \forall x \in K$; in fact, if $(\mu, \lambda) \in (C^* \times D^*)$, then

$$\langle \mu, f(x, x)\rangle + \langle \lambda, g(x)\rangle = \langle \lambda, g(x)\rangle \geq 0.$$

Suppose that x^* is a solution of VVI. Since $f(x, y)$ is a linear function in the variable y and g is a D–function, the set $\mathcal{E}(x^*)$ is convex (see Remark 2.1). Therefore $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. Without loss of generality we can suppose that the coefficients of the separating hyperplane $(\mu^*, \lambda^*) \in S$. From Proposition 3.1, we have that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y) := -[\langle \mu, f(x^*, y)\rangle \ + \ \langle \lambda, g(y)\rangle]$ on $(C^* \times D^*) \times X$ and the saddle value $L(x^*; \mu^*, \lambda^*, x^*) = 0$ (see Remark 3.1). Recalling that the saddle point condition can be characterized by suitable minimax problems [15], we have

$$\min_{(\mu,\lambda)\in C^*\times D^*} \sup_{y\in X} [\langle \mu, f(x, y)\rangle \ + \ \langle \lambda, g(y)\rangle] = L(x^*; \mu^*, \lambda^*, x^*) = 0. \tag{9}$$

Since $(\mu^*, \lambda^*) \in S$, taking into account (9), we obtain that $\psi(x^*) = 0$.

Vice–versa, suppose that $\psi(x^*) = 0$. Then $\exists(\mu^*, \lambda^*) \in S$, such that

$$\langle \mu^*, f(x^*, y)\rangle + \langle \lambda^*, g(y)\rangle \le 0, \quad \forall y \in X.$$

For Theorem 2.1, the possibility of the system $S_i(x^*)$ for $i = 1, \ldots, p$, implies that $\mu^* > 0$. Applying Proposition 2.4 , we obtain that x^* is a solution of VVI.
2. The proof is analogous to the one of *1* using (8) instead of the fact that $S_i(x^*)$ is possible for $i = 1, \ldots, p$ and replacing the condition $\mu^* > 0$ with $\mu^* \neq 0$. □

Remark 5.1 We observe that $h_x(\mu, \lambda) := \sup_{y\in X}[\langle \mu, f(x, y)\rangle + \langle \lambda, g(y)\rangle]$, being the supremum of a collection of linear functions, is a convex function, so that $\psi(x) = \min_{(\mu,\lambda)\in S} h_x(\mu, \lambda)$ is the optimal value of a parametric problem on a compact set, with a convex objective function.

The gap function ψ that we have analysed in this section, in general, is not differentiable. Following the line adopted in [5, 16], adding a suitable regularizing term $H(x, y) : X \times X \longrightarrow R$ to the function $\langle \mu, f(x, y)\rangle + \langle \lambda, g(y)\rangle$, it is possible to obtain a differentiable gap function for VVI. To this aim, scalarization methods (see e.g. [11]) for VVI can be a further useful tool to carry out the analysis.

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