The number of m-ary search trees on n keys

[short title: Number of m-ary search trees]

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Abstract

Problems associated with m-ary trees have been studied by computer scientists and combinatorialists. It is well known that a simple generalization of the Catalan numbers counts the number of m-ary trees on n nodes. In this paper we consider $\tau_{m,n}$, the number of m-ary search trees on n *keys,* a quantity that arises in studying the space of m-ary search trees under the uniform probability model. We prove an exact formula for $\tau_{m,n}$, both by analytic and by combinatorial means. We use uniform local approximations for sums of i.i.d. random variables to study the asymptotic development of $\tau_{m,n}$ for fixed m as $n \to \infty$.

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1 Introduction and summary

For integer $m \geq 2$, the m-ary search tree, or multiway tree, generalizes the binary search tree. Search trees are fundamental data structures in computer science. For background we refer the reader to Knuth (1973b), Mahmoud (1992), and Dobrow and Fill (1996).

An *m-ary tree* is a rooted tree with at most m "children" for each node (vertex), each of which is distinguished as one of m possible types. Recursively expressed, an m-ary tree either is empty or is a node (called the root) with m distinguished subtrees, each of which is an m-ary tree.

An *m-ary search tree* is an m-ary tree in which each node has the capacity to contain $m-1$ elements of some linearly ordered set, called the set of *keys*. In typical implementations, the keys at each node are stored in increasing order and at each node one has m pointers to the subtrees. By spreading the input data in m directions instead of only 2, as is the case in a binary search tree, one seeks to have shorter path lengths and thus quicker search times.

There is an extensive computer science literature on multiway trees. There is also a large combinatorics literature on m-ary trees. However, as far as we can determine, existing combinatorial work has dealt almost exclusively with m-ary trees on n *nodes*, whereas here we shall be concerned with m-ary search trees on n *keys*. We consider the space of m-ary search trees on n keys and for simplicity take the set of keys to be $[n] := \{1, 2, \ldots, n\}$.

Two common probability models on the space of m-ary search trees are the uniform model (every tree equally likely) and the random permutation, or random insertion, model. Dobrow and Fill (1996) treat certain aspects of the random permutation model; Mahmoud (1992) has much more. In this paper we consider the most fundamental question for the uniform case: Let $\tau_{m,n}$ be the number of m-ary search trees on n keys. How big is $\tau_{m,n}$?

This paper is organized as follows. In Section 2, we give an exact formula for $\tau_{m,n}$, proving this result by generating functions and also by a more direct combinatorial argument. In Section 3 and 4 we analyze the asymptotics of $\tau_{m,n}$ as $n \to \infty$ with m constant. In Sections 5 and 6 we give monotonicity results and large-m asymptotics for $\tau_{m,n}$.

2 Exact results

For an ordered r-tuple $(k_0, \ldots, k_{r-1})$, write k_+ for $\sum_{i=0}^{r-1} k_i$.

Theorem 1 *The number of m-ary search trees on n keys is given by*

$$\tau_{m,n} = \sum \binom{k_+}{k_0, \ldots, k_{m-2}} \frac{[(m/(m-1))(k_+ - 1)]!}{[(k_+ - 1)/(m-1)]! k_+!}, \tag{1}$$

where the sum is over all $(m-1)$-tuples $(k_0, \ldots, k_{m-2})$ such that: (i) $k_i \geq 0$ *for* $0 \leq i \leq m-2$, (ii) $m-1$ *divides* $k_+ - 1$, *and* (iii) $\sum_{j=0}^{m-2}(j+1)k_j = n+1$.

The following alternative form of (1) will be useful later:

$$\tau_{m,n} = \sum_{s=\left\lceil \frac{n-(m-2)}{(m-1)^2} \right\rceil}^{\left\lfloor \frac{n}{m-1} \right\rfloor} \sum \frac{(ms)!}{s! k_0! \cdots k_{m-2}!}, \tag{2}$$

where the inner sum is over all $(m-1)$-tuples $(k_0, \ldots, k_{m-2})$ such that: (i) $k_i \geq 0$ for $0 \leq i \leq m-2$, (ii) $k_+ = (m-1)s + 1$, and (iii) $\sum_{j=0}^{m-2} j k_j = n - (m-1)s$.

Proof *Generating function proof:* By the recursive definition of m-ary search trees, for $n \geq m-1$,

$$\tau_{m,n} = \sum \tau_{m,k_1} \cdot \cdots \cdot \tau_{m,k_m}, \tag{3}$$

where the sum is over all m-tuples $(k_1, \ldots, k_m)$ such that $k_i \geq 0$ for $1 \leq i \leq m$ and $k_+ = n - (m-1)$. Fixing m, let $A(z) := \sum_{n=0}^{\infty} \tau_{m,n} z^n$ denote the corresponding generating function. Then (3) gives

$$A(z) - \sum_{j=0}^{m-2} z^j = z^{m-1} A^m(z),$$

or

$$A(z) = z^{m-1} A^m(z) + \frac{1 - z^{m-1}}{1 - z}. \tag{4}$$

Equation (4) can be explicitly solved. [See Exercise 2.3.4.4–33 in Knuth (1973a).] The solution gives

$$\begin{aligned} A(z) &= \sum_{n_1, n_2 \,:\, (1-m)n_1 + n_2 = 1} \frac{(n_1 + n_2 - 1)!}{n_1! n_2!} (z^{m-1})^{n_1} \left(\frac{1 - z^{m-1}}{1 - z} \right)^{n_2} \\ &= \sum_{s=0}^{\infty} \frac{(ms)!}{s![(m-1)s+1]!} z^{(m-1)s} (1 + z + \cdots + z^{m-2})^{(m-1)s+1} \end{aligned} \tag{5}$$

Extracting the coefficient of z^n in (5) gives the result.

Combinatorial proof: We use the fact that the generalized Catalan number

$$C_{m,n} := \frac{(mn)!}{n![(m-1)n+1]!} = \frac{1}{(m-1)n+1}\binom{mn}{n}$$

gives the number of m-ary trees on n nodes, $n \geq 0$. [See Hilton and Pederson (1991) for much interesting material on generalized Catalan numbers.]

It will be convenient here to consider *extended* m-ary trees. We extend a tree by adding to each of its original (now internal) nodes 0, 1, ..., or m external nodes to make the outdegree of all internal nodes equal to m. We state the following well-known fact without proof.

Lemma 2.1 *Any m-ary tree with s internal nodes has $(m-1)s+1$ external nodes.*

From any m-ary tree S with s internal nodes, where

$$(m-1)s \leq n \leq (m-1)s + (m-2)[(m-1)s+1] = (m-1)^2 s + (m-2),$$

i.e., where

$$\left\lceil \frac{n-(m-2)}{(m-1)^2} \right\rceil \leq s \leq \left\lfloor \frac{n}{m-1} \right\rfloor,$$

one can build an m-ary search tree T on n keys whose full nodes are precisely the internal nodes of S by *partially* filling the external nodes of S according to the following two-step procedure:

Step 1. Choose $k_i, 0 \leq i \leq m-2$, to be the number of external nodes of S to be partially filled with i keys. This entails the restrictions on $k_0, \ldots, k_{m-2}$ as stated for the inner sum in (2).

Step 2. Label the $(m-1)s+1$ external nodes of S in some (arbitrary) fashion. Then choose k_i of these to be partially filled with i keys, $0 \leq i \leq m-2$.

The above argument shows that

$$\begin{aligned} \tau_{m,n} &= \sum_s C_{m,s} \sum \binom{(m-1)s+1}{k_0, \cdots, k_{m-2}} \qquad (6) \\ &= \sum_s \sum \frac{(ms)!}{s! k_0! \cdots k_{m-2}!}, \end{aligned}$$

in agreement with (2), where in each case the outer sum is over s satisfying

$$\left\lceil \frac{n-(m-2)}{(m-1)^2} \right\rceil \le s \le \left\lfloor \frac{n}{m-1} \right\rfloor$$

and the inner sum satisfies the restrictions that apply to equation (2). ■

Table 1 gives values of $\tau_{m,n}$ for $2 \le m \le 10$ and $0 \le n \le 10$.

Table 1. $\tau_{m,n}$

m	n										
	0	1	2	3	4	5	6	7	8	9	10
2	1	1	2	5	14	42	132	429	1,430	4,862	16,796
3	1	1	1	3	6	16	42	114	322	918	2,673
4	1	1	1	1	4	10	20	47	128	340	868
5	1	1	1	1	1	5	15	35	70	146	360
6	1	1	1	1	1	1	6	21	56	126	252
7	1	1	1	1	1	1	1	7	28	84	210
8	1	1	1	1	1	1	1	1	8	36	120
9	1	1	1	1	1	1	1	1	1	9	45
10	1	1	1	1	1	1	1	1	1	1	10

3 Asymptotics

The main result of this section, Theorem 2, gives an asymptotic expression for $\tau_{m,n}$. Our analysis is based on deriving a uniform local approximation for the distribution of a certain random sum. For completeness we first give an asymptotic expression for the number of m-ary trees on n nodes. The proof is straightforward using Stirling's approximation.

Lemma 3.1 *As $n \to \infty$,*

$$C_{m,n} = \left[1 + O\left(n^{-1}\right)\right] \left(\frac{m}{2\pi}\right)^{1/2} [(m-1)n]^{-\frac{3}{2}} \left[\frac{m^m}{(m-1)^{m-1}}\right]^n \qquad (7)$$

uniformly in $m \ge 2$.

Next we give asymptotics for $\tau_{m,n}$. We have not tried for and do not know a sharp remainder estimate in Theorem 2.

Theorem 2 *As $n \to \infty$,*

$$\tau_{m,n} \;=\; \left[1 + O\left(n^{-\frac{2}{5}}\right)\right] \left(\frac{m\alpha^*}{2\pi}\right)^{1/2} m^{-\frac{m}{m-1}} n^{-\frac{3}{2}} \left(\frac{1}{z^*}\right)^{n+1} \tag{8}$$

for fixed $m \geq 2$, where z^* is the unique solution in $(0,1)$ to

$$m^{\frac{m}{m-1}}(z + z^2 + \cdots + z^{m-1}) = m - 1 \tag{9}$$

and

$$\alpha^* := m - \left(m^{\frac{m}{m-1}} - 1\right)\left[(z^*)^{-1} - 1\right]^{-1} \in [1, m-1]. \tag{10}$$

Proof For $m = 2$, the result follows easily since $\tau_{2,n} = C_{2,n}$. Consider (6) for fixed $m \geq 3$. The lead order asymptotics for $C_{m,s}$ are provided by Lemma 3.1 uniformly in s over the range of summation. Moreover, we can give a probabilistic interpretation to the inner sum in (6):

$$\sum_{\mathbf{k}} \binom{(m-1)s+1}{k_0, \ldots, k_{m-2}} = (m-1)^{(m-1)s+1}\, P\left(\sum_{j=0}^{m-2} jK_j = n - (m-1)s\right),$$

where

$$(K_0, \ldots, K_{m-2}) \sim \text{Multinomial}\left((m-1)s+1; \frac{1}{m-1}, \ldots, \frac{1}{m-1}\right).$$

Putting $M := (m-1)^{(m-1)s+1}$ for abbreviation, observe next that

$$M\, P\left(\sum_{j=0}^{m-2} jK_j = n - (m-1)s\right) = M\, P\left(S_{(m-1)s+1} = n - (m-1)s\right),$$

where $S_\nu := \sum_{i=1}^{\nu} X_i$ for $\nu \geq 0$ and $X_1, X_2, \ldots$ are i.i.d. uniform over the set $\{0, 1, \ldots, m-2\}$. [To understand this in the context of the combinatorial proof of Theorem 1, note that both sides count the number of ways of partially filling the $(m-1)s+1$ external nodes of S, independently from node to node, subject only to the restriction that the total number of keys added be $n-(m-1)s$.] Putting together the pieces of our argument thus far,

$$\begin{aligned}\tau_{m,n} &= \left[1+O\left(n^{-1}\right)\right] m^{-\frac{m}{m-1}} \left[\frac{m}{2\pi(m-1)}\right]^{1/2} \qquad (11)\\ &\quad \times \sum_{s=\left\lceil \frac{n-(m-2)}{(m-1)^2}\right\rceil}^{\left\lfloor \frac{n}{m-1}\right\rfloor} s^{-\frac{3}{2}} \left[m^{\frac{m}{m-1}}\right]^{(m-1)s+1} P\left(S_{(m-1)s+1}=n-(m-1)s\right).\end{aligned}$$

As we show in Section 4,

$$P(S_{(m-1)s+1}=n-(m-1)s)=(m-1)^{-[(m-1)s+1]}, \qquad (12)$$

if

$$s=\frac{n-(m-2)}{(m-1)^2} \text{ or } s=\frac{n}{m-1};$$

$$P(S_{(m-1)s+1}=n-(m-1)s)\le \exp\left\{[(m-1)s+1][K(\theta_c)-c\theta_c]\right\}, \qquad (13)$$

if

$$\frac{n-(m-2)}{(m-1)^2}<s<\frac{n}{m-1};$$

and, for any $\delta>0$,

$$\begin{aligned}&P\left(S_{(m-1)s+1}=n-(m-1)s\right) \qquad (14)\\ &=\left[1+O\left(n^{-1}\right)\right]\left[2\pi(m-1)sK''(\theta_c)\right]^{-1/2}\exp\left\{[(m-1)s+1]\left[K\left(\theta_c\right)-c\theta_c\right]\right\}\end{aligned}$$

uniformly in s satisfying

$$(1+\delta)\frac{n}{(m-1)^2}\le s\le (1-\delta)\frac{n}{m-1}.$$

In (13) and (14),

$$c\equiv c(n,m,s):=\frac{n+1}{(m-1)s+1}-1$$

satisfies $0<c<m-2$, θ_c will be defined shortly, and

$$K(\theta)=\begin{cases}\log|e^{\theta(m-1)}-1|-\log|e^{\theta}-1|-\log(m-1) & \text{if } \theta\ne 0\\ 0 & \text{if } \theta=0.\end{cases}$$

Note that $K(\theta)$ increases from $-\log(m-1)$ to ∞ as θ increases over $(-\infty, \infty)$. Further,

$$K'(\theta) = \begin{cases} (m-1)\left[1-e^{-\theta(m-1)}\right]^{-1} - \left[1-e^{-\theta}\right]^{-1} & \text{if } \theta \neq 0 \\ \frac{1}{2}(m-2) & \text{if } \theta = 0 \end{cases}$$

increases strictly from 0 to $m-2$ for $\theta \in (-\infty, \infty)$, and

$$K''(\theta) = \begin{cases} e^{-\theta}\left(1-e^{-\theta}\right)^{-2} - (m-1)^2 e^{-\theta(m-1)}\left[1-e^{-\theta(m-1)}\right]^{-2} & \text{if } \theta \neq 0 \\ \frac{1}{12}\left[(m-1)^2-1\right] & \text{if } \theta = 0 \end{cases}$$

is strictly positive for all $\theta \in \mathbb{R}$. The value $\theta = \theta_c$ is defined as the unique real solution to $K'(\theta) = c$.

To see which terms contribute most to (11), we seek the value of $c \in (0, m-2)$ maximizing

$$\begin{aligned} &\frac{1}{c+1}\left[\frac{m}{m-1}\log m + K(\theta_c) - c\theta_c\right] \\ &= -\theta_c + \frac{1}{K'(\theta_c)+1}\left[\frac{m}{m-1}\log m + K(\theta_c) + \theta_c\right]. \end{aligned}$$

It follows from a little calculus that the function

$$f(\theta) := -\theta + \frac{1}{K'(\theta)+1}\left[\frac{m}{m-1}\log m + K(\theta) + \theta\right] \tag{15}$$

is unimodal and achieves its maximum at $\theta^* \in \mathbb{R}$ satisfying

$$\frac{m}{m-1}\log m + K(\theta^*) + \theta^* = 0.$$

Note that $\theta^* < 0$. Writing $z^* = e^{\theta^*} = e^{-|\theta^*|} \in (0,1)$, z^* is characterized as the solution of the polynomial equation

$$m^{m/(m-1)}\left(z + z^2 + \cdots + z^{m-1}\right) = m-1.$$

It then follows, omitting a few simple details, that

$$c^* := K'(\theta^*) = (m-1) - \left(m^{m/(m-1)} - 1\right)\frac{z^*}{1-z^*} = \alpha^* - 1 \in (0, m-2).$$

It now follows simply from (11), (12), (13), and (14) that, for any $\epsilon > 0$,

$$\begin{aligned}\tau_{m,n} &= \left[1+O\left(n^{-1}\right)\right] m^{-\frac{m}{m-1}} \left[\frac{m}{2\pi(m-1)}\right]^{1/2} [2\pi(m-1)]^{-1/2} \\ &\times \sum_{s=\left\lceil \frac{1-\epsilon}{m-1}\left(\frac{n+1}{c^*+1}-1\right)\right\rceil}^{\left\lfloor \frac{1+\epsilon}{m-1}\left(\frac{n+1}{c^*+1}-1\right)\right\rfloor} s^{-2} \left[K''\left(\theta_c\right)\right]^{-\frac{1}{2}} \exp\left\{(n+1)f(\theta_c)\right\},\end{aligned}$$

where the function f is defined at (15).

By expanding $f(\theta_c)$ about θ^*,

$$\begin{aligned}f(\theta_c) &= f(\theta^*) + (\theta_c-\theta^*)f'(\theta^*) + \frac{1}{2}(\theta_c-\theta^*)^2 f''(\theta^*) + \frac{1}{6}(\theta_c-\theta^*)^3 f'''(\tilde{\theta}) \\ &= |\theta^*| + \frac{1}{2}(\theta_c-\theta^*)^2 f''(\theta^*) + \frac{1}{6}(\theta_c-\theta^*)^3 f'''(\tilde{\theta})\end{aligned}$$

where $\tilde{\theta}$ is intermediate to θ^* and θ_c. Further,

$$f''(\theta^*) = \frac{-K''(\theta^*)}{K'(\theta^*)+1} < 0.$$

Also, it is clear that $f'''(\tilde{\theta}) = O(1)$ uniformly for s in the range of summation. Therefore, also using the fact that f is unimodal,

$$\begin{aligned}\tau_{m,n} &= \left[1+O\left(n^{-1}\right)\right] m^{-\frac{m}{m-1}} \left[\frac{m}{2\pi(m-1)}\right]^{1/2} [2\pi(m-1)]^{-1/2} \\ &\times \sum_{s:|\theta_c-\theta^*|\leq(n+1)^{-2/5}} s^{-2} \left(\frac{1}{z^*}\right)^{n+1} \left[K''\left(\theta_c\right)\right]^{-\frac{1}{2}} \\ &\times \exp\left\{(n+1)\left[-\frac{1}{2}(\theta_c-\theta^*)^2|f''(\theta^*)| + O(|\theta_c-\theta^*|^3)\right]\right\} \\ &= \left[1+O\left(n^{-1}\right)\right] m^{-\frac{m}{m-1}} \left[\frac{m}{2\pi(m-1)}\right]^{1/2} [2\pi(m-1)]^{-1/2} \left(\frac{1}{z^*}\right)^{n+1} \\ &\times \sum_{s:|\theta_c-\theta^*|\leq(n+1)^{-2/5}} s^{-2} \left[K''\left(\theta_c\right)\right]^{-\frac{1}{2}} \left[1+O\left(n|\theta_c-\theta^*|^3\right)\right] \\ &\times \exp\left\{-\frac{1}{2}|f''(\theta^*)|(n+1)(\theta_c-\theta^*)^2\right\}.\end{aligned}$$

Note that the O-estimates in the sums here are uniform over s satisfying $|\theta_c - \theta^*| \leq (n+1)^{-2/5}$. Straightforward calculations now show that, uniformly in such s, we have

$$\begin{aligned} s^{-2} &= \left(\frac{n+1}{m-1}\right)^{-2} (K'(\theta^*)+1)^2 \left[1 + 2|f''(\theta^*)|(\theta_c - \theta^*)\right] \\ &\quad \times \left[1 + O\left(n^{-1}\right) + O\left((\theta_c - \theta^*)^2\right)\right]. \end{aligned}$$

Also,

$$[K''(\theta_c)]^{-1/2} = [K''(\theta^*)]^{-1/2} \left\{1 - \frac{K'''(\theta^*)}{2K''(\theta^*)}(\theta_c - \theta^*) + O\left((\theta_c - \theta^*)^2\right)\right\}.$$

We now have

$$\begin{aligned} \tau_{m,n} = & \left[1 + O\left(n^{-1}\right)\right] m^{-\frac{m}{m-1}} \left[\frac{m}{2\pi(m-1)}\right]^{1/2} [2\pi(m-1)]^{-1/2} \left(\frac{m-1}{n+1}\right)^2 \\ & \times \ (K'(\theta^*)+1)^2 (z^*)^{-(n+1)} \\ & \times \sum_{s:|\theta_c - \theta^*| \leq (n+1)^{-2/5}} \left[1 + O\left(n^{-1}\right) + O\left((\theta_c - \theta^*)^2\right) + O\left(n|\theta_c - \theta^*|^3\right)\right] \\ & \times \ [K''(\theta^*)]^{-1/2} \left\{1 - \frac{K'''(\theta^*)}{2K''(\theta^*)}(\theta_c - \theta^*) + O\left((\theta_c - \theta^*)^2\right)\right\} \\ & \times \ \left[1 + 2|f''(\theta^*)|(\theta_c - \theta^*)\right] \exp\left\{-\frac{1}{2}|f''(\theta^*)|(n+1)(\theta_c - \theta^*)^2\right\}. \quad (16) \end{aligned}$$

We next consider the sum

$$\sum_{s:|\theta_c - \theta^*| \leq (n+1)^{-2/5}} \exp\left\{-\frac{1}{2}|f''(\theta^*)|(n+1)(\theta_c - \theta^*)^2\right\} \quad (17)$$

appearing in (16). Using the fact that successive values of s give values of θ_c separated by

$$\left[1 + O\left(n^{-2/5}\right)\right] [K''(\theta^*)]^{-1} [K'(\theta^*)+1]^2 \frac{m-1}{n+1},$$

which is of exact order $1/n$, and writing

$$h(n, \theta, \theta^*) := \frac{1}{2}|f''(\theta^*)|(n+1)(\theta - \theta^*)^2$$

for brevity, one concludes

$$
\begin{aligned}
&\text{expression (17)} \\
&= \left[1+O\left(n^{-2/5}\right)\right] K''(\theta^*)\left[K'(\theta^*)+1\right]^{-2}\frac{n+1}{m-1} \\
&\quad\times \int_{\theta^*-(n+1)^{-2/5}}^{\theta^*+(n+1)^{-2/5}} \exp\left\{-h(n,\theta,\theta^*)+O\left(n^{-1}\right)+O\left(|\theta-\theta^*|\right)\right\} d\theta \\
&= \left[1+O\left(n^{-2/5}\right)\right] K''(\theta^*)\left[K'(\theta^*)+1\right]^{-2}\frac{n+1}{m-1}(n+1)^{-1/2}|f''(\theta^*)|^{-1/2} \\
&\quad\times \int_{-(n+1)^{1/10}|f''(\theta^*)|^{1/2}}^{(n+1)^{1/10}|f''(\theta^*)|^{1/2}} \left[1+O\left(n^{-1/2}|u|\right)\right]\exp\left\{-\frac{1}{2}u^2\right\} du \\
&= \left[1+O\left(n^{-2/5}\right)\right] K''(\theta^*)\left[K'(\theta^*)+1\right]^{-2}(m-1)^{-1}(n+1)^{1/2} \\
&\quad\times |f''(\theta^*)|^{-1/2}(2\pi)^{1/2}.
\end{aligned}
$$

The other terms in (16) are easily managed, leading finally (after some cancellation) to

$$
\begin{aligned}
\tau_{m,n} &= \left[1+O\left(n^{-2/5}\right)\right] m^{-\frac{m}{m-1}}\left(\frac{m}{2\pi}\right)^{1/2} n^{-3/2}\left(K'(\theta^*)+1\right)^{1/2}\left(\frac{1}{z^*}\right)^{n+1} \\
&= \left[1+O\left(n^{-2/5}\right)\right] m^{-\frac{m}{m-1}}\left(\frac{m}{2\pi}\right)^{1/2} n^{-3/2}\left(\alpha^*\right)^{1/2}\left(\frac{1}{z^*}\right)^{n+1},
\end{aligned}
$$

as desired. ■

Examples:

(a) $m = 2$. Although the proof of this case was handled separately, the results fit the framework of Theorem 2. We have $z^* = 1/4$ and $\alpha^* = 1$. Thus

$$
\begin{aligned}
\tau_{2,n} &= \left(1+O\left(n^{-2/5}\right)\right)\left(\frac{2}{2\pi}\right)^{1/2} 2^{-2}n^{-3/2}4^{n+1} \\
&= \left(1+O\left(n^{-2/5}\right)\right)\pi^{-1/2}n^{-3/2}4^{n}.
\end{aligned}
$$

Note from Lemma 3.1 that $O(n^{-2/5})$ is even guaranteed to be $O(n^{-1})$ in this case. [Numerical computations suggest that when $m = 3$, the remainder

$O(n^{-2/5})$ is again $O(n^{-1})$; we have not examined this issue at all for larger values of m.]

(b) $m \geq 2$. It is easy to solve for z^* with a high degree of accuracy using a computation package like MAPLE. Having done so, we give explicit asymptotic formulas for $\tau_{m,n}$ and $C_{m,n}$ for selected values of m in Table 2. The values of w, x, y, z appearing there are given to five significant digits each.

Table 2.

	$\tau_{m,n} \sim wn^{-3/2}x^n$		$C_{m,n} \sim yn^{-3/2}z^n$	
m	w	$x = 1/z^*(m)$	y	$z = m^m/(m-1)^{m-1}$
2	.56419	4	.56419	4
3	.49667	3.3692	.24430	6.75
4	.44883	3.0413	.15355	9.4815
5	.41242	2.8405	.11151	12.207
6	.38355	2.7053	.087404	14.930
7	.36001	2.6085	.071818	17.651
8	.34041	2.5359	.060927	20.372
9	.32380	2.4795	.052893	23.092
10	.30952	2.4346	.046725	25.812
25	.20736	2.1912	.016965	66.593
50	.15135	2.1052	.0082243	134.55
100	.10931	2.0582	.0040500	270.47
250	.070267	2.0265	.0016054	678.21

Although the expansions in Lemma 3.1 and Theorem 2 are for *fixed* m as $n \to \infty$, Table 2 graphically reveals the large-m behavior of the constants in the asymptotic expressions for $\tau_{m,n}$ and $C_{m,n}$. As the table suggests, and as we will show in Section 6, the ratio $1/z^*(m)$ for $\tau_{m,n}$ decreases to 2 as $m \to \infty$, in sharp contrast to the ratio $m^m/(m-1)^{m-1}$ for $C_{m,n}$, which increases (linearly) to ∞.

4 Uniform local approximation

In this section we use standard large deviation techniques [cf. Lugannani and Rice (1980) and Daniels (1987)] to approximate $P(S_{(m-1)s+1} = n-(m-1)s)$, where

$$S_\nu = \sum_{i=1}^{\nu} X_i$$

and $X_1, X_2, \ldots$ are i.i.d. Uniform$\{0, 1, \ldots, m-2\}$. Put

$$c \equiv c(n,m,s) := \frac{n-(m-1)s}{(m-1)s+1} = \frac{n+1}{(m-1)s+1} - 1.$$

In order to establish (12) through (14), we need to approximate $P(S_\nu = c\nu)$, where

$$\nu = (m-1)s + 1 \geq \frac{n-(m-2)}{m-1} + 1 = \frac{n+1}{m-1}$$

is large and c satisfies $0 \leq c \leq m-2$. Since we can easily calculate

$$P(S_\nu = 0) = (m-1)^{-\nu} = P(S_\nu = (m-2)\nu), \tag{18}$$

equation (12) holds and we may assume $0 < c < m-2$. The assertions (13) and (14) are a consequence of the following result.

Lemma 4.1 *(a) For all $0 < c < m-2$ we have*

$$P(S_\nu = c\nu) \leq \exp\{\nu[K(\theta_c) - c\theta_c]\}.$$

(b) As $\nu \to \infty$,

$$P(S_\nu = c\nu) = \left[1 + O\left(\nu^{-1}\right)\right] [2\pi\nu K''(\theta_c)]^{-1/2} \exp\left\{\nu[K(\theta_c) - c\theta_c]\right\},$$

uniformly for c in any compact subinterval of $(0, m-2)$.

Proof Let $X \sim$ Uniform$\{0, 1, \ldots, m-2\}$. The cumulant generating function (cgf) for X is

$$\begin{aligned} K(\theta) &:= \log \mathrm{E}\, e^{\theta X} = \log\left(\sum_{j=0}^{m-2} e^{\theta j}\right) - \log(m-1) \\ &= \begin{cases} \log\left(1 - e^{\theta(m-1)}\right) - \log(1 - e^{\theta}) - \log(m-1) & \text{if } \theta < 0 \\ \log\left(e^{\theta(m-1)} - 1\right) - \log(e^{\theta} - 1) - \log(m-1) & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0. \end{cases} \end{aligned}$$

For $\theta \in \mathbb{R}$, define the "exponentially tilted" distribution

$$P_\theta(X = j) := e^{\theta j - K(\theta)} P(X = j), \quad j = 0, \ldots, m-2. \tag{19}$$

This distribution has cgf $K_\theta(\eta) = K(\theta + \eta) - K(\theta)$ and so has mean $K'_\theta(0) = K'(\theta)$ and variance $K''_\theta(0) = K''(\theta)$. Simple calculations give

$$\mathrm{E}_\theta X = \begin{cases} \frac{1}{2}(m-2) & \text{if } \theta = 0 \\ (m-1)\left[1 - e^{-\theta(m-1)}\right]^{-1} - \left(1 - e^{-\theta}\right)^{-1} & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} &\mathrm{Var}_\theta X \\ &= \begin{cases} \frac{1}{12}\left[(m-1)^2 - 1\right] & \text{if } \theta = 0 \\ e^{-\theta}\left(1 - e^{-\theta}\right)^{-2} - (m-1)^2 e^{-\theta(m-1)}\left[1 - e^{-\theta(m-1)}\right]^{-2} & \text{otherwise.} \end{cases} \end{aligned}$$

We will henceforth assume that $m \geq 3$. Note in this case that $\mathrm{Var}_\theta X > 0$.

We are particularly interested in the value of $\theta_c \equiv \theta_{c(n,m,s)}$ of θ satisfying $K'(\theta_c) = c$. Since $K'(\theta)$ increases (strictly) from 0 at $\theta = -\infty$ to $m - 2$ at $\theta = \infty$, such θ_c is well defined and finite. Unfortunately, it does not seem possible to solve explicitly for θ_c except in the cases $m = 3$ and $m = 4$.

According to (19),

$$P(S_\nu = c\nu) \;=\; e^{\nu[K(\theta_c) - c\theta_c]} P_{\theta_c}(S_\nu = c\nu),$$

from which part (a) of the lemma follows immediately. To prove part (b), we continue the calculation using the Fourier inversion formula for integer-valued random variables:

$$\begin{aligned} P_{\theta_c}(S_\nu = c\nu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{E}_{\theta_c} e^{itS_\nu}\right) e^{-ic\nu t}\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\left(\mathrm{E}_{\theta_c} e^{itX}\right) e^{-ict}\right]^\nu dt. \end{aligned}$$

Now

$$\begin{aligned} \left|\left(\mathrm{E}_{\theta_c} e^{itX}\right) e^{-ict}\right|^2 &= \left|\mathrm{E}_{\theta_c} e^{itX}\right|^2 = \mathrm{E}_{\theta_c} \exp\left[it\left(X_1 - X_2\right)\right] \\ &= \mathrm{E}_{\theta_c} \cos\left(t\left(X_1 - X_2\right)\right) = 1 - 2\mathrm{E}_{\theta_c} \sin^2\left(\frac{1}{2}t\left(X_1 - X_2\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\le \; 1 - 4\, P_{\theta_c}(X=0)\, P_{\theta_c}(X=1)\, \sin^2\left(\frac{1}{2}t\right) \\
&= \; 1 - \frac{4\exp(\theta_c)}{\left[\sum_{j=0}^{m-2}\exp(\theta_c j)\right]^2}\, \sin^2\left(\frac{1}{2}t\right) \\
&\le \; 1 - A\sin^2\left(\frac{1}{2}t\right) \le 1 - Bt^2
\end{aligned}
$$

for positive constants A and $B = A/\pi^2$, c in a compact subinterval of $(0, m-2)$, and $t \in (-\pi, \pi)$. Thus the contribution to $P_{\theta_c}(S_\nu = c\nu)$ from $|t| \ge \nu^{-2/5}$ is bounded by

$$
\left(1 - B\nu^{-4/5}\right)^{\nu/2} \le \exp\left(-\frac{1}{2}B\nu^{1/5}\right)
$$

and so is uniformly negligible (even for higher order expansions).

Next, extend K to complex arguments via the definition

$$
K(z) = \log\left(\sum_{j=0}^{m-2} e^{jz}\right) - \log(m-1);
$$

using the principal branch of the logarithm function, this gives an analytic function of those z with imaginary part less than $2\pi/(m-1)$ in absolute value. What remains of $P_{\theta_c}(S_\nu = c\nu)$ is

$$
\frac{1}{2\pi}\int_{-\nu^{-2/5}}^{\nu^{-2/5}} \exp\left[\nu\left\{K(\theta_c + it) - K(\theta_c) - c(it)\right\}\right] dt.
$$

Using Taylor's theorem, it is not hard to check that

$$
K(\theta_c + it) - K(\theta_c) - c(it) = -\frac{1}{2}K''(\theta_c)t^2 - \frac{i}{6}K'''(\theta_c)t^3 + O(t^4),
$$

uniformly for c in a compact subinterval of $(0, m-2)$ and in $|t| \le \nu^{-2/5}$ for large ν. Let Z be a random variable with the standard normal distribution. Then, as $\nu \to \infty$, we have, with the required uniformity,

$$
\frac{1}{2\pi}\int_{-\nu^{-2/5}}^{\nu^{-2/5}} \exp\left[\nu\left\{K(\theta_c + it) - K(\theta_c) - c(it)\right\}\right] dt
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\nu^{-2/5}}^{\nu^{-2/5}} \exp\left[\nu\left\{-\frac{1}{2}K''(\theta_c)t^2 - \frac{i}{6}K'''(\theta_c)t^3 + O(t^4)\right\}\right] dt \\
&= \frac{1}{2\pi}\int_{-\nu^{-2/5}}^{\nu^{-2/5}} \left[1 - \frac{i}{6}K'''(\theta_c)\nu t^3 + O(\nu t^4) + O(\nu^2 t^6)\right]\exp\left[-\frac{1}{2}K''(\theta_c)\nu t^2\right] dt \\
&= [\nu K''(\theta_c)]^{-1/2} \times \frac{1}{2\pi} \\
&\times \int_{-[\nu K''(\theta_c)]^{1/2}\nu^{-2/5}}^{[\nu K''(\theta_c)]^{1/2}\nu^{-2/5}} \left[1 - \frac{i}{6}\frac{K'''(\theta_c)}{(K''(\theta_c))^{3/2}}\nu^{-1/2}u^3 + O\left(\nu^{-1}(u^4+u^6)\right)\right] e^{-u^2/2}\, du \\
&= [\nu K''(\theta_c)]^{-1/2} \times \frac{1}{2\pi}\int_{-[\nu K''(\theta_c)]^{1/2}\nu^{-2/5}}^{[\nu K''(\theta_c)]^{1/2}\nu^{-2/5}} \left[1 + O\left(\nu^{-1}(u^4+u^6)\right)\right] e^{-u^2/2}\, du \\
&= [2\pi\nu K''(\theta_c)]^{-1/2} \times \left\{P\left(|Z| \le [K''(\theta_c)]^{1/2}\nu^{1/10}\right) + O\left(\nu^{-1}\right)\right\} \\
&= [2\pi\nu K''(\theta_c)]^{-1/2}\left[1 + O\left(\nu^{-1}\right)\right],
\end{aligned}
$$

and the result is proved. ■

Remark: We can show (but omit the details) that

$$K(\theta_c) - c\theta_c \to -\log(m-1)$$

as $c \to 0$ or $c \to m-2$. In other words, there is a certain amount of continuity in going from Lemma 4.1 to (18).

5 Monotonicity of τ

In this section we consider monotonicity of $\tau_{m,n}$ in m and n.

It is intuitively obvious, and easy to show by induction, that both $C_{m,n}$ and $\tau_{m,n}$ are increasing in $n \ge 0$ for fixed m. Further, $C_{m,n}$ is strictly increasing in $n \ge 1$, and $\tau_{m,n}$ is strictly increasing in $n \ge m-1$.

For fixed $n \ge 2$, it is clear that $C_{m,n}$ increases strictly in m, since an m-ary tree on n nodes can also be considered an $(m+1)$-ary tree on n nodes, but not conversely. In contrast, we conjecture the following.

Conjecture 5.1 *For fixed $n \geq 1$ and $2 \leq m < m'$,*

$$\tau_{m,n} \geq \tau_{m',n},$$

with strict inequality when $m' \leq n+1$.

Although we have been unable to prove this, we shall give a partial monotonicity result in Theorem 3, and Proposition 6.2 is further evidence in favor.

Note added in proof: Conjecture 5.1 is false. The first counterexample is the following:

$$\tau_{8,16} = 12112 < 12870 = \tau_{9,16}.$$

Before proceeding we switch notation to $t(c,n) := \tau_{c+1,n}$, $c \geq 1$ and $n \geq 0$. Here c denotes the capacity, or maximum number of keys that can be stored, at each node. We define a capacity-c tree to be a $(c+1)$-ary search tree.

Our main result (Theorem 3) is a consequence of the following three lemmas. We state the first easy lemma without proof.

Lemma 5.1 *A capacity-c tree on $n \geq 1$ keys has at most $\left\lfloor \frac{n-1}{c} \right\rfloor$ nodes with at least one nonempty subtree.*

Remark: The bound in Lemma 5.1 is achieved at (for example) the tree obtained by successively inserting the keys $1, 2, \ldots, n$ into an initially empty capacity-c tree.

Lemma 5.2 *If $1 \leq c < c'$ and $n \geq 1$, then*

$$t(c,n) \leq t\left(c', n + (c'-c)\left\lfloor \frac{n-1}{c} \right\rfloor\right), \tag{20}$$

with strict inequality if and only if $n \geq c+1$.

Proof If $1 \leq n \leq c$, then $t(c,n) = 1$ and

$$t\left(c', n + (c'-c)\left\lfloor \frac{n-1}{c} \right\rfloor\right) = t(c',n) = 1.$$

So we assume $n \geq c+1$ and build an injection from trees T counted by $t(c,n)$ to trees T' counted by the right side of (20). It will be easy to check that the injection is *not* a surjection, and the lemma will follow.

We use the notion of a *complete tree* [see Dobrow and Fill (1996) for further background]. Suppose first that $n = m^k - 1$ for integer k. Call the unique m-ary search tree on n keys with minimum height ($= k - 1$) the *perfect tree.* For general n, let $k = \lfloor \log_m(n+1) \rfloor$. The *complete tree* can be obtained by attaching to the perfect tree on $m^k - 1$ keys, and as far to the left as possible, $n - (m^k - 1)$ keys at distance k from the root.

To build T' from T, first observe that, since $n \geq c+1$, the root of T is full and has at least one nonempty subtree. To begin building T', replace *each* of the (at most $\left\lfloor \frac{n-1}{c} \right\rfloor$) nodes with at least one nonempty subtree by a full node of capacity c', and replace each of the other nodes with a node of capacity c' containing the same number of keys as the node has in T. The current tree is a tree with nodes of capacity c', but it may have strictly fewer than $n + (c' - c)\left\lfloor \frac{n-1}{c} \right\rfloor$ keys. Remedy this by replacing the currently empty $(c'+1)$-st subtree of the root by the complete capacity-c' tree on the remaining number of keys. The resulting tree is the desired T'. ■

Lemma 5.3 *If $1 \leq c < c'$ and c' is a multiple of c and $n \geq 0$, then $t(c', n) \leq t(c, n)$, with strict inequality if and only if $n \geq c+1$.*

Proof As in the previous proof, we exhibit an injection from capacity-c' trees to capacity-c trees that is not surjective when $n \geq c+1$. In Figure 1 we give a "Proof without Words" for the case when $c = 2$ and $c' = 6$. It is easy to see how this generalizes to give the result. ■

Theorem 3 *If $1 \leq c < c'$ and $n \geq 0$, then*

$$t(c', n) \leq t\left(c, \left\lfloor \frac{c\lceil c'/c \rceil}{c'} n \right\rfloor\right)$$

with strict inequality if $n \geq c+1$.

Remarks:

1. If c' is a multiple of c, then Theorem 3 is an immediate consequence of Lemma 5.3.

2. According to Lemma 5.3, $\tau_{m,n} \leq \tau_{2,n}$ for $m \geq 3$, with strict inequality if and only if $n \geq 2$.

3. The theorem "nearly" gives $t(c', n) \le t(c, n)$ when c' is much larger than c, since then $c\lceil c'/c\rceil/c'$ is not much larger than 1. Unfortunately the theorem fares badly when c' is not much larger than c. For example, if $c' = c + 1$, then

$$\frac{c\lceil c'/c\rceil}{c'} = \frac{2c}{c+1},$$

and the conclusion of the theorem is (for c large) not much better than $t(c+1, n) \le t(c, 2n)$.

Proof According to Remark 1, we may suppose that c' is *not* a multiple of c. Without loss of generality, assume $n \ge c+1$, since otherwise $t(c', n) = 1$. We apply Lemma 5.2, letting c' play the role of c and $c\lceil c'/c\rceil > c'$ play the role of c'. Thus

$$t(c', n) \le t\left(c\lceil c'/c\rceil, n + (c\lceil c'/c\rceil - c')\left\lfloor \frac{n-1}{c'} \right\rfloor\right).$$

The second argument on the right satisfies

$$\begin{aligned} n + (c\lceil c'/c\rceil - c')\left\lfloor \frac{n-1}{c'} \right\rfloor &\le n + (c\lceil c'/c\rceil - c')\frac{n-1}{c'} \\ &= \frac{c\lceil c'/c\rceil}{c'}n - \frac{c\lceil c'/c\rceil - c'}{c'} \\ &< \frac{c\lceil c'/c\rceil}{c'}n, \end{aligned}$$

and so is $\le \left\lfloor \frac{c\lceil c'/c\rceil}{c'}n \right\rfloor$. Thus, since $t(c, n)$ is nondecreasing in $n \ge 0$ for each fixed $c \ge 1$,

$$t(c', n) \le t\left(c\lceil c'/c\rceil, \left\lfloor \frac{c\lceil c'/c\rceil}{c'}n \right\rfloor\right). \tag{21}$$

Now apply Lemma 5.3, with the role of c' there played by $c\lceil c'/c\rceil$, giving

$$t\left(c\lceil c'/c\rceil, \left\lfloor \frac{c\lceil c'/c\rceil}{c'}n \right\rfloor\right) < t\left(c, \left\lfloor \frac{c\lceil c'/c\rceil}{c'}n \right\rfloor\right), \tag{22}$$

where strict inequality holds because $\left\lfloor \frac{c\lceil c'/c\rceil}{c'}n \right\rfloor \ge n \ge c+1$. Combining (21) and (22) completes the proof. ■

6 Large-m behavior of parameters

In typical computer science applications of m-ary search trees, m is often large (between 100 and 1,000). Thus it is of interest to derive asymptotics for parameters depending on m in our asymptotic expansions. In this section we describe the large-m behavior of the fundamental parameters z^* and α^* appearing in Theorem 2.

The defining equation for $z^* \equiv z^*(m)$ can be written

$$\left[(z^*)^{-1} - 1\right]^{-1} \left[1 - (z^*)^{m-1}\right] = \frac{m-1}{m^{m/(m-1)}} =: \gamma(m).$$

To get asymptotic expansions for $z^*(m)$ and $\alpha^* \equiv \alpha^*(m)$, we need to know the behavior of γ:

Lemma 6.1 *For any $k \geq 0$,*

$$\gamma(x) = \exp\left\{-\left[\sum_{j=1}^{k} x^{-j}\left(\log x + j^{-1}\right) + O\left(x^{-(k+1)}\log x\right)\right]\right\}$$

as $x \to \infty$.

In particular,

$$\begin{aligned} \gamma(x) &= \exp\left\{-\left[x^{-1}\log x + x^{-1} + O\left(x^{-2}\log x\right)\right]\right\} \\ &= 1 - x^{-1}\log x - x^{-1} + O\left(x^{-2}(\log x)^2\right) \end{aligned}$$

as $x \to \infty$.

We will be content with the following simple result:

Proposition 6.1 *As $m \to \infty$,*

$$\begin{aligned} z^*(m) &= \left[1 + \frac{1}{\gamma(m)}\right]^{-1} + \exp\{-(1+o(1))m\log 2\} \\ &= \frac{1}{2} - \frac{1}{4}m^{-1}\log m - \frac{1}{4}m^{-1} + O\left(m^{-2}(\log m)^2\right) \end{aligned}$$

and

$$\begin{aligned} \alpha^*(m) &= m - \left(m^{\frac{m}{m-1}} - 1\right)\left[(z^*(m))^{-1} - 1\right]^{-1} \\ &= 1 + \gamma(m) - \exp\{-(1+o(1))m\log 2\} \\ &= 2 - m^{-1}\log m - m^{-1} + O\left(m^{-2}(\log m)^2\right). \end{aligned}$$

Remark: The approximations

$$\left[1 + \frac{1}{\gamma(m)}\right]^{-1} \text{ and } 1 + \gamma(m)$$

for $z^*(m)$ and $\alpha^*(m)$, respectively, are very easily computed and remarkably accurate, even for small values of m.

Proposition 6.2 *$z^*(m)$ is strictly increasing in $m \geq 2$.*

Proof Observe that

$$z^*(2) = \frac{1}{4} < z^*(3) \doteq 0.30 < z^*(4) \doteq 0.33.$$

Suppose for the sake of contradiction that $z^*(m+1) \leq z^*(m)$ for some $m \geq 4$. Then

$$\begin{aligned}
(m+1)^{-\frac{m+1}{m}} &= \frac{1}{m}\sum_{k=1}^{m-1}[z^*(m+1)]^k + \frac{1}{m}[z^*(m+1)]^m \\
&\leq \frac{1}{m}\sum_{k=1}^{m-1}[z^*(m)]^k + \frac{1}{m}[z^*(m+1)]^m \\
&= \frac{m-1}{m}m^{-\frac{m}{m-1}} + \frac{1}{m}[z^*(m+1)]^m,
\end{aligned}$$

and so by Lemma 6.2 (to follow)

$$2^{-m} > [z^*(m+1)]^m \geq m(m+1)^{-\frac{m+1}{m}} - (m-1)m^{-\frac{m}{m-1}}.$$

But this contradicts Lemma 6.3 (also to follow), and the proposition is proved. ■

Lemma 6.2

$$z^*(m) < 1/2 \textit{ for all } m \geq 2.$$

Proof Consider the defining equation (9) for $z^*(m)$. Since the left side of (9) is strictly increasing in $z \in (0,1)$, it suffices to show that

$$m - 1 < m^{\frac{m}{m-1}}\sum_{k=1}^{m-1}2^{-k} = m^{\frac{m}{m-1}}\left[1 - 2^{-(m-1)}\right],$$

i.e., that

$$m \log m - (m-1)\log(m-1) + (m-1)\log\left[1 - 2^{-(m-1)}\right] > 0.$$

This follows from calculus, since the last expression is strictly increasing in real $m > 1$. ■

Lemma 6.3 *For $m \geq 4$,*

$$m(m+1)^{-\frac{m+1}{m}} - (m-1)m^{-\frac{m}{m-1}} > 2^{-m}. \tag{23}$$

Proof The result is easily checked for $m = 4$, so we assume $m \geq 5$. The left side of (23) equals

$$(m-1)m^{-\frac{m}{m-1}}\left[\exp\{f(m+1) - f(m)\} - 1\right],$$

where

$$f(x) := \log\left[(x-1)x^{-\frac{x}{x-1}}\right] = \log(x-1) - \frac{x}{x-1}\log x$$

for $x > 1$. After some calculation we find

$$f'(x) = (x-1)^{-2}\log x$$

and

$$f''(x) = (x-1)^{-3}x^{-1}[(x-1) - 2x\log x].$$

To treat this further, let $g(x) := (x-1) - 2x\log x$ and note $g(1+) = 0$. We have

$$g'(x) = -1 - 2\log x < -1 < 0$$

for $x > 1$, so g decreases and $g(x) < 0$ for $x > 1$. We conclude that f is concave. Therefore,

$$\begin{aligned} \exp\{f(m+1) - f(m)\} &\geq \exp\{f'(m+1)\} = \exp\{m^{-2}\log(m+1)\} \\ &\geq 1 + m^{-2}\log(m+1), \end{aligned}$$

and so

$$\begin{aligned} m(m+1)^{-\frac{m+1}{m}} - (m-1)m^{-\frac{m}{m-1}} &\geq (m-1)m^{-\frac{m}{m-1}}m^{-2}\log(m+1) \\ &= \left[m^{-2}\log(m+1)\right]\exp\{f(m)\}. \end{aligned}$$

Let

$$\begin{aligned} h(x) &:= \log\left(2^x x^{-2}\log(x+1)\exp\{f(x)\}\right) \\ &= x\log 2 - 2\log x + \log\log(x+1) + f(x); \end{aligned}$$

it suffices to show that $h(x) > 0$ for $x \geq 5$. Now $h(5) \doteq 0.20 > 0$ and

$$\begin{aligned} h'(x) &= \log 2 - 2x^{-1} + \frac{1}{(x+1)\log(x+1)} + (x-1)^{-2}\log x \\ &> \frac{1}{(x+1)\log(x+1)} + (x-1)^{-2}\log x > 0 \end{aligned}$$

for $x \geq 5$. Thus $h(x) > 0$ for $x \geq 5$. ■

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