Interactive Design Space Exploration and Optimization for CAD Models
Supplemental Material

Our approach takes advantage of how the centers of odd B-splines at different levels are distributed across the domain. In this supplemental material, we first describe this distribution making the necessary notations, then prove properties of the basis refinement step. Finally, we use the above results to prove the local point lemma and that locality is preserved when we define y as in Section 5.2 to ensure linear precision. We conclude this supplemental material with an example that illustrates the effects of the proposed refinement method in the resulting approximation function.

S1 Notation

As discussed in Section 5.2, we denote as c_i^j the center of the linear B-splines ϕ_i^j. Let $\mathcal{I}^j$ be the lattice of centers c_i^j of linear B-splines at the refinement level j. Observe that, if $t > j$, then $\mathcal{I}^j \subset \mathcal{I}^t$. Crucially, linear B-splines at refinement level j can be centered only at the lattice points $\mathcal{I}^j$; at each successive level of refinement, this lattice becomes twice as fine (i.e., the distance between adjacent points in the lattice is halved). This is illustrated in Figure S1.

The size of the support of a linear B-spline at level j is denoted by $\mathbf{s}_j$ (Figure S1). We let $\|\cdot\|_d$ denote the length of an element in direction d.

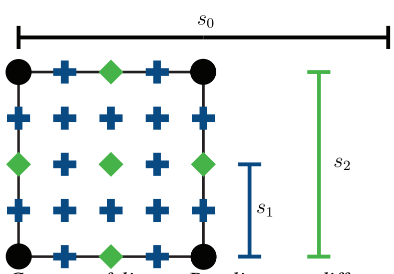

Figure S1: *Centers of linear B-splines at different levels. The set $\mathcal{I}^0$ consists only of the centers denoted by circles, the set $\mathcal{I}^1$ includes the centers denoted by circles and those marked by diamonds, and $\mathcal{I}^2$ includes all the centers denoted in the figure.*

Using the expression in Section 5.3, we say that ϕ_i^j is active if $\exists k$ such that $\alpha_k^{i,j} \neq 0$.

Using the refinement relations, we say that a B-spline ϕ_n^{j+1} is a child of ϕ_i^j (equivalently, that ϕ_i^j is a parent of ϕ_n^{j+1}) if ϕ_n^{j+1} results from the refinement of ϕ_i^j, i.e., if the refinement coefficient a_{in}^{j+1} (Equation 4) is not zero. From Figure 6, it is clear that one function can have multiple children and multiple parents and a function will have a single parent if and only if they have the same center.

S2 Properties of Step 2

Remark S1. For every active linear B-spline ϕ_i^j:

$$\mathbf{s}_j \leq 2\|e_l\|_d \quad \forall e_l, e_l \cup S(\phi_i^j) \neq \emptyset. \tag{S1}$$

Proof. In the initial configuration, defined in Section 4.2, we have a single element e_0 and a set of linear B-splines ϕ_i^0 at the coarsest level centered at corners of this element such that $\mathbf{s}_0 = 2\|e_0\|_d$ in all directions d.

In each iteration, when an element e_l is split, we refine all linear B-splines ϕ_i^j which overlap that split element and have $\mathbf{s}_j > \|e_l\|_d$. Therefore the above statement follows from an inductive argument. □

Remark S2. If ϕ_i^j is refined, then it has no active ancestors

Proof. In step 2, a linear B-spline ϕ_i^j is refined if $\mathbf{s}_j > \|e_l\|_d$. If it had an active parent, ϕ_n^{j-1}, then its support would be twice as large and therefore $\mathbf{s}_{j-1} > 2\|e_l\|_d$, violating the property proved in Remark S1. Any previous ancestor would have an even larger support, which concludes the proof. □

These remarks will be used in the following proofs of locality and we can also conclude from them that at each iteration step 2 needs to perform at most one level of refinement.

S3 Local Point Lemma

In this section we will prove that if c_i^j is the center of an active linear B-spline ϕ_i^j, then c_i^j is a local point of ϕ_i^j.

Remark S3. $S(\phi_i^j)$ overlaps with at most two elements in a given direction and if it overlap with two elements then c_i^j is at the boundary between these elements.

Proof. Since the locations of the centers of the linear B-splines c_i^j are fixed and given by $\mathcal{I}^j$, it is clear from Figure S1 that ϕ_i^j that overlaps e_l will overlap another element e_k if and only if e_l and e_k are adjacent and the center c_i^j lies on the boundary between e_l and e_k.

Since element refinement involves splitting an element halfway and basis functions are refined to guarantee $\|e_l\|_d \leq 2\mathbf{s}_j$ for all active ϕ_i^j which overlaps e_l, this property is preserved after every iteration of our algorithm. □

This remark implies that if $S(\phi_i^j)$ overlaps with e_l, then $e_l \in \mathcal{N}(c_i^j)$. From this we conclude that $S(\phi_i^j) \subset \mathcal{N}(c_i^j)$, and therefore c_i^j is a local point of ϕ_i^j from the definition (recall Section 4).

Since we have shown that the center c_i^j is a local point, this concludes the proof of the local point lemma.

S4 Locality proof with Linear Precision

If ϕ_i^j is active, let y_i^j be, as defined as in Section 5.2,

$$y_i^j = \sum_k \alpha_k^{i,j} x_k / \alpha_i^j, \quad \text{where} \quad \alpha_i^j = \sum_k \alpha_k^{i,j} \tag{S2}$$

In this section we will show that if ϕ_i^j is active, then then y_i^j is a local point of ϕ_i^j.

We will use an inductive argument. From the proof given in the previous section, in the initial configuration when $y_i^0 = c_i^0$, y_i^0 is a local point. We will show that this property is preserved at every iteration of our algorithm, i.e., it is preserved when basis functions are refined when elements are split.

S4.1 Preservation over Basis Refinement

When a B-spline ϕ_i^j is refined it becomes inactive and the the coefficients $\alpha_k^{j+1,n}$ of its children ϕ_n^{j+1} are updated. This results in updating the positions y_n^{j+1} to $\bar{y}_n^{j+1}$. To show that the above property is preserved over basis refinement, it is sufficient to prove that $\bar{y}_n^{j+1}$ is a local point of ϕ_n^{j+1}.

Let us first consider the case when the ϕ_n^{j+1} was inactive before refinement of ϕ_i^j. Let a_{in}^{j+1} be the refinement coefficient given by Equation 4. In this scenario, the refinement of ϕ_i^j updates the coefficients $\alpha_k^{n,j+1}$ as follows (Equation 8):

$$\bar{\alpha}_k^{n,j+1} = a_{in}^{j+1} \alpha_k^{i,j}, \quad \forall k. \tag{S3}$$

From the definition of y (Equation S2) that

$$\bar{y}_n^{j+1} = \frac{\cancel{a_{in}^{j+1}} \sum_k \alpha_k^{i,j} x_k}{\cancel{a_{in}^{j+1}} \sum_k \alpha_k^{i,j}} = y_i^j. \tag{S4}$$

From the refinement property $S(\phi_n^{j+1}) \subset S(\phi_i^j)$, the induction assumption $S(\phi_i^j) \subset \mathcal{N}(y_i^j)$, and the above result $\bar{y}_n^{j+1} = y_i^j$ it follows that $S(\phi_n^{j+1}) \subset \mathcal{N}(\bar{y}_n^{j+1})$. Therfore $\bar{y}_n^{j+1}$ is a local point of ϕ_n^{j+1}.

Let us now consider the case when ϕ_n^{j+1} was active before refinement of ϕ_i^j. We make the flowing remarks:

Remark S4. If ϕ_n^{j+1} is an active child of ϕ_i^j, then the updated position of y_n^{j+1}, given by $\bar{y}_n^{j+1}$, can be expressed as a convex combination of y_i^j and y_n^{j+1}.

Proof. Let a_{in}^{j+1} be the refinement coefficient given by Equation 4. The refinement of ϕ_i^j updates the coefficients $\alpha_k^{n,j+1}$ as follows (Equation 8):

$$\bar{\alpha}_k^{n,j+1} = \alpha_k^{n,j+1} + a_{in}^{j+1} \alpha_k^{i,j}, \quad \forall k. \tag{S5}$$

Therefore, if y is defined as in Equation S2, then

$$\bar{y}_n^{j+1} = \frac{\alpha_n^{j+1} y_n^{j+1} + a_{in}^{j+1} \alpha_i^j y_i^j}{\alpha_n^{j+1} + a_{in}^{j+1} \alpha_i^j}. \tag{S6}$$

□

Remark S5. If ϕ_n^{j+1} is active, $\mathcal{N}(y_i^j) \cap \mathcal{N}(y_n^{j+1}) \subset \mathcal{N}(\bar{y}_n^{j+1})$

Proof. If $e_l \in \mathcal{N}(y_i^j) \cap \mathcal{N}(y_n^{j+1})$, then $y_i^j, y_n^{j+1} \in e_l$. Since $\bar{y}_n^{j+1}$ is a convex combination of y_i^j and y_n^{j+1}, $\bar{y}_n^{j+1} \in e_l$ and therefore $e_l \in \mathcal{N}(\bar{y}_n^{j+1})$. □

From the induction assumption $S(\phi_n^{j+1}) \subset \mathcal{N}(y_n^{j+1})$ and $S(\phi_i^j) \subset \mathcal{N}(y_i^j))$. Since ϕ_n^{j+1} results from the refinement of ϕ_i^j, $S(\phi_n^{j+1}) \subset S(\phi_i^j)$ and therefore $S(\phi_n^{j+1}) \subset \mathcal{N}(y_i^j) \cap \mathcal{N}(y_n^{j+1})$. From Remark S5 we conclude that $S(\phi_n^{j+1}) \subset \mathcal{N}(\bar{y}_n^{j+1})$, showing that the property is preserved during basis refinement.

S4.2 Preservation over Element Refinement

Finally we will show that $S(\phi_i^j) \subset \mathcal{N}(y_i^j)$ is preserved when an element are split. We start by making the following remarks.

Remark S6. Let ϕ_i^j be an active linear B-spline, $j > 0$. If ϕ_n^{j-1} are the parents of ϕ_i^j which have been refined, then

$$\alpha_i^j = \sum_n a_{ni}^j$$

$$y_i^j = \sum_n a_{ni}^j c_n^{j-1} / (\sum_n a_{ni}^j) \tag{S7}$$

where the coefficients a are given by Equation 4.

Before we prove this Remark, we prove the following Remark that stems directly from it.

Remark S7. If ϕ_i^j is active and has no active ancestors, then $\alpha_i^j = 1$ and $y_i^j = c_i^j$.

Proof. For $j = 0$, this results directly from the initial configuration when all linear B-splines are at the coarsest level and $c_i^0 = y_i^0$ and $\alpha_i^0 = 1$.

For $j > 0$ we will use the result from Remark S6:

On the one dimensional case for linear B-splines, using the values of a from Equation 5, we can write Equation S7 when all parents are refined as

$$\begin{cases} y_{2i}^j = c_i^{j-1} \\ y_{2i+1}^j = \frac{1}{2} c_i^{j-1} + \frac{1}{2} c_{i+1}^{j-1}. \end{cases}$$

From the lattice structure and the symmetry of the a terms around the center, we see that $y_i^j = c_i^j$ when all parents are refined. From Equation 5, $\alpha_i^j = \sum_n a_{ni}^j = 1$, which is a result from the partition of unity property of refinement relations. This result is directly extended in the multi-dimensional case.

□

Proof. [Remark S6] We will prove this property by induction. At the initial configuration when all linear B-splines are at the coarsest level, $c_i^0 = y_i^0$ and $\alpha_i^0 = 1$. At this level, there are no active linear B-splines with $j > 0$ and therefore the Remark S6 holds.

We assume that Remark S6 holds and will show that after an iteration of the refinement algorithm it still holds.

We first show that it still holds after step 2. Let ϕ_i^j be a linear B-spline which whill be refined in this step. Only linear B-splines with no active ancestors can be refined (Remark S2) and therefore ϕ_i^j has no active ancestors. Since we assume that Remark S6 holds, Remark S7 also holds and therefore $y_i^j = c_i^j$ and $\alpha_i^j = 1$. Let ϕ_n^{j+1} be a child of ϕ_i^j.

If ϕ_n^{j+1} is inactive, then

$$\begin{aligned} \bar{\alpha}_n^{j+1} &= a_{in}^{j+1} \alpha_i^j = a_{in}^{j+1} \quad \text{(from Equation S3)} \\ \bar{y}_n^{j+1} &= y_i^j = c_i^j \quad \text{(from Equation S4)} \end{aligned} \tag{S8}$$

and Remark S6 holds.

Otherwise, if ϕ_n^{j+1} is not inactive, then

$$\begin{aligned} \bar{\alpha}_n^{j+1} &= \alpha_n^{j+1} + a_{in}^{j+1} \quad \text{(from Equation S5)} \\ \bar{y}_n^{j+1} &= \frac{\alpha_n^{j+1} y_n^{j+1} + a_{in}^{j+1} c_i^j}{\alpha_n^{j+1} + a_{in}^{j+1}} \quad \text{(from Equation S6).} \end{aligned} \tag{S9}$$

From the induction assumption $\alpha_n^{j+1} y_n^{j+1} = \sum_m a_{im}^{i+1} c_m^j$ and $\alpha_n^{j+1} = \sum_m a_{im}^i$, for m indexing all parents ϕ_m^{j+1} other than ϕ_n^{j+1} that have been refined. Therefore

$$\bar{\alpha}_n^1 = \sum_m a_{im}^i + a_{in}^{j+1}$$
$$\bar{y}_n^{j+1} = \frac{\sum_m a_{im}^{i+1} c_m^j + a_{in}^{j+1} c_t^j}{\sum_m a_{im}^i + a_{in}^{j+1}}$$

and Remark S6 holds. From this we conclude that the property is preserved after step 2.

In step 3, though the values $\alpha_k^{i,j}$ are updated, the values α_i^j do not change (Equation 10), concluding the proof.

□

We will now show that $S(\phi_i^j) \subset \mathcal{N}(y_i^j)$ is preserved when e_l is split into e_{lA} and e_{lB} across direction d. It is sufficient to validate the statement on the active linear B-splines ϕ_i^j that overlap e_l. From the induction assumption y_i^j is a local point of e_l. We must then prove that if $S(\phi_i^j)$ overlaps with e_{lA} (and/or e_{lB}), then y_i^j is a local point of e_{lA} (and/or e_{lB}). We will proceed to prove this considering the two possible cases: $S(\phi_i^j)$ overlaps with only one element (e_{lA} or e_{lB}) and $S(\phi_i^j)$ overlaps with both elements (e_{lA} and e_{lB}).

Case 1: ϕ_i^j overlaps one element Consider a linear B-spline ϕ_i^j that overlaps with e_l. From the induction assumption ($S(\phi_i^j) \in \mathcal{N}(y_i^j)$) $y_i^j \in e_l$. First, let us consider the case when ϕ_i^j overlaps only one of the elements that result from the split. Without loss of generality, we assume $S(\phi_i^j) \cap e_{lB} = \emptyset$. If $y_i^j \notin e_{lA}$, then this element refinement would make y_i^j no longer a local point. In what follows we will show that this is never the case, i.e., if $S(\phi_i^j) \cap e_{lB} = \emptyset$, then $y_i^j \in e_{lA}$, from which we can conclude that $S(\phi_i^j) \in \mathcal{N}(y_i^j)$.

Remark S8. If ϕ_i^j is active, then $y_i^j \in \overline{S(\phi_i^j)}$, where $\overline{S(\phi_i^j)}$ is the closure of $S(\phi_i^j)$.

Proof. This property holds in the initial configuration when $y_i^0 = c_i^0$. We will show that this property is preserved during basis refinement (step 2) and conclude the proof by induction. As in the previous paragraph, we will look at the updated positions $\bar{y}_n^{j+1}$ when ϕ_n^{j+1} results from the refinement of ϕ_i^j.

From Remark S2 and Remark S7, if ϕ_i^j is refined, then $y_i^j = c_i^j$. The refinement relations guarantee that for linear B-splines $c_i^j \in \overline{S(\phi_n^{j+1})}$ (see Figure 6) from which we conclude $y_i^j \in \overline{S(\phi_n^{j+1})}$.

If y_n^{j+1} is not active $\bar{y}_n^{j+1} = y_i^j$ (Equation S4) and therefore $\bar{y}_n^{j+1} \in \overline{S(\phi_n^{j+1})}$.

Otherwise, if y_n^{j+1} is active, $y_n^{j+1} \in \overline{S(\phi_n^{j+1})}$ from the induction assumption. Remark S4 allows us to express $\bar{y}_n^{j+1}$ and a convex combination of y_n^{j+1} and y_i^j (both in $\overline{S(\phi_n^{j+1})}$) from which we conclude that $\bar{y}_n^{j+1} \in \overline{S(\phi_n^{j+1})}$.

□

From the induction assumptions ($y_i^j \in e_l$) and Remark S8, $y_i^j \in e_l \cap \overline{S(\phi_i^j)}$. From the assumption $S(\phi_i^j) \cap e_{lB} = \emptyset$, $\overline{S(\phi_i^j)} \cap e_{lB} \subset \partial e_{lB}$, where ∂e_{lB} is the boundary of e_{lB}. From this we conclude that $y_i^j \in e_{lA} \cup \partial e_{lB}$. Since $S(\phi_i^j)$ is a K-dimensional cuboid and e_{lA} and e_{lB} are adjacent K-dimensional cuboids we conclude from $S(\phi_i^j) \cup e_{lA} \neq \emptyset$ that $\overline{S(\phi_i^j)} \cup \partial e_{lb} \setminus e_{lA} = \emptyset$. Therefore, $y_i^j \in e_{lA}$ and therefore $S(\phi_i^j) \subset \mathcal{N}(y_i^j)$ after element refinement.

Case 2: ϕ_i^j overlaps both elements Now let us consider the case when ϕ_i^j overlaps both elements that result from the split, i.e., $S(\phi_i^j) \cap e_{lA} \neq \emptyset$ and $S(\phi_i^j) \cap e_{lB} \neq \emptyset$. As previously discussed, $y_i^j \in e_l$. In what follows we will prove that y_i^j is also on the boundary between e_{lA} and e_{lB}, which will allow us to conclude that after the split the property $S(\phi_i^j) \in \mathcal{N}(y_i^j)$ is preserved.

Let us first consider the case when there are no active ancestors and therefore $\sum_k \alpha_k^{i,j} = 1$ and $y_i^j = c_i^j$ (Remark S7). From Remark S3, if ϕ_i^j overlaps both elements, then c_i^j is on the boundary between e_{lA} and e_{lB} and therefore $S(\phi_i^j) \in \mathcal{N}(c_i^j)$. Since in this case $y_i^j = c_i^j$, the property is preserved.

Let us now consider the case when there are active ancestors. Let $\mathcal{B}_i^j$ be the set of active ancestors.

Remark S9. $\mathcal{B}_i^j$ describes the set of active linear B-splines $\phi_n^m \neq \phi_i^j$ that do not vanish at c_i^j

Proof. The B-splines $\phi_n^m \neq \phi_i^j$ that do not vanish at c_i^j are its ancestors (all of them) or descendents that are centered at c_i^j (see Figure 6). All of the active ancestors are in $\mathcal{B}_i^j$. Since ϕ_i^j is active, there can be no active linear B-spline whose only ancestor is ϕ_i^j. Since the descendents that do not vanish at c_i^j have ϕ_i^j as a unique ancestor, none of them are active. □

From Remark S9 and the assumption that a partition of unity is guaranteed in all previous iterations, we conclude

$$\sum_k \alpha_k^{i,j} + \sum_{n,m \in \mathcal{B}_i^j} \sum_k \alpha_k^{n,m} \phi_n^m(c_i^j) = 1. \tag{S10}$$

If $S(\phi_i^j)$ overlaps e_{lA} and e_{lB} the same is true for all of its ancestors. Therefore, from Remark S3, they must all be centered on the boundary between e_{lA} and e_{lB} . We conclude that c_n^m is equal to c_i^j in direction d, $c_n^m|_d = c_i^j|_d$. We will use this to show that $y_i^j|_d = c_i^j|_d$.

Let m_0 be the coarsest level in $\mathcal{B}_i^j$. Any function $\phi_n^{m_0}$ must have no active ancestors since those would also be on $\mathcal{B}_i^j$. From Remark S7, $y_n^{m_0} = c_n^{m_0}$ and therefore, $y_n^{m_0}|_d = c_i^j|_d$.

Next, we take the next coarsest level on $\mathcal{B}_i^j$, m_1, and let $\phi_{n_1}^{m_1}$ be any linear B-spline at this level. $\mathcal{B}_{n_1}^{m_1}$ only contains the linear B-splines in $\mathcal{B}_i^j$ at level m_0. As in Equation S10, we can use Remark S9 and the partition of unity assumption to conclude

$$\sum_k \alpha_k^{n_1,m_1} + \sum_n \sum_k \alpha_k^{m_0,n} \phi_n^{m_0}(c_{n_1}^{m_1}) = 1$$

Using the assumption of linear precision on all previous iterations and letting the evaluation function $x_k \mapsto p_k$ be the identity, we conclude (Equation 7) that

$$c_{n_1}^{m_1} = \sum_k \alpha_k^{n_1,m_1} x_k + \sum_n \sum_k \alpha_k^{m_0,n} x_k \phi_n^{m_0}(c_{n_1}^{m_1}).$$

From this we can express $c_{n_1}^{m_1}$ as a convex combination of $y_{n_1}^{m_1}$ and $y_n^{m_0}$.

$$c_{n_1}^{m_1} = \alpha_{n_1}^{m_1} y_{n_1}^{m_1} + \sum_n \phi_n^{m_0}(c_{n_1}^{m_1}) \alpha_{m_0}^n y_{m_0}^n.$$

Since $c_{n_1}^{m_1}|_d = y_n^{m_0}|_d = c_i^j|_d$, it follows that $y_{n_1}^{m_1}|_d = c_i^j|_d$. We can continue this process to the finer levels to achieve that $y_i^j|_d = c_i^j|_d$.

From this we conclude that y_i^j is a local point after an element is split which concludes the proof that y_i^j is a local point.

S5 Example

Let $q_i^j = \sum \alpha_k^{i,j} p_k$, where the coefficients $\alpha_k^{i,j}$ are given by Equation 6. Then, Equation 7 can be expressed as:

$$P(x) = \sum_{i,j} q_i^j \phi_i^j(x). \tag{S11}$$

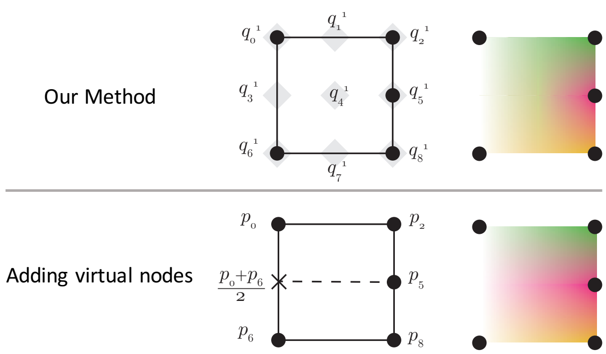

Figure S2: *Contrasting the interpolation solution from our method with the interpolation solution adding virtual nodes. Top row: the interpolation solution from our method. The resulting interpolation is equivalent to a uniform basis function at the coarsest level j such that every sample on the element boundary belongs to $\mathcal{I}^j$ (in this case $j = 1$). The values q_i^j can be computed hierarchically at points for which samples p_i^j do not exist. In this example, q_3^1 is based on the average of the adjacent samples $x_0, x_6 \in \mathcal{I}^0$, and q_4^1 is the average of the four corner samples, also contained in $\mathcal{I}^0$. Bottom row: interpolation solution adding virtual nodes. The color display on the right illustrates how our method restricts the impact of a sample in $\mathcal{I}^j$ to $\mathbf{s}_j$.*

To further illustrate the result of our refinement strategy for high dimensions, we show a two-dimensional example on the top row of Figure S2. Given an element and the samples x_k on its boundary, we can determine the solution of our approximation. We use Equation S11 with uniform basis functions at the coarsest level j at which $x_k \in \mathcal{I}^j, \forall k$. The coefficients q_i^j can be determined at each successive level j as follows. At $j = 0$, since there are guaranteed to be samples at $x_k = x_i^0$, we set $q_i^0 = p_k$. At level $j > 0$, the coefficient q_j^i at a point $x_k = y_i^j$ for which a sample does not exist is given by a multi-linear combination of coefficients $q_i^{(j-1)}$ at the coarser level $j - 1$.

Figure S2 compares our method (depicted in the top row) with the approach of creating virtual nodes and then using bilinear interpolation on each sub-element (shown in the bottom row). The figure highlights the difference in the effect of p_5. In our technique, if j is the coarsest level such that $x_k \in \mathcal{I}^j$, then the influence of the sample x_k is limited to the support of basis functions at level j. As we have shown in this supplemental material, this property is used to prove locality. Therefore, this property gives us the advantage of being able to define a simple refinement algorithm that updates the approximation while constructing a k-d tree.

Notice that the same result on the top row of Figure S2 can be achieved by the following steps: first, use a set of basis functions at the coarsest level, setting $P(x) = \sum_{i=0,2,6,8} \phi_i^0 p_i$; then, use the refinement relations to rewrite this expression as $\sum_{i=0}^{8} \phi_i^1 q_i^1$; finally, replace q_5^1 in this expression, which was originally $(p_2 + p_8)/2$, with p_5. The second advantage of expressing this interpolation as basis functions with refinement relations is that this method can be extended to higher-order basis functions, such as cubic B-splines.