Operator-valued Kernels and Control of Infinite dimensional Dynamic Systems

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Motivation: Green kernels to solve PDEs

To solve the heat equation $\partial_s u(s,y) = \Delta u(s,y)$ on $\mathbb{R}^d$ with $u(t,x\cdot) = f_t(\cdot)$ for a given t, one just has to find the Green kernel $k(s,t,y,x)$ s.t.

$$\partial_s k(s,t,y,x) = \Delta_y k(s,t,y,x),\ \forall s,y \text{ and } k(t,t,y,x) = \delta_y(x),\ \forall y$$

then the solution is obtained through a kernel integral operator $u = Kf$, i.e.

$$u(s,y) = \int_x k(s,t,y,x) f_t(x) dx,$$

and we know that actually this is the heat kernel

$$k(s-t,x,y) = \frac{1}{(4\pi(s-t))^{\frac{d}{2}}} e^{-\frac{\|x-y\|_d^2}{4(s-t)}} \quad \text{for } s \geq t.$$

What about $\partial_s u = \Delta u + v$ where $v(s,y)$ is a control? Is it possible to find a notion of Green kernel for Linear-Quadratic optimal control problems?

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Yes! This is what we are going to see in this talk by focusing on the Hilbert space of controllable trajectories.

Time-varying infinite-dimensional LQ optimal control

Let $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|_H)$ be two separable Hilbert spaces, and U a Hilbert space. We assume that $V \subset H$, with continuous injection. Identifying H to its dual, we have also the inclusion $H \subset V'$ with continuous injection, where V' is the dual of V.

$$\min_{y(\cdot),u(\cdot)} \quad \chi_{y_0}(y(t_0)) + g(y(T))$$

$$+ (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt$$

$$\text{s.t.} \qquad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ a.e. in } [t_0, T]$$

- state $y(t) \in V$, control $u(t) \in U$, $\exists \alpha > 0, \beta \in \mathbb{R}, \forall z \in V, \langle A(t)z, z\rangle_{V' \times V} + \beta \|z\|_H^2 \geq \alpha \|z\|_V^2$
- $A(t) \in \mathcal{L}(V, V')$, $B(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, H))$, $M(\cdot) \in L^\infty(t_0, T; \mathcal{L}(H, H))$, $N(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, U))$, $M(t) \geq 0$ and $N(t) \geq \nu \, \mathrm{Id}_U$ $(\nu > 0)$, $J_0 \succ 0$,
- differentiable terminal cost $g : V \to \mathbb{R}$, indicator function χ_{y_0},

- $y(\cdot) : [t_0, T] \to V$ absolutely continuous, $N(\cdot)^{1/2} u(\cdot) \in L^2([t_0, T])$

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$$\min_{y(\cdot),u(\cdot)} \quad \chi_{y_0}(y(t_0)) + g(y(T)) \qquad \rightarrow L(y(t_j)_{j\in[J]})$$

$$+ (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt \qquad \rightarrow \|y(\cdot)\|^2_{\mathcal{H}_K}$$

$$\text{s.t.} \quad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ a.e. in } [t_0, T] \qquad \rightarrow y(\cdot) \in \mathcal{H}_K$$

- state $y(t) \in V$, control $u(t) \in U$, $\exists \alpha > 0, \beta \in \mathbb{R}, \forall z \in V, \langle A(t)z, z\rangle_{V'\times V} + \beta\|z\|_H^2 \geq \alpha\|z\|_V^2$
- $A(t) \in \mathcal{L}(V, V')$, $B(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, H))$, $M(\cdot) \in L^\infty(t_0, T; \mathcal{L}(H, H))$, $N(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, U))$, $M(t) \geq 0$ and $N(t) \geq \nu \, \mathrm{Id}_U$ $(\nu > 0)$, $J_0 \succ 0$,
- differentiable terminal cost $g : V \to \mathbb{R}$, indicator function χ_{y_0}, “loss function“ $L : (\mathbb{R}^Q)^J \to \mathbb{R} \cup \{\infty\}$,
- $y(\cdot) : [t_0, T] \to V$ absolutely continuous, $N(\cdot)^{1/2}u(\cdot) \in L^2([t_0, T])$

LQ optimal control is a kernel regression!

By rewriting the LQ problem, we can turn it into a loss+regularizer problem in a "machine learning" (regression) fashion.

$$\min_{y(\cdot),u(\cdot)} \quad \chi_{y_0}(y(t_0)) + g(y(T)) + (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U]dt$$

$$\text{s.t.} \quad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ a.e. in } [t_0, T]$$

$$\min_{y(\cdot),u(\cdot)} \quad L(y(t_j)_{j\in[J]}) + \|y(\cdot)\|^2_{\mathcal{H}_K}$$

$$\text{s.t.} \quad y(\cdot) \in \mathcal{H}_K$$

We will see that the regression is over a reproducing kernel Hilbert space (RKHS) $\mathcal{H}_K$ with a kernel K depending on $[t_0, T], A, B, M, N$. The space $\mathcal{H}_K$ plays the role of a Sobolev space for LQ optimal control (similarly to Poisson's equation).

The classical way of solving LQ optimal control: the Riccati equation

The functional $u(\cdot) \mapsto J(u(\cdot)) = \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt$ is quadratic and strictly convex. It has a unique minimum $u(\cdot)$, which is computed as follows: the forward-backward system of equations

$$\begin{aligned} \frac{dy}{dt} + A(t)y(t) + B(t)N^{-1}(t)B^*(t)p(t) &= 0, & y(t_0) &= y_0 \\ -\frac{dp}{dt} + A^*(t)p(t) - M(t)y(t) &= 0, & p(T) &= 0, \end{aligned} \tag{1}$$

has a unique solution. Moreover, we have the decoupling property

$$p(t) = P(t)y(t) \tag{2}$$

in which $P(t) \in \mathcal{L}(H; H)$ is symmetric and positive semidefinite. The operator $P(t)$ is defined by solving a system similar to (1) for each $t \in [t_0, T]$ and $h \in H$

$$\begin{aligned} \frac{d\xi}{ds} + A(s)\xi(s) + B(s)N^{-1}(s)B^*(s)\eta(s) &= 0, & \xi(t) &= h, \\ -\frac{d\eta}{ds} + A^*(s)\eta(s) - M(s)\xi(s) &= 0, & \eta(T) &= 0 \ \forall s \in (t, T), \end{aligned} \tag{3}$$

and then setting $\eta(t) = P(t)h$.

The classical way of solving LQ optimal control: the Riccati equation (cont.)

If $\varphi(\cdot) \in L^2(t_0, T; H)$ satisfies $\frac{d\varphi}{dt} + A(t)\varphi(t) \in L^2(t_0, T; H)$, then $\Psi(t) = P(t)\varphi(t)$ satisfies $-\frac{d\Psi}{dt} + A^*(t)\Psi(t) \in L^2(t_0, T; H)$, and

$$-\frac{d\Psi}{dt} + A^*(t)\Psi(t) + P(t)\Big[\frac{d\varphi}{dt} + A(t)\varphi(t) + B(t)N^{-1}(t)B^*(t)\Psi(t)\Big] = M(t)\varphi(t).$$

This formally can be written as

$$-\frac{dP}{dt} + P(t)A(t) + A^*(t)P(t) + P(t)B(t)N^{-1}(t)B^*(t)P(t) = M(t),\ P(T) = 0. \tag{4}$$

The optimal state $y(\cdot)$ for the LQR control problem is solution of the equation

$$\frac{dy}{dt} + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))y(t) = 0,\ y(t_0) = y_0. \tag{5}$$

and the optimal control $u(\cdot)$ is given by $u(t) = -N^{-1}(t)B^*(t)P(t)y(t)$.
We will use in the sequel the semi-group (a.k.a. evolution family)

$$\partial_t \Phi_{A,P}(t,s) + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))\Phi_{A,P}(t,s) = 0, \quad \Phi_{A,P}(s,s) = \mathrm{Id}_H\,. \tag{6}$$

Reproducing kernel Hilbert spaces (RKHS)

A RKHS $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$ is a Hilbert space of real-valued functions over a set $\mathcal{T}$ if one of the following equivalent conditions is satisfied

$\exists\, k : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ s.t. $k_t(\cdot) = k(\cdot, t) \in \mathcal{H}_k$ and $f(t) = \langle f(\cdot), k_t(\cdot)\rangle_{\mathcal{H}_k}$ for all $t \in \mathcal{T}$ and $f \in \mathcal{H}_k$ (reproducing property)

the topology of $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$ is stronger than pointwise convergence
i.e. $\delta_t : f \in \mathcal{H}_k \mapsto f(t)$ is continuous for all $t \in \mathcal{T}$.

$$|f(t) - f_n(t)| = |\langle f - f_n, k_t\rangle_{\mathcal{H}_k}| \leq \|f - f_n\|_{\mathcal{H}_k} \|k_t\|_{\mathcal{H}_k} = \|f - f_n\|_{\mathcal{H}_k} \sqrt{k(t,t)}$$

For $\mathcal{T} \subset \mathbb{R}^d$, Sobolev spaces $\mathcal{H}^s(\mathcal{T}, \mathbb{R})$ satisfying $s > d/2$ are RKHSs.

$$\begin{cases} H_0^1 = \{f \mid f(0) = 0,\ \exists f' \in L^2(0,\infty)\} \\ \langle f, g\rangle_{H_0^1} = \int_0^\infty f' g' \mathrm{d}t \end{cases} \longleftrightarrow k(t,s) = \min(t,s).$$

Other classical kernels

$$k_{\text{Gauss}}(t,s) = \exp\left(-\|t-s\|_{\mathbb{R}^d}^2 / (2\sigma^2)\right) \quad k_{\text{poly}}(t,s) = (1 + \langle t, s\rangle_{\mathbb{R}^d})^2.$$

Two essential tools for computations

Representer Theorem (e.g. [Schölkopf et al., 2001])

Let $L : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$, strictly increasing $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and

$$\bar{f} \in \underset{f \in \mathcal{H}_k}{\operatorname{argmin}}\, L\left((f(t_n))_{n\in[N]}\right) + \Omega\left(\|f\|_k\right)$$

Then $\exists\, (a_n)_{n\in[N]} \in \mathbb{R}^N$ s.t. $\bar{f}(\cdot) = \sum_{n\in[N]} a_n k(\cdot, t_n)$

$\hookrightarrow$ Optimal solutions lie in a finite dimensional subspace of $\mathcal{H}_k$.

Finite number of evaluations $\implies$ finite number of coefficients

Kernel trick

$$\langle \sum_{n\in[N]} a_n k(\cdot, t_n), \sum_{m\in[M]} b_m k(\cdot, s_m) \rangle_{\mathcal{H}_k} = \sum_{n\in[N]} \sum_{m\in[M]} a_n b_m k(t_n, s_m)$$

$\hookrightarrow$ On this finite dimensional subspace, no need to know $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$.

Vector-valued reproducing kernel Hilbert space (vRKHS)

Let $\mathcal{T}$ be a non-empty set. A Hilbert space $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$ of V-vector-valued functions defined on $\mathcal{T}$ is a vRKHS if there exists a matrix-valued kernel $K : \mathcal{T} \times \mathcal{T} \to \mathcal{L}(V', V)$ such that the reproducing property holds:

$$K(\cdot, t)p \in \mathcal{H}_K, \quad p^\top f(t) = \langle f, K(\cdot, t)p \rangle_K, \qquad \text{for } t \in \mathcal{T},\ p \in V', f \in \mathcal{H}_K$$

There is a one-to-one correspondence between K and $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$, so changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_K$ changes K. We also have a representer theorem for

$$\mathcal{J}(y(\cdot)) = L((y(t_n))_{n=1}^N, \|y(\cdot)\|^2_{\mathcal{H}_K}) \tag{7}$$

for a given extended-valued function $L : H^N \times [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$.

[Micchelli and Pontil, 2005, Theorem 4.2]
If for every $z \in H^N$ the function $h : \xi \in \mathbb{R}_+ \mapsto L(z, \xi) \in \mathbb{R}_+ \cup \{+\infty\}$ is strictly increasing and $\hat{y}(\cdot) \in \mathcal{H}_K$ minimizes the functional (20), then $\hat{y}(\cdot) = \sum_{n=1}^N K(\cdot, t_n) z_n$ for some $\{z_n\}_{n=1}^N \subseteq H$. In addition, if L is strictly convex, the minimizer is unique.

Hilbert space of trajectories

We consider the subset $\mathcal{H}$ of $L^2(t_0, T; H)$ defined as follows

$$\mathcal{H} = \{y(\cdot) \in L^2(t_0, T; H) \mid \tfrac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ with } u(\cdot) \in L^2(t_0, T; U)\}.$$

There is not necessarily a unique choice of $u(\cdot)$ for a given $y(\cdot) \in \mathcal{H}$ (for instance if $B(t)$ is not injective for some t). Therefore, with each $y(\cdot) \in \mathcal{H}$, we associate the control $u(\cdot)$ having minimal norm based on the pseudoinverse of $B(t)^{\ominus}$ of $B(t)$ for the U-norm $\|\cdot\|_{N(t)} := \|N(t)^{1/2}\cdot\|_U$:

$$u(t) = B(t)^{\ominus}[\tfrac{dy}{dt} + A(t)y(t)] \text{ a.e. in } [t_0, T], \rightarrow \text{ we get rid of the control!} \tag{8}$$

whence $u(\cdot)$ minimizes $\int_{t_0}^{T} (N(t)u(t), u(t))_U \, dt$ among the controls admissible for $y(\cdot) \in \mathcal{H}$. We consequently equip $\mathcal{H}$ with the norm

$$\|y(\cdot)\|_{\mathcal{H}}^2 = (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U]dt,$$

with J_0 s.t. $(J_0 + P(t_0))$ invertible. Then $\mathcal{H}$ has the structure of a Hilbert space.

Hilbert space of trajectories is a RKHS with explicit kernel!

$$\mathcal{H} = \{y(\cdot) \in L^2(t_0, T; H) \mid \tfrac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ with } u(\cdot) \in L^2(t_0, T; U)\}. \quad (9)$$

$$\|y(\cdot)\|_{\mathcal{H}}^2 = (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U]dt, \quad (10)$$

Theorem (Main result)

We assume the coercivity of the drift, the strong convexity of the objective, and the invertibility of $(J_0 + P(t_0))$ conditions. Set $K(s,t) \in \mathcal{L}(H,H)$ as

$$K(s,t) = \Phi_{A,P}(s,0)(J_0 + P(t_0))^{-1}\Phi_{A,P}^*(t,0) + \int_{t_0}^{\min(s,t)} \Phi_{A,P}(s,\tau)B(\tau)N^{-1}(\tau)B^*(\tau)\Phi_{A,P}^*(t,\tau)d\tau.$$

Then the space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ defined by (9),(10) is a RKHS associated with the kernel K.

where $\partial_t \Phi_{A,P}(t,s) + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))\Phi_{A,P}(t,s) = 0, \quad \Phi_{A,P}(s,s) = \mathrm{Id}_H$.

Proof is mostly integration by parts (if we guess the form of the kernel).

Decomposition of the kernel into null-control and null-initial condition

From now on, we denote $\mathcal{H}$ by $\mathcal{H}_K$. We split the kernel K into

$$K(s,t) = K^0(s,t) + K^1(s,t) \tag{11}$$

$$K^0(s,t) := \Phi_{A,P}(s,0)(J_0 + P(t_0))^{-1}\Phi^*_{A,P}(t,0), \tag{12}$$

$$K^1(s,t) := \int_{t_0}^{\min(s,t)} \Phi_{A,P}(s,\tau)B(\tau)N^{-1}(\tau)B^*(\tau)\Phi^*_{A,P}(t,\tau)d\tau.$$

The kernel K^1 is instrumental for the LQR. Consider the Hilbert subspace of $\mathcal{H}^1_K$ of functions with initial value equal to 0, equipped with $\|\cdot\|_{\mathcal{H}_K}$,

$$\mathcal{H}^1_K = \{y(\cdot) \mid \tfrac{dy}{dt} + A(t)y(t) = B(t)u(t), y(t_0) = 0, \text{ with } u(\cdot) \in L^2(t_0,T;U)\}. \tag{13}$$

Proposition

The Hilbert space $\mathcal{H}^1_K$ is a RKHS associated with the operator-valued kernel $K^1(s,t)$.

Example of heat equation with distributed control

We here focus on bounded $B(\cdot) \in L^\infty$ and parabolic equations (unbounded/hyperbolic would require a few changes). Take $V = H^1(\mathbb{R}^d, \mathbb{R})$, $H = L^2(\mathbb{R}^d, \mathbb{R})$, $A(\cdot) \equiv -\Delta$ and $B(\cdot) \equiv \mathrm{Id}_H$, then the heat equation with distributed control writes as

$$\frac{dy}{dt} = \Delta y(t) + u(t),\ y(t_0) = y_0 \in H. \tag{14}$$

As objective, take $J_0 = \lambda \, \mathrm{Id}_H$ with $\lambda > 0$, $M(\cdot) \equiv 0$ and $N(\cdot) \equiv \mathrm{Id}_H$, thus $P(\cdot) \equiv 0$, and $\Phi_{A,P}(t,s) = \Phi_A(t,s)$. In this well-known context, the (integral) operator $\Phi_A(t,s) = e^{-A(t-s)}$ is merely the heat semi-group associated to the heat kernel, for $t > s$,

$$k(t-s,x,y) = \frac{1}{(4\pi(t-s))^{d/2}} e^{-\|x-y\|_d^2/4(t-s)}.$$

Using that A is self-adjoint and the known expression of the Fourier transform of a normalized Gaussian, one can show that $\int_0^{2s} k(\tau,x,y)d\tau = k(s^2,x,y)$ and consequently that, for $t > s$, $K^1(s,t) = \frac{1}{2}[\int_0^{2s} e^{-A\tau} d\tau] \circ e^{-A(t-s)}$ is a kernel integral operator with kernel $k_1 = k(t-s+s^2,x,y)/2$. On the other hand $K^0(s,t) = e^{-A(t+s)}/\lambda$ has for kernel $k_0 = k(t+s,x,y)/\lambda$. This allows for explicit handling of the kernel K in applied cases with various objective functions.

Solving control problems: Final nonlinear term - Mayer problem

We consider the dynamic system

$$\frac{dy}{dt} + A(t)y(t) = B(t)u(t), \quad y(t_0) = y_0. \tag{15}$$

We want to find the pair $y_0, u(\cdot)$ in order to minimize

$$J(u(\cdot), y_0) := g(y(T)) + \frac{1}{2}\,(y(t_0), J_0 y(t_0))_H + \frac{1}{2}\int_{t_0}^{T} \left[\,(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U\,\right] dt,$$

where $h \mapsto g(h)$ is a Gâteaux differentiable function on H. Using the norm $\|\cdot\|_{\mathcal{H}_K}$ defined in (10), this problem can be formulated as minimizing a functional on $\mathcal{H}_K$, namely

$$\mathcal{J}(y(\cdot)) := g(y(T)) + \frac{1}{2}\|y(\cdot)\|^2_{\mathcal{H}_K}. \tag{16}$$

If $\hat{y}(\cdot)$ is a minimizer, it satisfies the Euler equation

$$(Dg(\hat{y}(T)), \zeta(T))_H + (\hat{y}(\cdot), \zeta(\cdot))_{\mathcal{H}_K} = 0,\ \forall \zeta(\cdot) \in \mathcal{H}_K. \tag{17}$$

By the reproducing property $(Dg(\hat{y}(T)), \zeta(T))_H = (K(\cdot, T)Dg(\hat{y}(T), \zeta(\cdot))_{\mathcal{H}_K}$ and (17) yields immediately the equation for $\hat{y}(\cdot)$

$$K(\cdot, T)Dg(\hat{y}(T)) + \hat{y}(\cdot) = 0. \tag{18}$$

Solving control problems: recovering the standard solution of the LQR

We can now go back to the standard LQR problem, where the initial state y_0 is known. The state $y(\cdot)$ can be written as follows $y(s) = \Phi_A(s,0)y_0 + \zeta(s)$ where $\zeta(\cdot)$ satisfies

$$\frac{d\zeta}{ds} + A(s)\zeta(s) = B(s)u(s), \quad \zeta(t_0) = 0.$$

Therefore $\zeta(\cdot) \in \mathcal{H}_{K^1}$. We write $y_0(s) = \Phi_A(s,0)y_0$ and

$$J(u(\cdot)) = \int_{t_0}^{T} (M(t)y_0(t), y_0(t))_H \, dt + \int_{t_0}^{T} (M(t)\zeta(t), \zeta(t))_H \, dt + 2\int_{t_0}^{T} (M(t)y_0(t), \zeta(t))_H \, dt$$
$$+ \int_{t_0}^{T} (N(t)u(t), u(t))_U \, dt.$$

The problem amounts to minimizing $\mathcal{J}(\zeta(\cdot)) = \|\zeta(\cdot)\|^2_{\mathcal{H}_K} + 2\int_{t_0}^{T} (M(t)y_0(t), \zeta(t))_H \, dt$ on the Hilbert space $\mathcal{H}_{K^1}$. Since

$$\mathcal{J}(\zeta(\cdot)) = \|\zeta(\cdot)\|^2_{\mathcal{H}_K} + 2\left(\zeta(\cdot), \int_{t_0}^{T} K^1(\cdot,t)M(t)y_0(t)dt\right)_H, \tag{19}$$

the minimizer is obtained immediately by the formula $\hat{\zeta}(s) = -\int_{t_0}^{T} K^1(s,t)M(t)y_0(t)dt.$

More general objectives: state constraints and intermediary points

More generally one may consider several constrained time points:

$$\mathcal{J}(y(\cdot)) = L((y(t_n))_{n=1}^N, \|y(\cdot)\|_{\mathcal{H}_K}^2) \tag{20}$$

for a given extended-valued function $L : H^N \times [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$.

[Micchelli and Pontil, 2005, Theorem 4.2]
If for every $z \in H^N$ the function $h : \xi \in \mathbb{R}_+ \mapsto L(z, \xi) \in \mathbb{R}_+ \cup \{+\infty\}$ is strictly increasing and $\hat{y}(\cdot) \in \mathcal{H}_K$ minimizes the functional (20), then $\hat{y}(\cdot) = \sum_{n=1}^N K(\cdot, t_n) z_n$ for some $\{z_n\}_{n=1}^N \subseteq H$.
In addition, if L is strictly convex, the minimizer is unique.

Conclusion

In a nutshell

- finding an RKHS somewhere allows for simpler computations
- in LQ optimal control, RKHSs come from vector spaces of trajectories
- in linear estimation, kernels come from covariances of optimal errors (explains the duality between estimation & control), *The RKHSs underlying linear SDE Estimation, Kalman filtering and their relation to optimal control*, Aubin-Frankowski & Bensoussan, 2022, Pure and Applied Functional analysis (to appear, available on arXiv)

Objective:

- re-read known optimal control/estimation problems through kernel lens
- use nonlinear embeddings on the state, apply it to stochastic optimal control, and optimization over measures
- Koopman operator and Model Predictive Control as possible applications

Thank you for your attention!

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