Discovery of Intrinsic Primitives on Triangle Meshes

Supplemental Material

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Proof of Proposition 1

1 Introduction

In this supplement, we supply a full proof of Proposition 1 of the *Discovery of Intrinsic Primitives on Triangle Meshes* paper for the sake of completeness. The proof is based on the max-min characterization of the spectral data of elliptic operators that possess a variational formulation. The AKVF operator is one such operator; and the Laplace-Beltrami operator is another. There are many theorems of this kind in the mathematical literature and the ideas which are used to prove the theorem are fairly well known. Examples can be found in the field of singular perturbation theory for the spectrum of the Laplace operator on dumb-bell surfaces [2] or dumb-bell domains [4, 6]; also in the field of gluing constant mean curvature surfaces [10] and other geometric problems. In particular, the technique of the forthcoming proof is based on [8, Appendix B].

Let Σ be a composite surface which decomposes as a union of "large" surfaces with boundary Ω_i connected by a number of "small" transition regions whose size is governed by a parameter ε, and furthermore that each Ω_i can be viewed as a large open subset of a surface Σ_i, obtained for instance by removing a small ball from Σ_i. We consider the eigenvalue problem for a linear, elliptic partial differential system of equations of the form $P^*P(\omega) = \lambda\omega$ where ω is a sections of a *vector bundle*, and we assume that we can view P either as a first-order partial differential operator over Σ or as such an operator over the disjoint union $\dot{\bigcup}\Sigma_i$. We prove here a Spectral Comparison Theorem that compares the spectral data of P^*P on Σ to the spectral data of P^*P on each Σ_i. This theorem is thus more general than what we need (this level of generality comes at no extra price) and so we show that it applies specifically when P^*P is the AKVF operator.

The Spectral Comparison Theorem shows that up to a threshold $M(\varepsilon)$ satisfying $\lim_{\varepsilon\to 0} M(\varepsilon) = \infty$, the eigenvalues of Σ_i and the eigenvalues of Σ below this threshold are close; and that the corresponding eigenspaces are close as well. The reason for this threshold is that the transition regions themselves begin to contribute to the spectrum of P^*P when the eigenvalues we are considering are sufficiently large. But because of the geometric constraints we'll impose on the size of the transition region, however, we can say that the threshold increases to infinity with ε. All of this can be intuitively interpreted in terms of oscillations. On a composite surface like the one considered here, eigenvectors corresponding to low eigenvalues are akin to low-frequency oscillations that can not "tunnel" through small transition regions. Thus these eigenvectors tend to de-couple and reflect only the geometry of their immediate surroundings. By contrast, eigenvectors corresponding to high eigenvalues are akin to high-frequency oscillations that are unimpeded by the small size of the transition regions and are truly global in nature. The smallest eigenvalue that allows for this effect, and the smallest eigenvalue whose eigenvector has non-negligible contribution from a transition region, thus depends directly on the size of the transition regions.

The remainder of this document is arranged as follows. In Section 2 we state carefully our assumptions about the surfaces whose spectra we'll compare. In Section 3, we state and prove the general Spectral Comparison Theorem. In Section 4 we state precisely how we form composite surfaces from simpler parts, and verify that the assumptions of Section 2 hold for these surfaces. Finally, in Section 5 we remove a simplifying assumption that was made in Section 3.

2 Assumptions

Let M_1, M_2 be two compact surfaces without boundary and with metrics g_1, g_2. Suppose that we have second-order, linear, uniformly elliptic, partial differential operators $\mathcal{L}_1$, $\mathcal{L}_2$ with smooth coefficients defined on M_1, M_2, respectively. We will assume for simplicity that each $\mathcal{L}_i$ is a *scalar* operator, meaning that $\mathcal{L}_i$ acts on functions defined on M_i. The analysis that follows holds for vector operators acting on sections of a *vector bundle* over M_i (such as one-forms) provided that a suitable generalization of the coercivity condition (defined below) holds. We will address this point again later. We now state the main assumptions relevant to the comparison of spectral data of these operators. In what follows, we use $\|\cdot\|_2$ for the L^2 norm and $\|\cdot\|_\infty$ for the L^∞ norm.

I. Surfaces. We'll assume that there exists $c > 0$ so that the following hold. Each M_i is "non-collapsed" i.e. there is a point where the exponential map is injective onto a ball of radius at least $1/c$ and where the curvature is bounded above by c. Furthermore, the volume of each M_i is bounded above by c.

II. Operators. We will assume that each $\mathcal{L}_i$ is self-adjoint with respect to the L^2-inner product of M_i associated to g_i. Furthermore we'll assume that we can write

$$\int_{M_i} \mathcal{L}_i(u) \cdot v \,\mathrm{dVol}_{M_i} = \mathcal{A}_i(u, v) = \int_{M_i} u \cdot \mathcal{L}_i(v) \,\mathrm{dVol}_{M_i} \qquad \forall\, u, v \in H^1(M_i)$$

where

$$\mathcal{A}_i(u, v) := \int_{M_i} \Big(\sum\nolimits_{s,t} A^{(i)}_{st} \nabla^s u \nabla^s v + a^{(i)} uv \Big) \mathrm{dVol}_{M_i}$$

and $A^{(i)}_{st}$ and $a^{(i)}$ are C^∞ functions. We are stating that each $\mathcal{L}_i$ is a *geometric* operator and is in *divergence form*. Moreover, we'll assume that there exist positive constants C_1, C_2 so that

$$C_1 \Big(\|\nabla u\|_2^2 - \|u\|_2^2 \Big) \leq \mathcal{A}_i(u, u) \leq C_2 \Big(\|\nabla u\|_2^2 + \|u\|_2^2 \Big) \quad \forall\, u \in H^1(M_i)\,.$$

This makes $\mathcal{A}_i$ a bounded, symmetric and now *coercive* quadratic form on $H^1(M_i)$. Note that the Laplace-Beltrami operator of M_i (defined as $\mathcal{L}_i := -g_i^{st} \nabla_s \nabla_t$) satisfies all of the above conditions. Its associated quadratic form is $\mathcal{A}_i(u, v) = \int_{M_i} g_i(\nabla u, \nabla v) \,\mathrm{dVol}_{M_i}$.

III. Variational Problems. Next, we describe how we intend to compare the quadratic forms $\mathcal{A}_1$ and $\mathcal{A}_2$ acting on functions spaces over M_1 and M_2. We will assume that for every sufficiently small $\varepsilon > 0$ there are linear maps $F_1 : C^\infty(M_1) \to C^\infty(M_2)$ and $F_2 : C^\infty(M_2) \to C^\infty(M_1)$ such that the following properties hold. We use the notation $i' = (2, 1)$ when $i = (1, 2)$.

1. Each F_i is bounded in the L^∞ norm. In other words, there is a constant C so that

$$\|F_i(u)\|_\infty \leq C \|u\|_\infty \quad \forall\, u \in C^\infty(M_i)\,.$$

2. The maps F_1 and F_2 are almost inverses of each other. In other words,

$$\|u - F_{i'} \circ F_i(u)\|_2 \leq \varepsilon \|u\|_\infty \quad \forall\, u \in C^\infty(M_i)\,.$$

3. Each F_i almost preserves the L^2-inner product. In other words,

$$\big|\langle u, v\rangle_{L^2(M_i)} - \langle F_i(u), F_i(v)\rangle_{L^2(M_{i'})}\big| \leq \varepsilon \|u\|_\infty \|v\|_\infty \quad \forall\, u, v \in C^\infty(M_i)\,.$$

4. Each F_i almost preserves the quadratic form $\mathcal{A}_i$ in the following sense:

$$\mathcal{A}_{i'}(F_i(u), F_i(u)) \leq \varepsilon\big(\|u\|_\infty^2 + \|\nabla u\|_2^2\big) + \mathcal{A}_i(u,u) \quad \forall\, u \in C^\infty(M_i)\,.$$

IV. A priori estimates. Finally, we state the technical analytical assumption about the solutions of the equation $\mathcal{L}_i(u) = \lambda u$ for each $i = 1, 2$ that must hold in order to compare the spectral data of $\mathcal{L}_1$ and $\mathcal{L}_2$. We will assume that we have the estimate

$$\|u\|_\infty \leq C\|u\|_2$$

for any solution of $\mathcal{L}_i(u) = \lambda u$ where C depends only on M_i and λ, but not on the value of ε appearing in Assumptions (III.1) – (III.4) above.

Remark: We actually have another a priori estimate that follows from the coercivity condition in Assumption (II). That is, if $\mathcal{L}_i(u) = \lambda u$ then $\|\nabla u\|_2^2 \leq C_1^{-1}\mathcal{A}_i(u,u) + \|u\|_2^2 = C\|u\|_2^2$ where C also depends only M_i and λ. We will make use of this estimate in the sequel as well.

3 Spectral Comparison Theorems

Our spectral comparison theorem for partial differential operators on two different, yet "comparable" surfaces is in the style of Kapouleas [8, Appendix B], though differs from it in several respects. In the remainder of this supplement, let $f_1^{(i)}, f_2^{(i)}, \ldots$ denote the orthonormalized eigenfunctions of $\mathcal{L}_i$ on M_i for $i = 1, 2$ with corresponding eigenvalues $\lambda_1^{(i)} \leq \lambda_2^{(i)} \leq \cdots$ counted with multiplicity. Also, let $P_{k,\delta}^{(i)} : L^2(M_i) \to L^2(M_i)$ be the L^2-orthogonal projection operator onto the subspace spanned by the eigenfunctions $f_\ell^{(i)}$ with ℓ such that $|\lambda_\ell^{(i)} - \lambda_k^{(i)}| < \delta$. Also, let $\tilde{f}_k^{(1)} := F_2(f_k^{(2)})$ and $\tilde{f}_k^{(2)} := F_1(f_k^{(1)})$ for each k be the *proxy eigenfunctions*. In the calculations that follow, C always denotes a constant (not always the same one) depending only on c and k.

Proposition 1. *Let the assumptions (I) – (IV) be in force. The spectral data of $\mathcal{L}_1$, $\mathcal{L}_2$ has the following properties. If ε is sufficiently small, then for every k there is a constant C depending only on c and k so that*

$$|\lambda_k^{(1)} - \lambda_k^{(2)}| \leq C\varepsilon\,.$$

Proof. We use the variational characterization of the spectral data of $\mathcal{L}_i$. That is, if we let $\mathcal{S}_k$ denote the set of k-dimensional subspaces of $H^1(M_i)$, then

$$\lambda_k^{(i)} = \inf_{S \in \mathcal{S}_k} \left(\sup_{u \neq 0 \text{ and } u \in S_k} \frac{\mathcal{A}_i(u,u)}{\|u\|_{L^2(M_i)}^2} \right).$$

This infimum is attained if $S_k = span\{f_1^{(i)}, \ldots, f_k^{(i)}\}$. The supremum is then attained if $u = f_k^{(i)}$. By applying this characterization with $i = 2$ we can therefore say

$$\lambda_k^{(2)} \leq \sup_{\tilde{f} \in S_k} \frac{\mathcal{A}_2(\tilde{f}, \tilde{f})}{\|\tilde{f}\|_2^2}$$

where $S_k = span\{\tilde{f}_1^{(2)}, \ldots, \tilde{f}_k^{(2)}\}$. Let $\tilde{f} := \sum_{i=1}^k b_i \tilde{f}_i^{(2)} = F_1(f)$ where $f := \sum_{i=1}^k b_i f_i^{(1)}$ for some $b_i \in \mathbb{R}$. We obtain an estimate for the denominator by

$$\Big|\|\tilde{f}\|_2^2 - \|b\|^2\Big| \leq \sum_{i,j} |b_i|\,|b_j|\,\Big|\langle F_1(f_i^{(1)}), F_1(f_j^{(1)}\rangle - \delta_{ij}\Big|$$

$$
\begin{aligned}
&\leq \varepsilon \sum_{i,j} |b_i|\,|b_j|\,\|f_i^{(1)}\|_\infty \|f_j^{(1)}\|_\infty \\
&\leq C\varepsilon \|b\|^2
\end{aligned}
$$

using Assumption (III) and (IV). (We are also implicitly using Assumption I since this guarantees that the eigenvalues of $\mathcal{L}_1$, and hence the constant appearing in the supremum norm bound of the eigenfunctions, are bounded below by a constant depending only on c and k). Consequently $\|\tilde{f}\|_2^2 \geq (1 - C\varepsilon)\|b\|^2$. Next, we obtain an estimate for the numerator by

$$
\begin{aligned}
\mathcal{A}_2(\tilde{f}, \tilde{f}) &\leq \mathcal{A}_1(f, f) + \varepsilon\big(\|f\|_\infty^2 + \|\nabla f\|_2^2\big) \\
&\leq \sum_{i,j} b_i b_j \mathcal{A}_1\big(f_i^{(1)}, f_j^{(1)}\big) + \varepsilon \sum_{i,j} |b_i||b_j| \big(\|f_i^{(1)}\|_\infty \|f_j^{(1)}\|_\infty + \|\nabla f_i^{(1)}\|_2 \|\nabla f_j^{(1)}\|_2\big) \\
&\leq \sum_i b_i^2 \lambda_i^{(1)} + C\varepsilon \|b\|^2 \\
&\leq \|b\|^2 \big(\lambda_k^{(1)} + C\varepsilon\big)\,.
\end{aligned}
$$

Here we have again used Assumption (IV) to deal with $\|f_i^{(1)}\|_\infty$ and $\|\nabla f_i^{(1)}\|_2$. Now, the two estimates we have just found together yield

$$
\frac{\mathcal{A}_2(f,f)}{\|f\|_2^2} \leq \frac{\lambda_k^{(1)} + C\varepsilon}{1 - C\varepsilon} \leq \lambda_k^{(1)} + C\varepsilon
$$

so that $\lambda_k^{(2)} - \lambda_k^{(1)} \leq C\varepsilon$. Reversing the roles of M_1 and M_2 in the preceding calculations yields the opposite inequality, thereby establishing the desired result. □

Proposition 2. *Let the assumptions about M_1 and M_2 be in force and let $\delta > 0$ be given. If ε is sufficiently small, then for every k there is a constant C depending only on c, k and δ so that*

$$
\big\|\tilde{f}_k^{(i)} - P_{k,\delta}^{(i)}\big(\tilde{f}_k^{(i)}\big)\big\|_2 \leq C\sqrt{\varepsilon}\,.
$$

Proof. We use the alternate variational characterization of the spectral data of $\mathcal{L}_i$. That is, with the notation from the previous proposition in force, we have

$$
\lambda_k^{(i)} = \sup_{S \in \mathcal{S}_{k-1}} \left(\inf_{u \neq 0 \text{ and } u \perp S_{k-1}} \frac{\mathcal{A}_i(u,u)}{\|u\|_{L^2(M_i)}^2} \right).
$$

This supremum is attained if $S = span\{f_1^{(i)}, \ldots, f_{k-1}^{(i)}\}$. The infimum is then attained if $u = f_k^{(i)}$.

Now let $\ell_1 < \ell_2$ be, respectively, the largest integer less than k and the smallest integer greater than k so that $|\lambda_\ell^{(i)} - \lambda_k^{(i)}| > \delta$. Write $\tilde{f}_k^{(i)} = f^\circ + f' + f''$ where

$$
f^\circ := P_{k,\delta}^{(i)}(\tilde{f}_k^{(i)}) \quad \text{and} \quad f' \in span\{f_j^{(i)} : j = 1, \ldots, \ell_1\} \quad \text{and} \quad f'' \in span\{f_j^{(i)} : j \geq \ell_2\}\,.
$$

Our goal is to show $\|f'\|_2 + \|f''\|_2 \leq C\sqrt{\varepsilon}$. We proceed by induction. We must first prove the result for $i = 1, 2$ and $k = 1$. We use the notation $i' = (2,1)$ if $i = (1,2)$ as usual. In this case, we have $\tilde{f}_1^{(i)} = P_{1,\delta}^{(i)}(\tilde{f}_1^{(i)}) + f''$ with $f'' \perp f_1^{(i)}$. Our variational characterization implies

$$
\begin{aligned}
(\lambda_1^{(i)} + \delta)\|f''\|_2^2 &\leq \mathcal{A}_i(f'', f'') \\
&= \mathcal{A}_i(\tilde{f}_1^{(i)} - f^\circ, \tilde{f}_1^{(i)} - f^\circ) \\
&= \mathcal{A}_i(\tilde{f}_1^{(i)}, \tilde{f}_1^{(i)}) - \mathcal{A}_i(f^\circ, f^\circ)
\end{aligned}
$$

where we have used the symmetry of $\mathcal{A}_i$ and the fact that $\mathcal{A}_i(f^\circ, f^\circ) = \mathcal{A}_i(\tilde{f}_1^{(i)}, f^\circ)$ since f° and f'' belong to orthogonal eigenspaces of $\mathcal{L}_i$ and thus $\mathcal{A}_i(f^\circ, f'') = 0$. Now

$$
\mathcal{A}_i(f^\circ, f^\circ) = \langle \tilde{f}_1^{(i)}, f_1^{(i)} \rangle_{L^2}^2 \mathcal{A}_1(f_1^{(i)}, f_1^{(i)}) = \lambda_1^{(i)} \|f^\circ\|_2^2\,.
$$

Furthermore by our assumptions about the variational problems,

$$\mathcal{A}_i(\tilde{f}_1^{(i)}, \tilde{f}_1^{(i)}) \leq \mathcal{A}_{i'}(f_1^{(i')}, f_1^{(i')}) + C\varepsilon \leq \lambda_1^{(i')} + C\varepsilon\,.$$

Therefore combining these two estimates yields

$$(\lambda_1^{(i)} + \delta)\|f''\|_2^2 \leq \lambda_1^{(i')} - \lambda_1^{(i)}\|f^\circ\|_2^2 + C\varepsilon\,.$$

Finally, observe that $\|f^\circ\|_2^2 + \|f''\|_2^2 = \|\tilde{f}_1^{(i)}\|_2^2 \geq (1 - C\varepsilon)$ by our assumptions about the variational problem and the fact that $\|f_1^{(i)}\| = 1$. Hence

$$(\lambda_1^{(i)} + \delta)\|f''\|_2^2 \leq \lambda_1^{(i')} - \lambda_1^{(i)}\big(1 - C\varepsilon - \|f''\|_2^2\big) + C\varepsilon$$

from which we deduce

$$\delta\|f''\|_2^2 \leq (\lambda_1^{(i)} - \lambda_1^{(i')}) + C\varepsilon(1 + \lambda_1^{(i)})\,.$$

By Proposition 1 the inequality $\|f''\| \leq C\sqrt{\varepsilon}$ follows once ε is sufficiently small.

To continue with the induction argument, we now assume that the result is true for $i = 1, 2$ and $k = 1, \ldots, n-1$ and we must prove the result for $k = n$. Consider first the component f' in the orthogonal decomposition of $\tilde{f}_n^{(i)}$. This is

$$f' = \sum_{j=1}^{\ell_1} \langle \tilde{f}_n^{(i)}, f_j^{(i)} \rangle_{L^2} f_j^{(i)} = \sum_{j=1}^{\ell_1} \langle f_n^{(i')}, \tilde{f}_j^{(i')} \rangle_{L^2} f_j^{(i)} + h$$

where $\|h\|_2 \leq C\varepsilon$ by the assumptions about the variational problem. But now we can use the induction hypothesis applied to $\tilde{f}_j^{(i')}$, which we write as $\tilde{f}_j^{(i')} = f_j^\circ + f_j' + f_j''$ to conclude

$$\langle f_n^{(i')}, \tilde{f}_j^{(i')} \rangle_{L^2} \leq \|f_j''\| \leq C\sqrt{\varepsilon}$$

and thus that $\|f'\|_2 \leq C\sqrt{\varepsilon}$. We now estimate $\|f''\|_2 \leq C\sqrt{\varepsilon}$ by performing the same kind of calculation as in the $k = 1$ part of the induction argument, except with additional f' terms in several places. But given what we have just computed, these terms all contribute an amount $\mathcal{O}(\sqrt{\varepsilon})$ to the inequality for $\|f''\|_2$. Thus we obtain the desired estimate for $\|f''\|_2$. $\square$

4 Spectral Comparison of Composite Surfaces

We will construct a composite surface M_1 consisting of two "large" surfaces connected by a small neck contained in a ball of radius $\alpha \ll 1$. We will show that Assumptions (I) – (IV) hold for M_1 and for a model surface M_2 consisting of the disjoint union of the two large surfaces, provided the eigenvalue λ is smaller than a threshold of size $o(\alpha)$. Consequently, the spectral comparison results contained in Propositions 1 and 2 hold for these surfaces when α is sufficiently small, provided the eigenvalue λ is not too large.

4.1 Construction of a Composite Surface

Let $M_2 := \Sigma_+ \dot{\cup} \Sigma_-$ be the disjoint union of two compact, non-collapsed surfaces $\Sigma_\pm$ carrying a differential operator $\mathcal{L}_2$ of the kind described in the previous section. We will suppose that M_1 is constructed by "gluing together" $\Sigma_\pm$ at points $p_\pm \in \Sigma_\pm$ by means of a small, cylindrical neck in the following manner. For simplicity, choose $p_\pm$ so that the principal curvature vectors at $p_\pm$ point into $\Sigma_\pm$ (that is, $\Sigma_\pm$ looks "convex" at $p_\pm$.

The specific details of the construction of M_1 don't really matter as far as the spectral comparison theorem is concerned. This is because it can be shown that a spectral comparison theorem also holds when M_1 and M_2 are small C^∞ perturbations of each other. But since we must be concrete in order to verify the various assumptions that must be satisfied for Propostion 1 and Propostion 2 to hold, we proceed as follows. First, define $Cyl_r := \{y^2 + z^2 < r^2\}$ and $H_\theta := \{(x, y, z) : |x| \leq \theta\alpha_1\}$ and $H := \{(x, y, z) : |x| \leq \sqrt{\alpha_1}\}$.

- Let α_1, α_2 be small numbers (to be further specified below).
- By means of translations and rotations, place Σ_+ and Σ_- in the $x < 0$ and $x > 0$ half-spaces, respectively, so that $p_\pm = (\pm\alpha_1, 0, 0)$ and $T_{p_\pm}\Sigma_\pm = span\{(0,1,0),(0,0,1)\}$.
- Delete all points of $\Sigma_\pm \cap Cyl_{2\alpha_2}$ in the same connected component as $p_\pm$.
- Replace the points in $\Sigma_\pm \cap \left[Cyl_{3\alpha_2} \setminus Cyl_{2\alpha_2}\right]$ by a collar that interpolates smoothly between $\Sigma_\pm \cap \partial Cyl_{3\alpha_2}$ and surfaces of revolution about the x-axis making contact with the circles $\partial Cyl_{2\alpha_2} \cap \partial H_{1/2}$ at an angle of $\pi/4$. Assume the surface begins deviating from $\Sigma_\pm$ within H_1.
- Construct a surface of revolution about the x-axis that transitions smoothly between $Cyl_{\alpha_2} \cap H_{1/4}$ and a surface that makes contact with the circles $\partial Cyl_{2\alpha_2} \cap \partial H_{1/2}$ at an angle of $\pi/4$.
- Let M_1 be the union of the cylindrical neck together with Σ_+ and Σ_- where the appropriate points have been removed or replaced.

The surface M_1 we have just constructed is pictured in Figure 1.

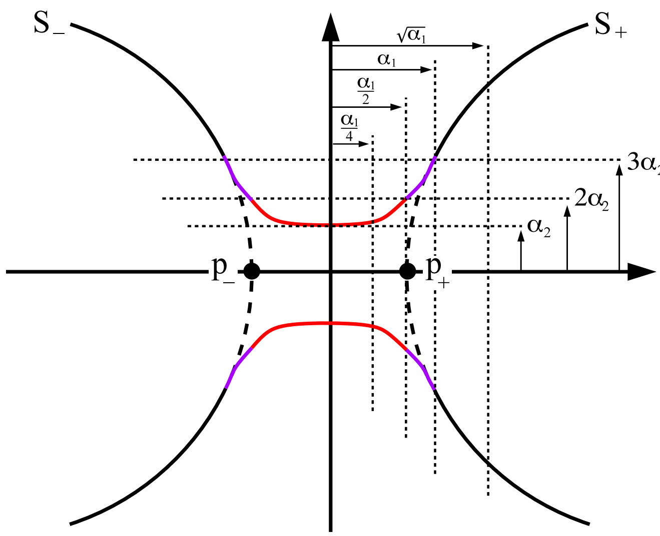

Figure 1: A schematic of the connected sum of Σ_+ and Σ_-.

4.2 Verification of the Assumptions

We'll assume that the operator $\mathcal{L}_2$ is a geometric operator acting on functions $f : M_2 \to \mathbb{R}$ that is built in some natural way from the covariant derivatives of f up to order 2. Furthermore, we'll assume that $\mathcal{L}_2$ has a natural, smooth extension to M_1 and that $\mathcal{L}_1 = \mathcal{L}_2$ in the region where M_1 coincides with M_2. Also, we'll assume that the coefficients of $\mathcal{L}_2$ in are bounded independently of neck size in the L^∞ norm induced by the Riemannian metric g_2. Finally, we'll assume that $\mathcal{L}_1$ remains in divergence form so that the associated quadratic form $\mathcal{A}_1$ can be defined such that it equals $\mathcal{A}_2$ when acting on functions supported on the part of M_2 that coincides with M_1.

The next step is to carefully define the comparison mappings F_1 and F_2, beginning with the cut-off function. Let $\mathring{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth, monotone function that equals one on $(-\infty, 1]$ and equals zero on $[2, \infty)$ and let $x(p)$ denote the x-coordinate of a point $p \in \Sigma_\pm$. Define

$$\chi(p) := \mathring{\chi}\left(\frac{\log(|x(p)|)}{\log(\sqrt{\alpha_1})}\right).$$

This function transitions from zero to one in $M_1 \cap \big[H \setminus int(H_0)\big]$ and equals on one $M_1 \cap H$. Note we'll abuse notation and view χ as being defined on either M_1 or M_2 as needed, where the definition on M_2 is simply obtained by pre-composing with nearest point projection from $M_1 \cap [H \setminus H_0]$ to $M_2 \cap [H \setminus H_0]$. Finally, let $F_1 : C^\infty(M_1) \to C^\infty(M_2)$ be given by $F_1(f) := \chi f$ which is then viewed as a function on $\Sigma_+ \dot\cup \Sigma_-$ that equals zero near $p_\pm$. Let $F_2 : C^\infty(M_2) \to C^\infty(M_1)$ be given by $F_2(f) := \chi f\big|_{\Sigma_+} + \chi f\big|_{\Sigma_-}$ we first view $f\big|_{\Sigma_+}$ and $f\big|_{\Sigma_-}$ as functions on the two components of $M_1 \setminus H_0$ before combining them into one function on all of M_1 that vanishes in $M_1 \cap H_0$.

We are now ready to check the various assumptions for M_1 and M_2. The Assumptions (I) and (II) are clear. For Assumption (III) we have four statements to check for each surface.

Assumption III.1. Straightforward since $\|\chi\|_\infty \leq 1$.

Assumption III.2. Start with $u \in C^\infty(M_1)$. Then

$$\|u - F_2 \circ F_1(u)\|_2^2 = \int_{M_1} (1-\chi^2)u^2 \leq C\, Vol(supp(1-\chi)) \cdot \|u\|_\infty^2 \leq \varepsilon^2 \|u\|_\infty^2$$

if α_1 and α_2 are sufficiently small. The inequality for $u \in C^\infty(M_2)$ is similar.

Assumption III.3. Another similar computation.

Assumption III.4. This is a more involved computation since we must estimate norms of $\nabla\chi$. First, we show that $\|\nabla\chi\|_2$ can be made as small as desired by choosing the parameters of our construction appropriately. To this end, note that

$$\int_{M_2} \|\nabla\chi\|^2 \leq C \int_0^{2\pi} \int_{\alpha_1}^{\sqrt{\alpha_1}} \|\nabla\chi(r,\theta)\|^2\, r\mathrm{d}r\mathrm{d}\theta$$

once we choose geodesic polar coordinates in the small annular region containing the support of $\nabla\chi$ and bound the metric there by the Euclidean metric. Thus we obtain

$$\int_{M_2} \|\nabla\chi\|^2 \leq C \int_{\alpha_1}^{\sqrt{\alpha_1}} \frac{1}{r(\log(\sqrt{\alpha_1}))^2}\mathrm{d}r = \frac{C\log(r)}{(\log(\sqrt{\alpha_1}))^2}\Bigg|_{\alpha_1}^{\sqrt{\alpha_1}} = \frac{C}{|\log{(\alpha_1)}|}$$

and this can be made as small as desired by choosing α_1 sufficiently small. Now let $\varepsilon > 0$ be given and compute

$$\begin{aligned}
\mathcal{A}_2(F_1(u), F_1(u)) &= \int_{M_2} \Big(a_{ij}^{(2)} \nabla^i(\chi u)\nabla^j(\chi u) + a^{(2)}\chi^2 u^2 \Big) \\
&= \int_{M_2 \setminus H_0} \Big(a_{ij}^{(2)} \left(u^2 \nabla^i\chi\nabla^k\chi + 2\chi u \nabla^i u \nabla^j\chi + \chi^2 \nabla^i u \nabla^j u \right) + a^{(2)}\chi^2 u^2 \Big) \\
&\leq C \int_{M_2 \setminus H_0} \Big(|u|^2 \|\nabla\chi\|^2 + |u|\|\nabla u\|\|\nabla\chi\| \Big) + \int_{M_2\setminus H_0} \chi^2 \Big(a_{ij}^{(2)} \nabla^i u \nabla^j u + a^{(2)} u^2 \Big) \\
&\leq \frac{C\|u\|_\infty^2}{|\log(\alpha_1)|} + \frac{C\|u\|_\infty \|\nabla u\|_2}{|\log(\alpha_1)|^{1/2}} + \int_{M_2\setminus H_0} a_{ij}^{(2)} \nabla^i u \nabla^j u + \int_{M_2 \setminus H_0} \chi^2 a^{(2)} u^2 \\
&\leq \frac{C}{|\log(\alpha_1)|^{1/2}} \left(\|u\|_\infty^2 + \|\nabla u\|_2^2 \right) + \mathcal{A}_1(u,u) + \int_{M_1} (\chi^2 - 1) a^{(1)} u^2 \\
&\leq \frac{C}{|\log(\alpha_1)|^{1/2}} \left(\|u\|_\infty^2 + \|\nabla u\|_2^2 \right) + C\sqrt{\alpha_1}\alpha_2 \|u\|_\infty^2 + \mathcal{A}_1(u,u) \\
&\leq \varepsilon \big(\|u\|_\infty^2 + \|\nabla u\|_2^2 \big) + \mathcal{A}_1(u,u)
\end{aligned}$$

if α_1 and α_2 are sufficiently small. We have used the fact that $\chi^2 a^{(2)}_{ij} \nabla^i u \nabla^j u \leq a^{(2)}_{ij} \nabla^i u \nabla^j u$ since $a^{(2)}_{ij} \nabla^i u \nabla^j u$ is positive and $\chi \leq 1$. Also we have used $Vol(M_1 \cap H) = \mathcal{O}(\sqrt{\alpha_1}\alpha_2)$. Hence our estimate follows. The estimate $\mathcal{A}_1(F_2(u), F_2(u)) \leq \varepsilon\big(\|u\|^2_\infty + \|\nabla u\|^2_2\big) + \mathcal{A}_2(u,u)$ holds by similar calculations.

It remains to prove that Assumption (IV) holds for each surface. We can of course assume that all the desired bounds hold for M_2 since these follow by standard elliptic theory on M_2 along with the compactness and non-collapsedness of M_2. Details of this theory can be found in [5] with extensions to surfaces in [1, 7, 9]. But it is a non-trivial question whether these estimates remain valid for M_1 due to the presence of the neck region in M_1.

As a matter of fact, the desired estimate is not true if λ is too large compared to the parameters specifying the size of the neck region of M_1. The reason is that there exist eigenfunctions of $\mathcal{L}_1$ that are *concentrated* in the neck region — namely these eigenfunctions have unit L^2 norm but are mostly supported in the neck region, which forces their L^∞ norm to be very large there. Such eigenfunctions exist because the Dirichlet problem for $\mathcal{L}_1$ restricted to $M_1 \cap H$ has eigenfunctions and these are a good approximation of the problematic eigenfunctions in question here. (It will become clear below in what sense we can expect the approximation to be valid.) However, the first eigenvalue of the Dirichlet problem for $\mathcal{L}_1$ restricted to $M_1 \cap H_0$ is of size $\mathcal{O}(\alpha_1^{-2})$ since $M_1 \cap H_0$ can be scaled up by a factor of α_1 to a surface with uniform geometry. Thus we expect that the problematic eigenfunctions won't cause us any trouble provided we seek only to estimate functions solving $\mathcal{L}_i(u) = \lambda u$ with λ sufficiently small. The following lemma validates this discussion.

Lemma 3. *There is a number $\mu(\varepsilon)$ with $\lim_{\varepsilon\to 0}\mu(\varepsilon) = \infty$ so that if $\lambda < \mu(\varepsilon)$ then any solution $u \in C^{2,\beta}(M_1)$ of the equation $\mathcal{L}_1(u) = \lambda u$ satisfies the estimate $\|u\|_\infty \leq C\|u\|_2$ where C is a constant depending only on λ and the geometry of $\Sigma_\pm$.*

Proof. Suppose that $u \in C^{2,\beta}(M_1)$ satisfies $\mathcal{L}_1(u) = \lambda u$. Let χ be a cut-off function that vanishes in $M_1 \cap H_0$ and transitions monotonically to one in $M_1 \cap H_1$ where $H_1 := \{(x,y,z) : \alpha_1 \leq |x| \leq 2\alpha_1\}$. Note that this is a different cut-off function that the one we have used before. Write $u = \chi u + (1-\chi)u := u_{ext} + u_{neck}$. The function $u\big|_{M_1 \setminus H_0}$ can be viewed as a function defined on $\Sigma_- \dot\cup \Sigma_+$ and thus can be estimated using standard interior elliptic estimates as $\|u\|_{L^\infty(M_1\setminus H_0)} \leq C\|u\|_2$ where C depends only on λ and $\Sigma_\pm$. Hence u_{ext} can be so estimated as well. Now observe that

$$\mathcal{L}_1(u_{neck}) - \lambda u_{neck} = -[\mathcal{L}_1, \chi](u) := \mathcal{L}(\chi u) - \chi \mathcal{L}_1(u)\ .$$

Note that since $[\mathcal{L}_1, \chi](u)$ contains $\nabla\chi \cdot \nabla u$ and $\nabla^2\chi \cdot u$ terms only, it is supported in $M_1 \cap [H_1 \setminus H_0]$.

Let us now consider u_{neck} as a function of compact support on the interior of $M_1 \cap H_1$. The Dirichlet eigenvalue problem $\mathcal{L}_1(u) - \sigma u = 0$ with $u\big|_{\partial M_1 \cap H_1} = 0$ has a first eigenvalue σ_0. And because we can re-scale $M_1 \cap H_1$ by the factor α_1^{-1} and obtain a surface with uniformly bounded geometry, this eigenvalue satisfies $\sigma = \mathcal{O}(\alpha_1^{-2})$. Therefore so long as $\sigma - \lambda$ is positive and uniformly bounded away from zero, the operator $\mathcal{L}_1 - \lambda$ is uniformly invertible. Thus we have the estimate

$$|u_{neck}|^*_{C^{2,\beta}} \leq C\alpha_1^2 |\mathcal{L}_1(u_{neck}) - \lambda u_{neck}|^*_{C^{0,\beta}} \leq C\alpha_1^2 \big|[\mathcal{L}_1,\chi](u)\big|^*_{C^{0,\beta}}$$

where the $|\cdot|^*_{C^{k,\beta}}$ norm is the *scale invariant* version of this norm in which the k^{th} derivative is weighted by α_1^k and the Hölder seminorm receives an extra factor of α_1^β. The reason for this weighting by factors of α_1 is because if we re-scale $M_1 \cap H_1$ by the factor α_1^{-1} we obtain a surface with uniformly bounded geometry for which a version of the above estimate holds but where all constants are independent of α_1. Reversing the scaling then yields the above estimate with the factors of α_1 as given. Now,

$$\begin{aligned} \alpha_1^2 \big|[\mathcal{L}_1,\chi](u)\big|^*_{C^{0,\beta}} &= \alpha_1^2 \big| a^{(1)}_{ij} \big(\nabla^j\nabla^i(\chi u) - \chi\nabla^j\nabla^i u\big)\big|^*_{C^{0,\beta}} \\ &\leq C\alpha_1^2 \big| |u| \|\nabla^2\chi\| + \|\nabla u\| \|\nabla\chi\|\big|^*_{C^{0,\beta}} \end{aligned}$$

$$\leq C|u|^*_{C^{1,\beta}(M_1\cap[H\setminus H_0])}$$

since it is easy to show that the pointwise norm of χ satisfies $|\chi|^*_{C^{2,\beta}} \leq C$ and both $\nabla\chi$ and thus $\nabla^2\chi$ are supported in the transition region of M_1. Our a priori estimate for M_2 now implies that $|u|_{C^0(M_1\cap[H\setminus H_0])} \leq C\|u\|_2$. Scaling and the local Schauder estimate for $\mathcal{L}_1 - \lambda$ then together imply that $\alpha_1\|\nabla u\|_{C^0(M_1\cap[H\setminus H_0])} + \alpha^{1+\beta}[\nabla u]_{\beta,M_1\cap[H\setminus H_0]} \leq C\|u\|_2$ as well. We thus find that

$$|u_{neck}|^*_{C^{2,\beta}} \leq C\|u\|_2$$

which implies that $\|u\|_{L^\infty(M_1\cap H)} \leq C\|u\|_2$. This estimate together with the previous estimate for u in the rest of M_1 yields the lemma. $\square$

5 Remarks on Vector Equations

We must still verify that the above analysis holds for the operator $\mathcal{L} := P^*P$ where P is the AKVF operator defined as the symmetric part of the covariant derivative tensor of X, namely $P(X) := Sym(\nabla X)$. Clearly, the various assumptions on the smoothness of the coefficients of the operator hold. What we must check is that the above analysis carries over to *systems* of second-order partial differential equations — since after all $\mathcal{L}$ is not a scalar operator (in local coordinates, the equation $\mathcal{L}(X) = \lambda X$ is a *system* of equations for the vector quantity X). At first glance, this distinction could potentially matter a great deal since the a number of the special yet very important analytic features of second-order, scalar equations (such as the maximum principle and any a priori estimates related to it) can fail to hold for elliptic systems.

However, a careful examination of the proof of the spectral comparison theorems above shows that we are in fact not using any special features of scalar equations, provided we generalize the coercivity condition from Assumption II properly. But coercivity of the quadratic form $\mathcal{A}_i$ is a consequence of the behaviour of its *principal symbol*, namely the algebraic operator $\xi \mapsto \sum_{s,t} a^{(i)}_{st}\xi^s\xi^t$ where $\xi \in \mathbb{R}^2$, since coercivity follows if we have the algebraic inequality $\sum_{s,t} a^{(i)}_{st}\xi^s\xi^t \geq C\|\xi\|^2$ for some $C > 0$. It is the notion of principal symbol and the algebraic coercivity inequality which must be properly generalized to elliptic systems.

This is done as follows. Let $\mathcal{L}$ be a vector operator on a surface Σ whose expression in terms a local coordinate frame for the tangent bundle of Σ is given by $\mathcal{L}(X) := -\sum_{stj}\nabla^t\big(a^i_{stj}\nabla^s X^j\big)$. The associated quadratic form is now given by

$$\mathcal{A}(X,Y) := \int_\Sigma \sum_{stjk} a_{stjk}\nabla^s X^j \nabla^t X^k$$

where $a_{stjk} := \sum_i g_{ik}a^i_{stj}$ and g_{ik} are the components of induced metric of Σ. The appropriate generalization of the above algebraic inequality is called the *strong Legendre condition*, which states that

$$\sum_{stjk} a_{stjk}\xi^{sj}\xi^{tk} \geq C\|\xi\|^2 := C\sum_{si}(\xi^{si})^2 \qquad \forall\, \xi \in \mathbb{R}^4\,.$$

It is straightforward to verify that the strong Legendre condition holds for the operator P^*P. In fact, the principal symbol of P^*P has coefficients $a_{1111} = 4$, $a_{1122} = 2$, $a_{2211} = 2$, $a_{1221} = 2$, $a_{2112} = 2$, $a_{2222} = 4$ and all others zero. Then $\sum_{stjk} a_{stjk}\xi^{sj}\xi^{tk} \geq 4(\xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2)$ so that the condition holds.

Once we have the strong Legendre condition, then the a priori estimates for solutions of $P^*P(X) = \lambda X$ follow by standard elliptic estimates for systems. The details can be found in [3]. The remaining steps of the spectral comparison theorems can still be carried out as before, but with extra book-keeping to keep track of the vectorial nature of the objects involved.

References

[1] Thierry Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[2] I. Chavel and E. Feldman, *Spectra of manifolds with small handles*, Comment. Math. Helv. **56** (1981), no. 1, 83–102.

[3] Y. Chen and L. Wu, *Second order elliptic equations and elliptic systems*, Translations of Mathematical Monographs, vol. 174, American Mathematical Society, Providence, RI, 1998, Translated from the 1991 Chinese original by Bei Hu.

[4] Kenji Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), no. 3, 517–547.

[5] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[6] S. Jimbo and Y. Morita, *Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels*, Comm. Partial Differential Equations **17** (1992), no. 3-4, 523–552.

[7] Jürgen Jost, *Riemannian geometry and geometric analysis*, Universitext, Springer-Verlag, Berlin, 1995.

[8] Nicolaos Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2) **131** (1990), no. 2, 239–330.

[9] Peter Li, *Lecture notes on geometric analysis*, Lecture Notes Series, vol. 6, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1993.

[10] Rafe Mazzeo, *Recent advances in the global theory of constant mean curvature surfaces*, Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math., vol. 350, Amer. Math. Soc., Providence, RI, 2004, pp. 179–199.