Inferring Strange Behavior from Connectivity Pattern in Social Networks

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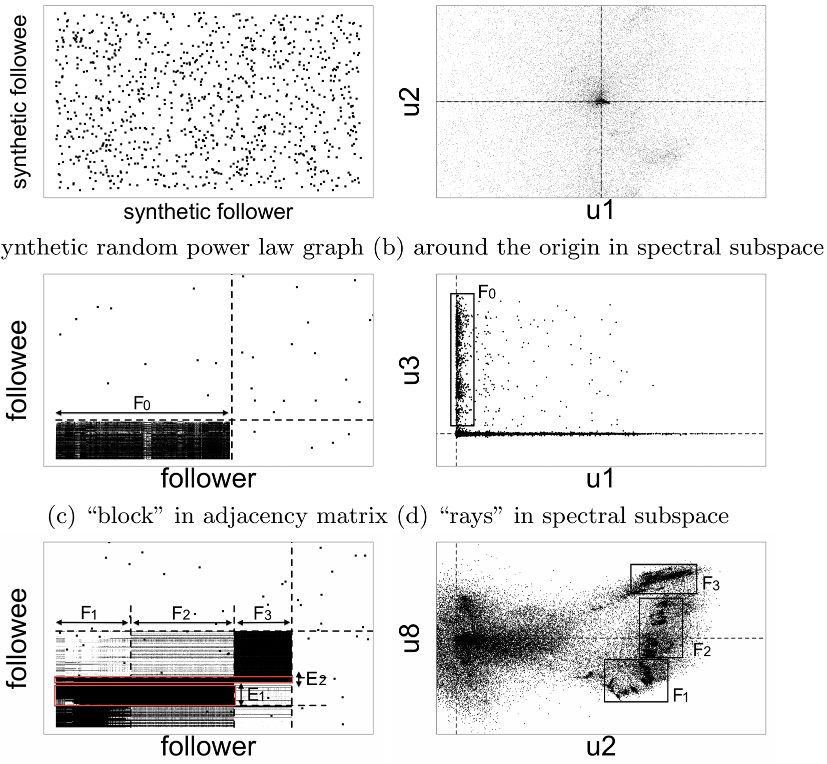
Abstract. Given a multimillion-node social network, how can we summarize connectivity pattern from the data, and how can we find unexpected user behavior? In this paper we study a complete graph from a large who-follows-whom network and spot lockstep behavior that large groups of followers connect to the same groups of followees. Our first contribution is that we study strange patterns on the adjacency matrix and in the spectral subspaces with respect to several flavors of lockstep. We discover that (a) the lockstep behavior on the graph shapes dense "block" in its adjacency matrix and creates "ray" in spectral subspaces, and (b) partially overlapping of the behavior shapes "staircase" in the matrix and creates "pearl" in the subspaces. The second contribution is that we provide a fast algorithm, using the discovery as a guide for practitioners, to detect users who offer the lockstep behavior. We demonstrate that our approach is effective on both synthetic and real data.

1 Introduction

Given a large social network, how can we catch strange user behaviors, and how can we find intriguing and unexpected connectivity patterns? While the strange behaviors have been documented across services ranging from telecommunication fraud [1] to deceptive Ebay's reviews [2] to ill-gotten Facebook's page-likes [3], we study here a complete graph of more than *117 million* users and *3.33 billion* edges in a popular microblogging service Tencent Weibo (Jan. 2011). Several recent studies have used social graph data to characterize connectivity patterns, with a focus on understanding the community structure [4–6] and the cluster property [7, 8]. However, no analysis was presented to demonstrate what strange connectivity pattern we can infer strange behavior from and how.

In this paper, we investigate *lockstep behavior* pattern on Weibo's "who-follows-whom" graph, that is, groups of followers acting together, consistently following the same group of followees, often with little other activity. Therefore, though the followees are not popular, they could have a large number of followers. We study different types of lockstep behavior, characterize connectivity patterns in the adjacency matrix of the graph, and examine the associated patterns in



(a) synthetic random power law graph (b) around the origin in spectral subspace

(c) "block" in adjacency matrix (d) "rays" in spectral subspace

(e) "staircase" in adjacency matrix (f) "pearls" in spectral subspace

Fig. 1. Lockstep behavior shows interesting connectivity patterns and spectral patterns: On synthetic graph, followers are around the origin in all spectral subspaces. On WEIBO, non-overlapping lockstep behaviors of followers in group F_0 shape a dense "block" in adjacency matrix and create "rays" in spectral subspace. Overlapping lockstep behaviors of followers in group F_1-F_3 create a "staircase" and "pearls".

spectral subspaces. Fig.1.(a,c,e) plot connections in the matrix, in which a black point shows the follower on the X-axis connecting to the followee on the Y-axis. Fig.1.(b,d,f) plot each follower node by its values in a pair of the left-singular vectors of the adjacency matrix. These figures visualize the spectral subspaces, and the dashed lines are X- and Y-axis. Specifically, we show that

- *No lockstep behavior:* According to the Chung-Lu model [9], we generate a random power law graph where no lockstep behavior exists. The adjacency matrix in Fig.1.(a) has no large, dense components. We study every 2-dimensional spectral subspace of this synthetic graph and observe that follower nodes are around the original point, as shown in Fig.1.(b).
- *Non-overlapping lockstep behavior:* On WEIBO, there is a group of followers in F_0 connecting to the same group of followees. Thus, the adjacency matrix shows a large, dense "block" (83,208 followers, 81.3% dense) in Fig.1.(c). Fig.1.(d) plots the spectral subspace formed by the 1^{st} and 3^{rd} left-singular vectors. The followers in group F_0 neatly align the Y-axis. We name this pattern "ray" according to the shape of the points.

- *Partially overlapping lockstep behavior:* A more surprising connectivity pattern we discover in the adjacency matrix is a "staircase" (10,052 followers, 43.1% dense), as shown in Fig.1.(e). Groups of followers in F_1-F_3 behave in lockstep, forming three more than 89% dense blocks. However, different from the non-overlapping case, F_1-F_2 have the same large group of followees E_1, and F_1-F_3 share a small group E_2. The overlapping lockstep behaviors of the followers create multiple micro-clusters of points that deviate from the origin and lines in the 2^{nd} and 8^{th} left-singular vector subspace. Fig.1.(f) shows the spherical micro-clusters, roughly on a circle, so called "pearls" pattern.

Motivated by this investigation, we further propose a novel approach, which include effective and efficient techniques that can learn the connectivity patterns and infer following behaviors in lockstep. The contributions are as follows:

- *Insights:* We offer new insights into the fingerprints on the singular vectors left by different types of synthetic lockstep behaviors. This gives us a set of rules that data scientists and practitioners can use to discover strange connectivity patterns and strange user behaviors.
- *Algorithm:* We propose an efficient algorithm that exploits the insights above, and automatically find the followers that behave in lockstep. We demonstrate the effectiveness on both synthetic data and a real social graph.

The rest of the paper is organized as follows: Section 2 discusses related work. Section 3 provides insights from strange connectivity patterns and Section 4 introduces our algorithm inferring lockstep behaviors. We give experimental results in Section 5 and conclude in Section 6.

2 Related work

A great deal of work has been devoted to mining connectivity patterns. For finding social communities, Leskovec *et al.* [4] capture the intuition of a cluster as set of users with better internal connectivity than external connectivity. Clauset *et al.* [10] and Wakita *et al.* [11] infer community structure from network topology by optimizing the modularity. It is desirable that user of a community have a dense internal links and small number of links connected to users of other communities. For graph clustering and partitioning, Ng *et al.* [12] present a spectral clustering algorithm using eigenvectors of matrices derived from the data. Huang *et al.* [13] devise a spectral bi-partitioning algorithm using the second eigenvector of the normalized Laplacian matrix.

The properties of spectral subspaces have recently received much attention. Prakash *et al.* [14] show that the singular vectors of mobile call graphs, when plotted against each other, have separate lines along specific axes, which is associated with the presence of tightly-knit communities. The authors propose SpokEn to chip the communities embeded in the graphs. Ying *et al.* [15] suggest that the lines formed by nodes in well-structured communities are not necessarily axes aligned. Wu *et al.* [16] give theoretical studies to explain the existence of orthogonal lines in the spectral subspaces.

However, none of the above approaches provided a guide for practitioners to understand real settings, namely, non-overlapping and partially overlapping lockstep behaviors, with an explanation for the strange spectral patterns we observe ("staircase" and "pearl"), and strange connectivity patterns.

3 Guide for lockstep behavior inference

In this section, we first introduce how to plot spectral subspaces. We then study different types of lockstep behavior, show the connectivity patterns. and give a list of rules on which type of behavior the spectral patterns represent.

3.1 Spectral-subspace plot

The concept of "spectral-subspace plot" is fundamental. The intuition behind it is that it is a visualization tool to help us see strange patterns. Let A be the $N \times N$ adjacency matrix of our social graph. Each user can be envisioned as an N-dimensional point; a spectral-subspace plot is a projection of those points in N dimensions, into a suitable 2-dimensional subspace. Specifically, the subspace is spanned by two singular vectors.

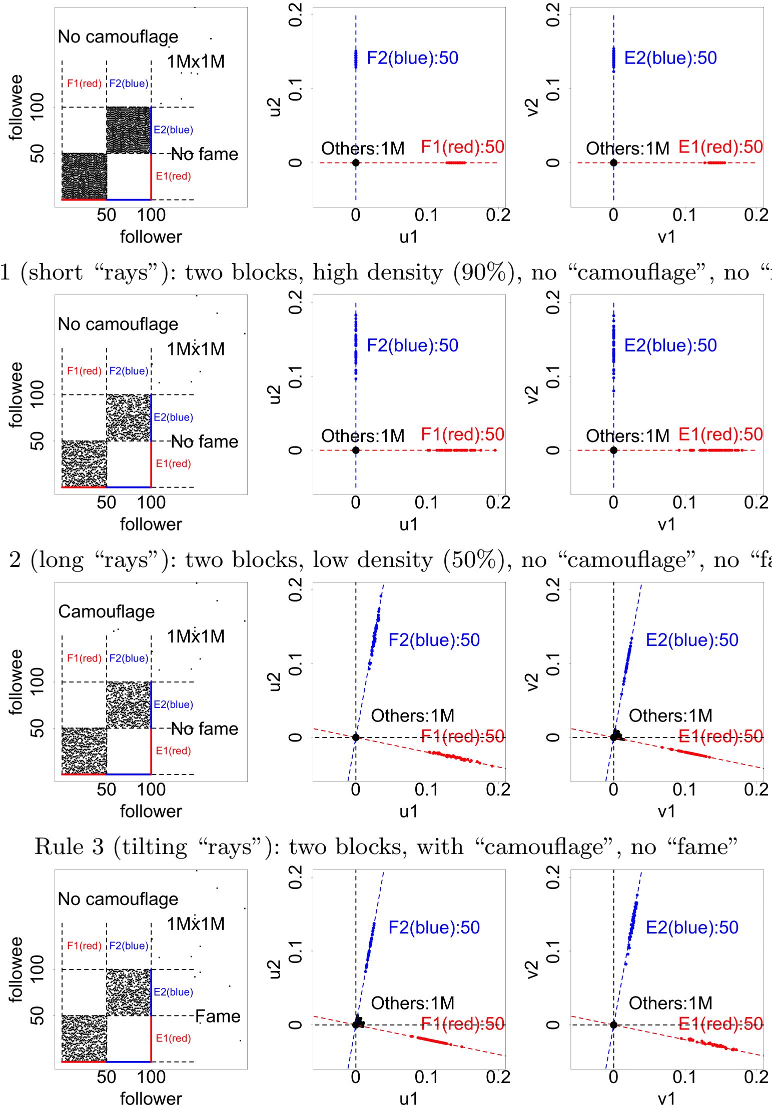
More formally, the k-truncated singular value decomposition (SVD) is a factorization of the form $A = U\Sigma V^T$, where Σ is a $k \times k$ diagonal matrix with the first k singular values, and U and V are orthonormal matrices of dimensions $N \times k$. U and V contain as their columns the left- and right- singular vectors, respectively. Let $u_{n,i}$ be the (n,i) entry of matrix U, and similarly, $v_{n,i}$ is the entry of matrix V. The score $u_{n,i}$ is the coordinate of n-th follower on the i-th left-singular vector. Thus, we define (i,j)-left-spectral-subspace plot as the scatter plot of the points $(u_{n,i}, u_{n,j})$, for $n = 1, \ldots, N$. This plot is exactly the projection of all N followers on the i-th and j-th left-singular vectors. We have the symmetric definition for the N users as followees: (i,j)-right-spectral-subspace plot is the scatter plot of the points $(v_{n,i}, v_{n,j})$, for $n = 1, \ldots, N$. Clearly, it is easy to visualize such 2-dimensional plots; if used carefully, the plots can reveal a lot of information about the adjacency matrix, as we will show shortly.

As we had shown in Fig.1.(a-b), normally, given a random power law graph, we would expect to find a cloud of points around the origin in all the spectral subspaces. However, we find strange shapes ("ray" and "pearl") in some left-spectral-subspace plots of WEIBO data. The question we want to answer here is: *What kind of user behavior could cause "rays" and "pearls" in spectral subspaces?*

The short answer is different types of lockstep behavior. We explain below in more detail what type of lockstep behavior generates such the odd patterns.

3.2 "Ray" for non-overlapping lockstep behavior

In order to enumerate all the types of lockstep behavior, we introduce concepts of "camouflage" and "fame". If a group of followers F had monetary incentives to follow the same group of followees E in lockstep, they could follow additional followees who are not in E, which is called "camouflage" that helps look normal. Similarly, the group of followees E could have additional followers who are not in F, which we succinctly call "fame".



Rule 1 (short "rays"): two blocks, high density (90%), no "camouflage", no "fame"

Rule 2 (long "rays"): two blocks, low density (50%), no "camouflage", no "fame"

Rule 3 (tilting "rays"): two blocks, with "camouflage", no "fame"

Rule 3 (tilting "rays"): two blocks, no "camouflage", with "fame"

Fig. 2. Rule 1-3 ("rays"): non-overlapping blocks in adjacency matrix.

With these concepts, we can now study users' lockstep behavior with synthetic datasets. We first generate a $1M \times 1M$ random power law graph and then inject two groups of followers that separately operate in lockstep. In detail, we create 50 new followers in group F_1 to consistently follow 50 followees in group E_1. Similarly, we create another new follower group F_2 to follow a followee group E_2. Thus, if we plot black dots for non-zero entries in the adjacency matrix in the left side of Fig.2, we spot two 50×50 non-overlapping, dense blocks. Properties of the non-overlapping lockstep behavior are discussed as follows:

– *Density:* High, if a new follower connects to 90% of the related followee group; low, if the ratio is as small as 50%.
– *Camouflage:* With camouflage, if the follower connects to 0.1% of other followees; no camouflage, if he follows only the new followees and no one else.
– *Fame:* With fame, if a new followee is also followed by 0.1% of other followers; no fame, if the followee is followed by no one else.

The spectral subspaces formed by left- and right-singular vectors are plotted in the middle and right of Fig.2, respectively. We spot footprints left in these plots by the different types of non-overlapping lockstep behavior and summarize the following rules:

– *Rule 1 (short "rays"):* If the lockstep behavior of followers is compact on the graph, the adjacency matrix contains one or more non-overlapping blocks of high density like 90%. The spectral-subspace plots show short rays: a set of points that densely fall along a line that goes through the origin.
– *Rule 2 (long "rays"):* If a group of followers and a group of followees are consistently but loosely connected, the adjacency matrix contains blocks of low density like 50%. The plots show long rays: the rays stretch into lines aligned with the axes and elongate towards the origin.
– *Rule 3 (tilting "rays"):* If the follower group has "camouflage" or the followee group has "fame", the adjacency matrix shows sparse external connections outside the blocks. Different from Rule 1-2, a more messy set of rays come out of the origin at different angles, called tilting rays.

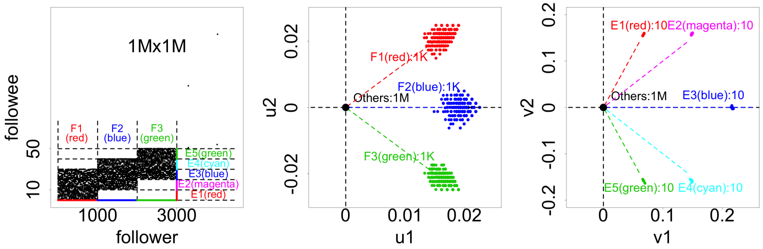
In summary, we find that non-overlapping lockstep behavior creates rays on the spectral-subspace plots: as the density decreases, the rays elongate; as the followers add camouflage or the followees add fame, the rays tilt.

3.3 "Pearl" for partially overlapping lockstep behavior

If a group of followers consistently follows their related group of followees, and partially connect to other groups of followees, we say they have partially overlapping lockstep behavior.

Here we inject the random power law graph with three follower groups F_i, for $i = 1, \ldots, 3$, and five followee groups E_i, for $i = 1, \ldots, 5$. Each follower group has 1,000 fans and each followee group has 10 idols. Followers in F_1 connect to followees in E_1-E_3; followers in F_2 connect to followees in E_2-E_4; and followers in F_3 connect to followees in E_3-E_5; Fig.3.(a) plots the adjacency matrix and (b) plots the left- and right-spectral subspaces. We summarize a new rule here.

– *Rule 4 ("pearls"):* Overlapping lockstep behavior creates "staircase" in the matrix, that is, multiple dense blocks that are overlapping due to followers from each block also connecting to some followees in some other blocks. The spectral-subspace plots show "pearls" as a set of points that form spherical-like high density regions within roughly a same radius from the origin, reminiscent of pearls in a necklace.



(a) adjacency matrix (b) left-spectral subspace plot (c) right-spectral subspace plot

Fig. 3. Rule 4 (“pearls”): a “staircase” of three partially overlapping blocks.

In our case, Fig.3.(b) shows “pearls” of three clusters, each having 1,000 followers in groups from F_1 to F_3. Fig.3.(c) shows five clusters, each having 10 followees in E_1 to E_5. If the follower groups share some followees, or followee groups have the same followers, their clusters are close on these plots.

With the insights into patterns on spectral-subspace plots (Rule 1-4), it is now easy for a practitioner to predict connectivity patterns in the adjacency matrix and infer different types of lockstep behavior.

4 Lockstep behavior inference algorithm

Our lockstep behavior inference algorithm has two steps:

- *Seed selection:* Following Rule 1-4 in Sect.3, select nodes as seeds of followers that behave in lockstep, simiply called “lockstep” followers.
- *“Lockstep” propagation:* Propagate “lockstep” score between followers and followees, and thus catch the lockstep behaviors.

4.1 Seed selection

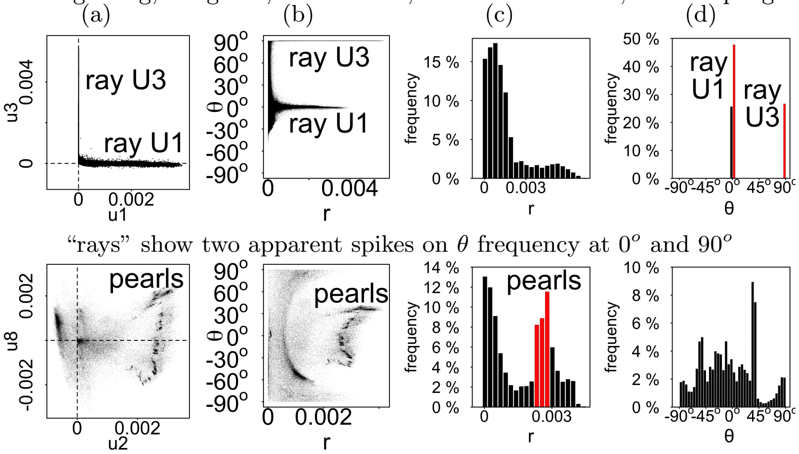
The algorithm can start with any kind of seeds, even randomly selected ones. However, careful selection of seeds obviously accelerates the response time. Fig.4 shows how we conduct the seed selection.

First, generate a range of spectral-subspace plots. We compute the top k left-singular vectors $u_1, \ldots, u_k$, and plot all the follower points in the subspace formed by each pair of the singular vectors. For example, Fig.4.(a) shows “rays” and “pearls” in (1,3)- and (2,8)-left-spectral-subspace plot, respectively.

Second, use the points as input to Hough transform and plot them in polar coordinates (r,θ), where r is the perpendicular distance and θ is the rotation angle. As shown in Fig.4.(b), for “rays”, it shows two straight lines at $\theta = 0^o$ and $\theta = 90^o$; for “pearls”, it shows a set of micro-clusters at some big r values.

Third, divide r and θ axes into bins and plot node frequencies in each bin. Therefore, for “rays”, the θ-bin plot shows two apparent spikes at 0^o and 90^o; for “pearls”, the r-bin plot shows a single spike apart from $r = 0$. With median filtering, we can detect the spikes and then catch the related nodes as seeds.

Notice that if there is no lockstep behavior, no dense block in the adjacency matrix, the spectral-subspace plots show a cloud of points around the origin, as shown in Fig.1.(a-b). The node frequency of angle θ should be almost a constant, and the node frequency of distance r should decrease smoothly with the value increasing. The r- and θ-bin plots are omitted for saving space.



"rays" show two apparent spikes on θ frequency at 0^o and 90^o

"pearls" show a spike on r frequency at a much-greater-than-zero value

Fig. 4. Find "rays" and "pearls": (a) spectral-subspace plot (b) hough transform (c) bin plot of perpendicular distance r frequency (d) bin plot of rotation angle θ frequency.

4.2 "Lockstep" propagation

We now interpret how we start with the seeds and refine a group of followers and followees with lockstep behavior. The "lockstep" value of a followee is defined as the percentage of the seeds or "lockstep" followers who are its followers. Similarly, the "lockstep" value of a follower is defined as the percentage of the "lockstep" followees who are its followees. We need a threshold to decide which users are new "lockstep" followers/followees and here we use 0.8 as default.

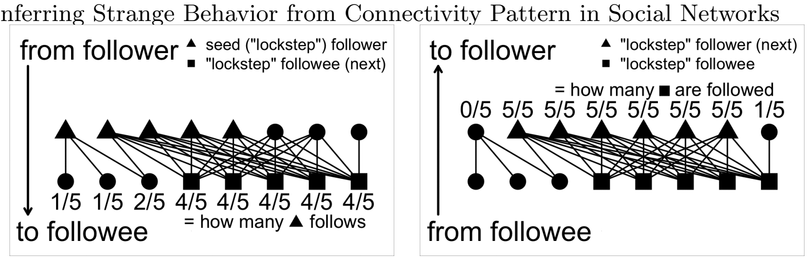
The algorithm recursively propagates this value from followers to followees, and vice versa, like what Belief Propagation method does. In more detail, we explain the steps as follows.

- *From follower to followee:* Fig.5.(a) shows an example of a directed graph with followers at the top and followees at the bottom. We start with 5 "lockstep" followers as seeds for propagation. For each followee, we count how many its followers are in the seed set. We select the group of "lockstep" followees who have too many "lockstep" followers.
- *From followee to follower:* Next for each follower, we count how many its followees are "lockstep". Fig.5.(b) shows how we select new "lockstep" followers and exonerate those innocent with zero or one "lockstep" followee.
- *Repeat until convergence:* Report the groups of "lockstep" followers and followees if they are not empty.

Note that our algorithm is linear to the scale of the social graph and thus scalable to be applied in real applications.

5 Experimental results

In this section we present our empirical evaluation, first on a large, real-world graph, and then on synthetic graphs where the ground truth is known.



(a) select "lockstep" followees: from (seed) followers to followees

(b) select "lockstep" followers: from followees to followers

Fig. 5. Find lockstep behavior of users by propagating "lockstep" value: select followers (followees) who have too many "lockstep" followees (followers).

5.1 Real-world graph

We operate our algorithm on the 100-million-node social graph Weibo. Table 1 report the statistics of strange connectivity patterns that we find on the network.

- *"Blocks" and "staircase":* With the proposed rules and algorithm, we catch a dense block with the "ray" pattern and a staircase of three overlapping blocks with the "pearl" pattern on spectral-subspace plots. Fig.1.(c,e) have show the adjacency matrix and their sets of followers F_0 and F_1-F_3.
- *High density, small "camouflage" and small "fame":* The density of every block is greater than 80%, while the density of the "staircase" is only 43%. It proves that the staircase consists of partially overlapping blocks. The camouflage, that is the connectivity between "lockstep" followers and other followees, is as small as 0.2% dense. The fame is smaller than 2%.

The above numbers validate the existence of non-overlapping and partially overlapping lockstep behavior and also the effectiveness of our method. Further, we give additional evidence of the similar personalities of the "lockstep" followers.

	"ray" F_0	"pearl" F_1	"pearl" F_2	"pearl" F_3	"pearl" Total
Num. seeds	100	1,239	107	990	—
Size of block	$83,208 \times 30$	$3,188 \times 135$	$7,210 \times 79$	$2,457 \times 148$	$10,052 \times 270$
Density	81.3%	91.3%	92.6%	89.1%	43.1%
Camouflage	0.14%	0.06%	0.10%	0.05%	0.07%
Fame	0.05%	1.93%	1.94%	1.72%	1.73%
Out-degree	231±109	310±7	312±7	304±5	310±7
In-degree	2.0±1.4	9±6	10±6	17±13	12±9

Table 1. Statistics of connectivity patterns formed by groups of users with lockstep behavior: The density of the "block" and blocks in "staircase" is greater than 80%, while the Weibo followers have little "camouflage" and followees have little "fame".

- *Strange profiles:* The login-names of 10,787 accounts from the "lockstep" users are like "a#####" (# is a digital number, for example, "a27217"). Their self-declared dates of birth are in lockstep the Jan. 1^{st}. They were probably created by a script, as opposed to natural users.

- *Small in-degree values of followers:* The average in-degree value of followers in the single "block" is as small as 2.0, while that of followers in the "staircase" is smaller than 20. The "lockstep" followers actively connect to their followees but they have little reputation themselves.
- *Similar out-degree values of followers:* The out-degree values of "lockstep" followers in the "staircase" are similarly around 300. In Fig.6, we plot the out-degree distribution of the graph in log-log scale and spot two spikes, which means abnormally high frequency of nodes who have around 300 followees. After we remove the "lockstep" followers, we find out that the spikes disappear and the distribution becomes smoother.

For the last point, we want to say, most graphs exhibit smooth degree distributions, often obeying a heavy-tailed distribution (power law, lognormal, etc). Deviations from smoothness are strange: Border *et al.* [17] said that in the case of the web graph, the spikes were due to link farms. Thus, if the removal of some "lockstep" users makes the degree plots smoother, then we have one more reason to believe that indeed those users were strange.

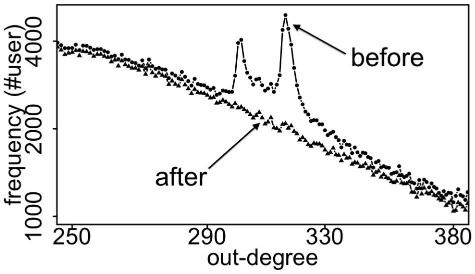
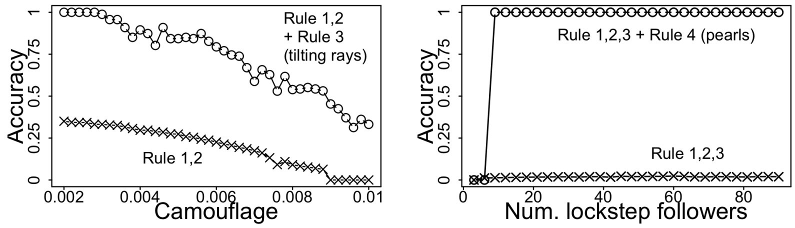


Fig. 6. The out-degree distribution (in log-log scale) becomes smoother after the removal of "lockstep" followers. The followers have similar out-degree values, i.e., similar numbers of followees from the same group.

5.2 Synthetic data

Here we want to validate the effectiveness of Rule 3 (tilting "rays") and 4 ("pearls"). We inject a group of followers and followees operating in lockstep on a 1-million-node random power law graph. The goal is to predict who are the injected nodes. We adopt *Accuracy* to qualify the performance, which is the ratio of correct predictions.

First, we add camouflage to the followers, i.e., we increase the density of connections between the followers and other followees on the graph from 0 to 0.01. We compare the performance of different versions of our algorithm: one considers Rule 3 when it selects seeds from spectral-subspace plots, and the other does not. Rule 3 says when the followers have camouflage, the rays tilt. Fig.7.(a) shows that both accuracy values decrease with the camouflage increasing, and the algorithm that considers Rule 3 performs much better.



(a) Rule 3 (tilting "rays") helps when lockstep followers have "camouflage"

(b) Rule 4 ("pearls") helps when lockstep followers form "staircase"

Fig. 7. Effectiveness of Rule 3-4. If the accuracy is higher, the performance is better.

Second, we inject partially overlapping lockstep behavior. In other words, we put a "staircase" in the adjacency matrix. We change the size of the staircase, i.e., the number of followers. One of the algorithms compared here considers Rule 4 and the other does not. Rule 4 says when there is a staircase, some spectral-subspace plots have "pearls". Fig.7.(b) shows that our algorithm that fully considers all the rules is sensitive to the number of "lockstep" followers. When it is bigger than 7, which is big enough for the behaviors to show footprints in the eigenspaces, we can catch over 95% of the followers, while the version that does not consider Rule 4 fails to predict them.

6 Conclusion

In this paper, we have proposed a novel method to infer users' lockstep behaviors from connectivity patterns on large "who-follows-whom" social graphs. We offer new understanding into the plots of spectral subspaces. The suspicious "ray" and "pearl" patterns are created by different types of lockstep behaviors. Using the insights, we design a fast algorithm to detect such behavior patterns. We demonstrate the effectiveness of our method on both a large real-world graph and synthetic data with injected lockstep behaviors.

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