Laplacian and Random Walks on Graphs

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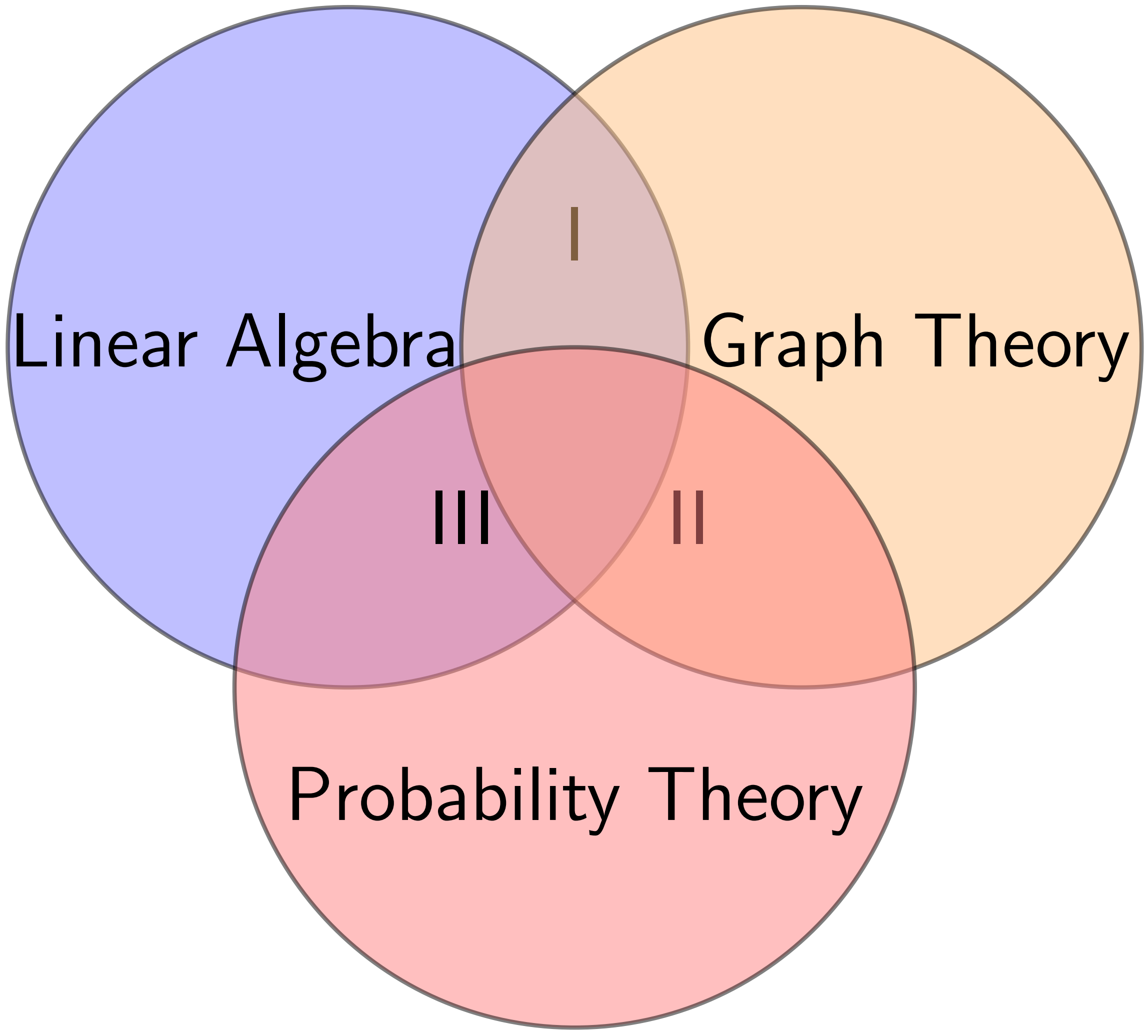
Selected Topics on Spectral Graph Theory (II)
Nankai University, Tianjin, May 22, 2014

Five talks

Selected Topics on Spectral Graph Theory

1. Graphs with Small Spectral Radius
   Time: Friday (May 16) 4pm.-5:30p.m.
2. Laplacian and Random Walks on Graphs
   Time: Thursday (May 22) 4pm.-5:30p.m.
3. Spectra of Random Graphs
   Time: Thursday (May 29) 4pm.-5:30p.m.
4. Hypergraphs with Small Spectral Radius
   Time: Friday (June 6) 4pm.-5:30p.m.
5. Lapalacian of Random Hypergraphs
   Time: Thursday (June 12) 4pm.-5:30p.m.

Backgrounds



I: Spectral Graph Theory II: Random Graph Theory
III: Random Matrix Theory

Outline

- Combinatorial Laplacian
- Normalized Laplacian
- An application

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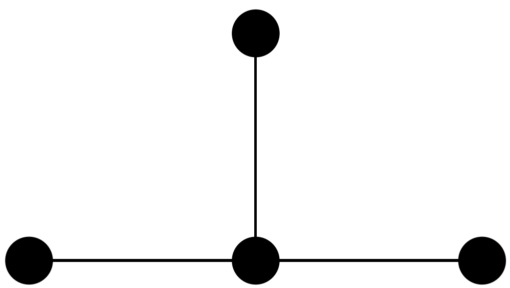

S_4

$$L(S_4) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

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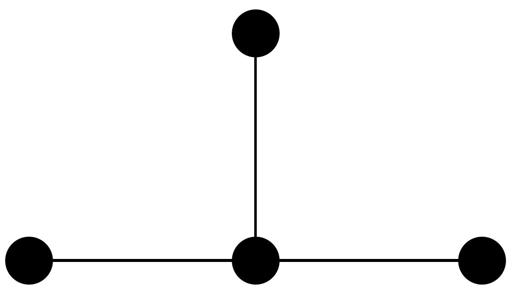

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Combinatorial Laplacian eigenvalues of S_4: $0, 1, 1, 4$.

Matrix-tree Theorem

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Proof: Fix an orientation of G, let B be the incidence matrix of the orientation, i.e., $b_{ve} = 1$ if v is the head of the arc e, $b_{ve} = -1$ if i is the tail of e, and $b_{ve} = 0$ otherwise. Let L_{11} be the sub-matrix obtained from L by deleting the first row and first column, and B_1 be the matrix obtained from B by deleting the first row. Then $L_{11} = B_1 B_1'$.

$$\begin{aligned} \det(L_{11}) &= \det(B_1 B_1') \\ &= \sum_S \det(B_S)^2 \qquad \text{By Cauchy-Binet formula} \\ &= \text{the number of Spanning Trees.} \qquad \square \end{aligned}$$

An application

Corollary: If G is connected, and $\lambda_1, \ldots, \lambda_{n-1}$ be the non-zero eigenvalues of L. Then the number of spanning tree is

$$\frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}.$$

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Chung-Yau [1999]: The number of spanning trees in any d-regular graph on n vertices is at most

$$(1+o(1))\frac{2\log n}{dn\log d}\left(\frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}}\right)^n.$$

This is best possible within a constant factor.

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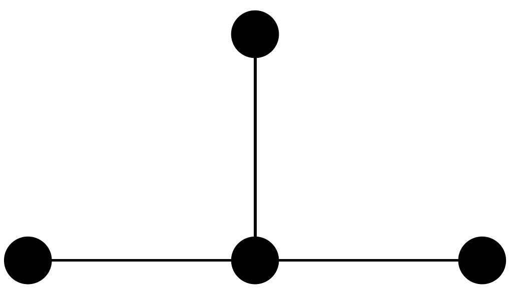

$$\mathcal{L}(S_4) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 1 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & 1 \end{pmatrix}$$

(Normalized) Laplacian eigenvalues of S_4:
$\lambda_0 = 0$, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$.

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- $\lambda_{n-1} = 2$ if and only if G is bipartite.
- $\lambda_1 > 1$ if and only if G is the complete graph.

Rayleigh quotients

The Laplacian eigenvalues can also be computed by Rayleigh quotients: for $0 \le i \le n-1$,

$$\lambda_i = \sup_{\dim(M)=n-i} \inf_{f \in M} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f(x)^2 d_x}.$$

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- many other applications.

Random walks

A walk on a graph is a sequence of vertices together a sequence of edges:

$$v_0, v_1, v_2, v_3, \ldots, v_k, v_{k+1}, \ldots$$

$$v_0v_1, v_1v_2, v_2v_3, \ldots, v_kv_{k+1}, \ldots$$

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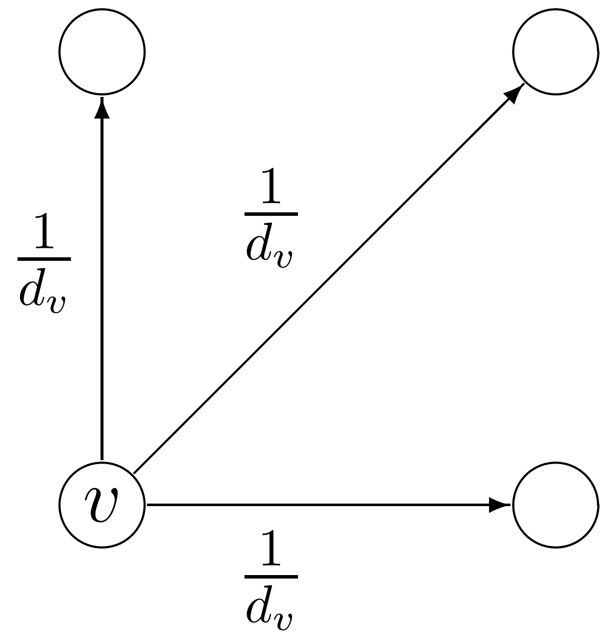
Random walks on a graph G:

$$f_{k+1} = f_k D^{-1} A.$$

$$D^{-1}A \sim D^{-1/2} A D^{-1/2} = I - \mathcal{L}.$$

$\bar{\lambda}$ determines the mixing rate of random walks.



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- Mixing:

$$\|(f_k - \pi)D^{-1/2}\| \leq \bar{\lambda}^k \|(f_0 - \pi)D^{-1/2}\|.$$

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If G is bipartite, then the random walk does not mix. In this case, we will use the α-lazy random walk with the transition matrix $\alpha I + (1-\alpha)D^{-1}A$.

Diameter

Suppose that G is not a complete graph. Then the diameter of G satisfies

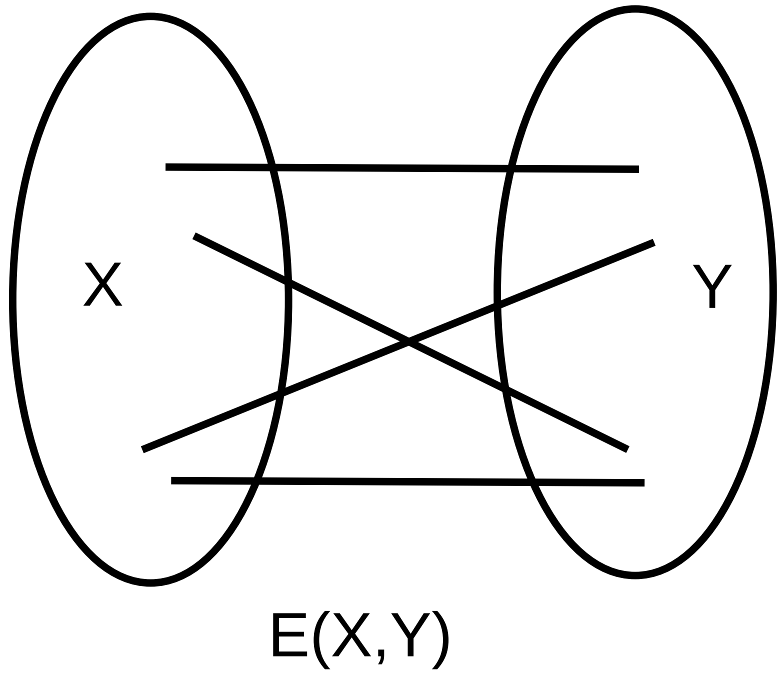
$$\text{diam}(G) \leq \left\lceil \frac{\log(\text{vol}(G)/\delta)}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil,$$

where δ is the minimum degree of G.

Edge discrepancy

Let $\mathrm{vol}(X) = \sum_{x \in X} d_x$ and $\bar{\lambda} = \max\{1 - \lambda_1, \lambda_{n-1} - 1\}$.

Then $\left| |E(X,Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}$.



Proof

Let $\mathbf{1}_X$ and $\mathbf{1}_Y$ be the indicated vector of X and Y respectively. Let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ be orthogonal unit eigenvectors of $\mathcal{L}$. Write $D^{1/2}\mathbf{1}_X = \sum_{i=0}^{n-1} x_i\alpha_i$ and $D^{1/2}\mathbf{1}_Y = \sum_{i=0}^{n-1} y_i\alpha_i$. Then

$$\begin{aligned}
|E(X,Y)| &= \mathbf{1}'_X A \mathbf{1}_Y \\
&= (D^{1/2}\mathbf{1}_X)'(I-\mathcal{L})D^{1/2}\mathbf{1}_Y \\
&= \sum_{i=0}^{n-1}(1-\lambda_i)x_i y_i.
\end{aligned}$$

continue

Note $x_0 = \frac{\mathrm{vol}(X)}{\sqrt{\mathrm{vol}(G)}}$ and $y_0 = \frac{\mathrm{vol}(Y)}{\sqrt{\mathrm{vol}(G)}}$. Hence,

$$\left| |E(X,Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right|$$

$$= \sum_{i=1}^{n} (1-\lambda_i) x_i y_i$$

$$\leq \bar{\lambda} \sqrt{\sum_{i=1}^{n-1} x_i^2} \sqrt{\sum_{i=1}^{n-1} y_i^2}$$

$$= \bar{\lambda} \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}. \qquad \square$$

Cheeger Constant

For a subset $S \subset V$, we define

$$h_G(S) = \frac{|E(S, \bar{S})|}{\min(\mathrm{vol}(S), \mathrm{vol}(\bar{S}))}.$$

The Cheeger constant h_G of a graph G is defined to be $h_G = \min_S h_G(S)$.

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Cheeger's inequality:

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}.$$

d-regular graph

If G is d-regular graph, then adjacency matrix, combinatorial Laplacian, and normalize Laplacian are all equivalent.

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Suppose A has eigenvalues $\mu_1, \ldots, \mu_n$. Then

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The theories of three matrices apply to the d-regular graphs.

An application

Constructing Small Folkman Graphs.

Ramsey's theorem

For integers $k, l \geq 2$, there exists a least positive integer $R(k,l)$ such that no matter how the complete graph $K_{R(k,l)}$ is two-colored, it will contain a blue subgraph K_k or a red subgraph K_l.

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$$R(3,3) = 6$$

$$R(4,4) = 18$$

$$43 \leq R(5,5) \leq 49$$

$$102 \leq R(6,6) \leq 165$$

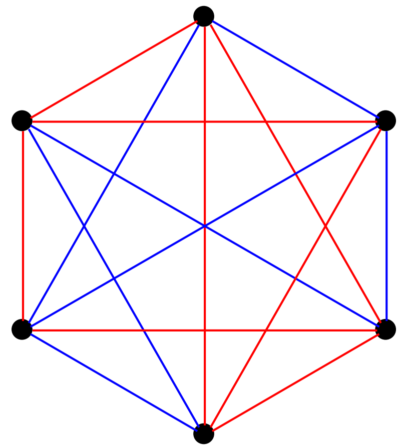
$$\vdots$$

$$(1+o(1))\frac{\sqrt{2}}{e} n 2^{n/2} \leq R(n,n) \leq (n-1)^{-C\frac{\log(n-1)}{\log\log(n-1)}} \binom{2(n-1)}{n-1}.$$

Spencer[1975] **Colon [2009]**

Ramsey number $R(3,3) = 6$

- If edges of K_6 are 2-colored then there exists a monochromatic triangle.



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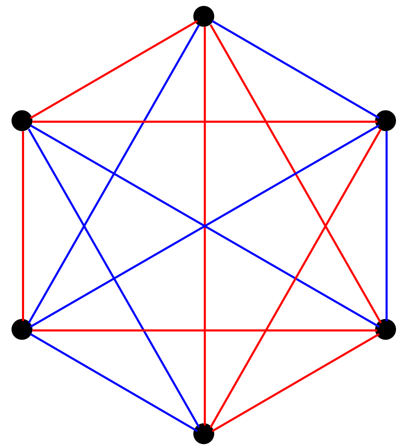
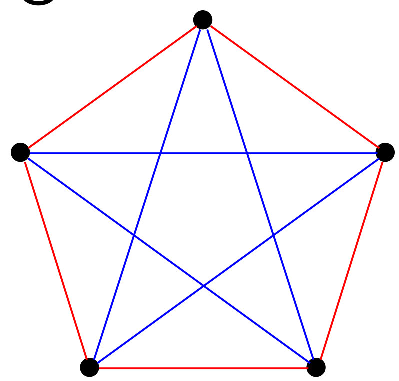
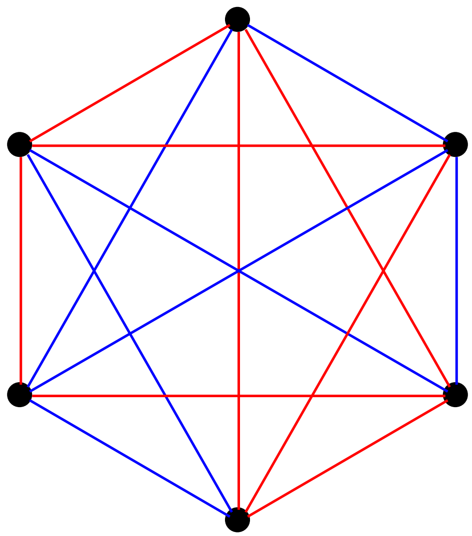


- There exists a 2-coloring of edges of K_5 with no monochromatic triangle.

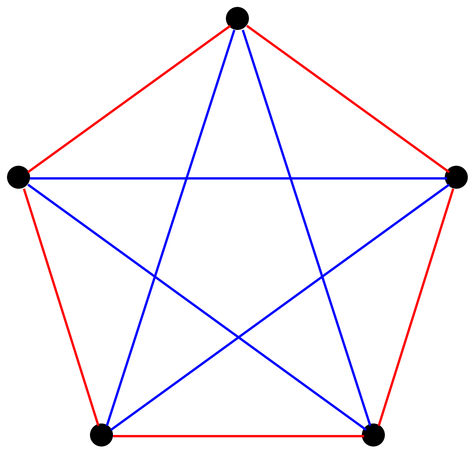


Rado's arrow notation

$G \to (H)$: if the edges of G are 2-colored then there exists a monochromatic subgraph of G isomorphic to H.



$K_6 \to (K_3)$



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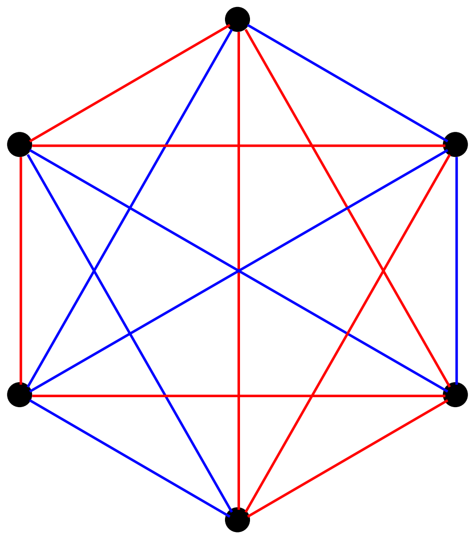

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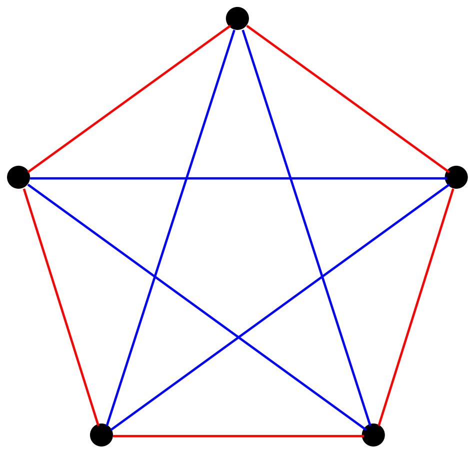

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Fact: If $K_6 \subset G$, then $G \to (K_3)$.

An Erdős-Hajnal Question

Is there a K_6-free graph G with $G \to (K_3)$?

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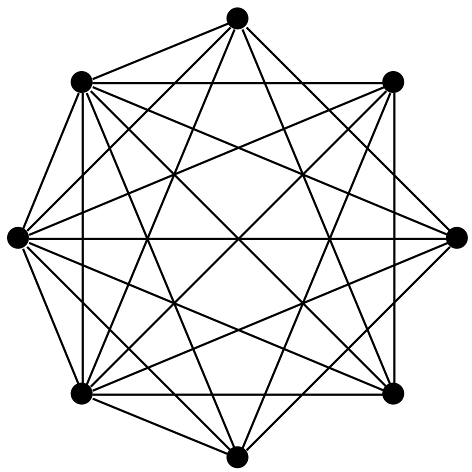
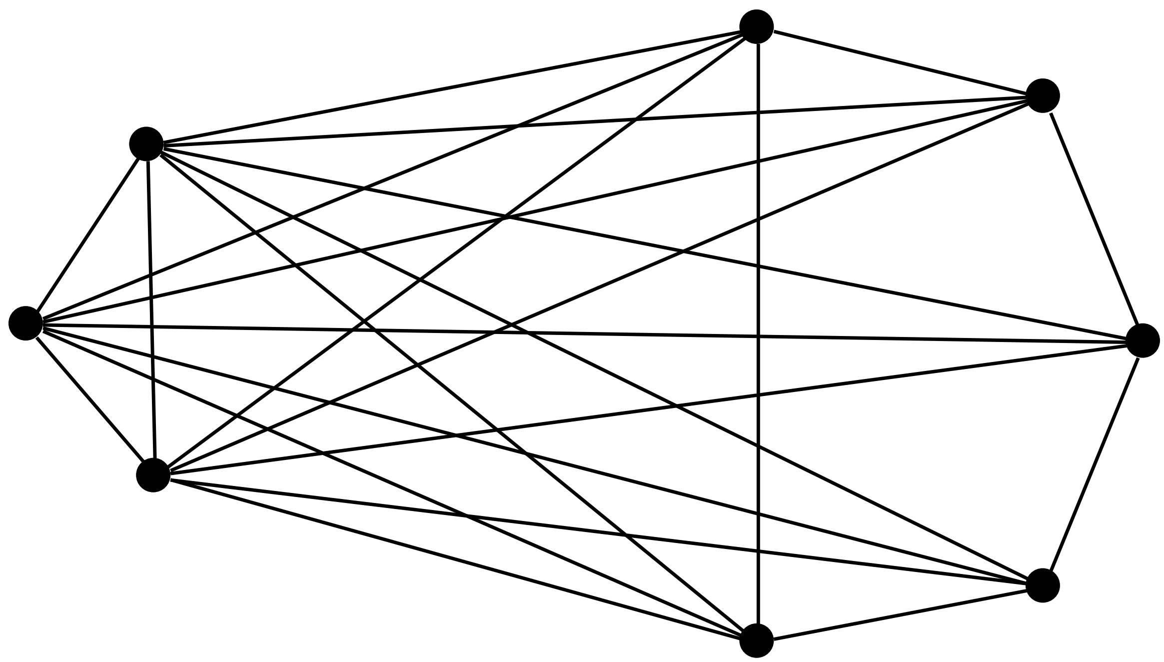
Graham (1968): Yes!



$K_8 \setminus C_5$

Graham's graph $K_8 \setminus C_5$

Suppose G has no monochromatic triangle.



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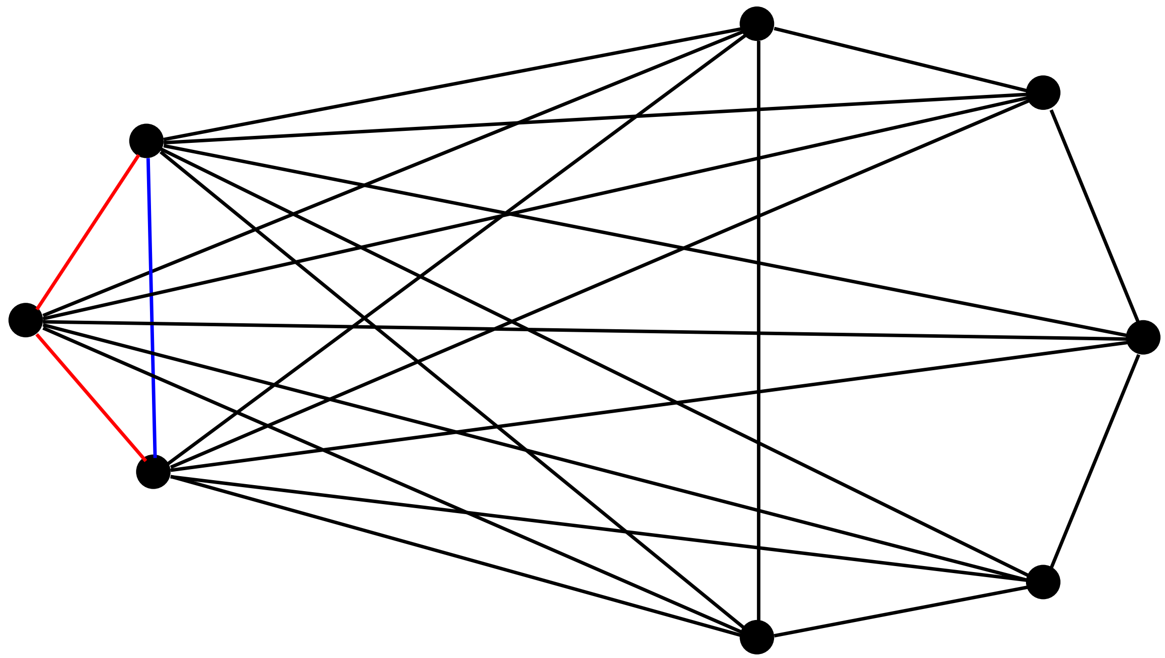

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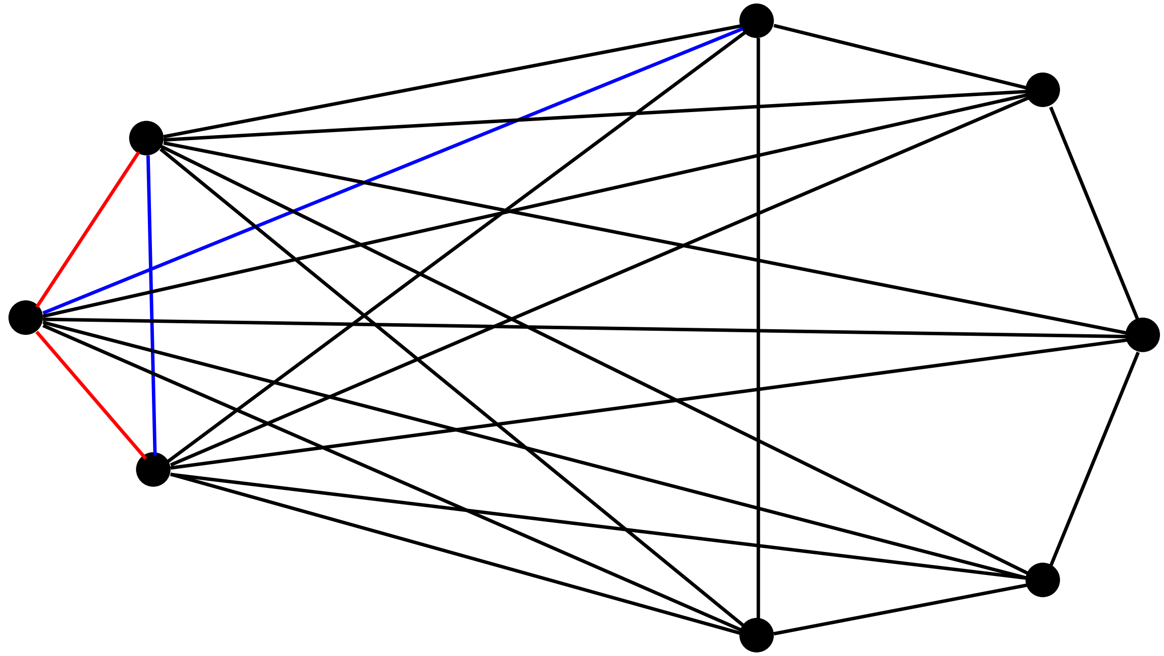

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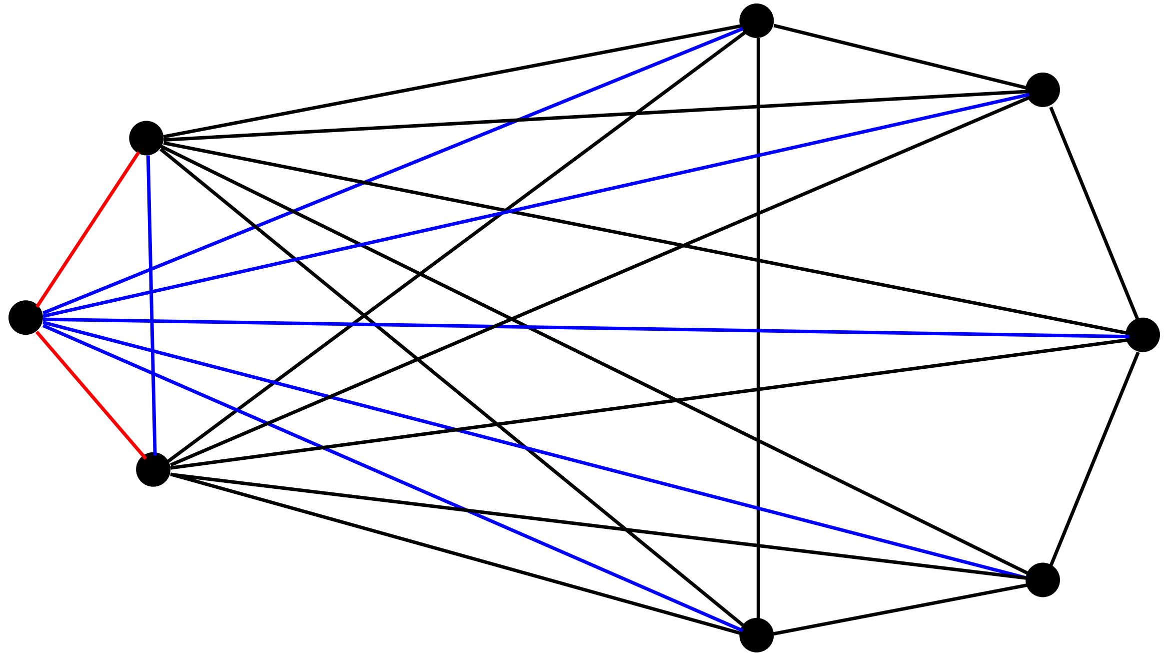

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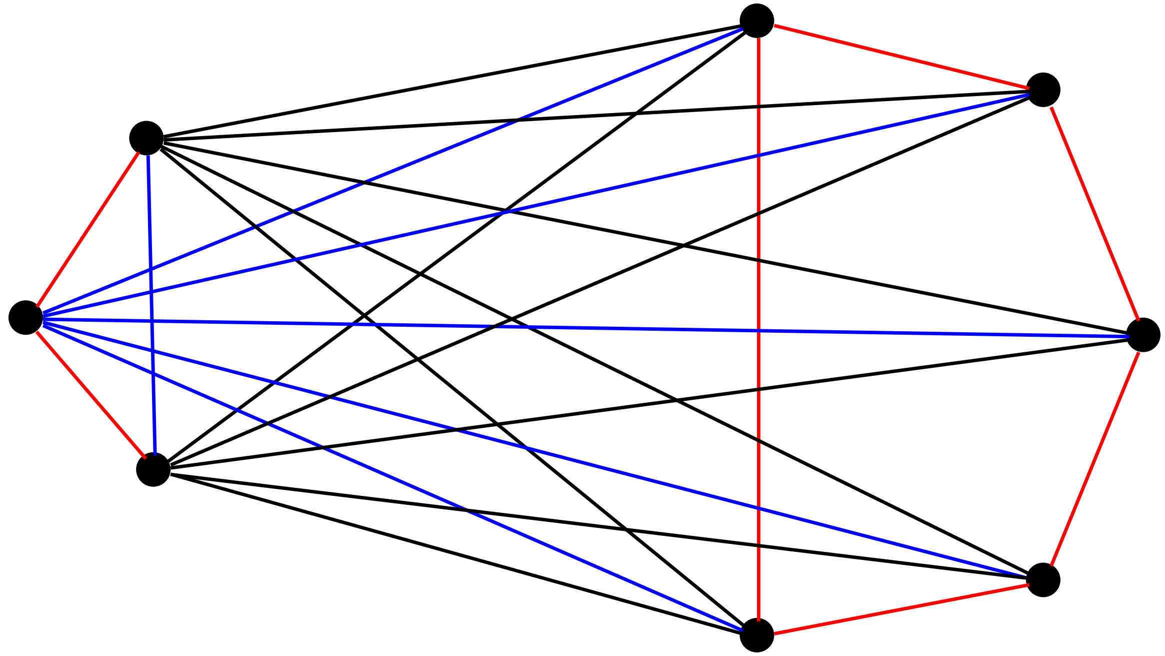

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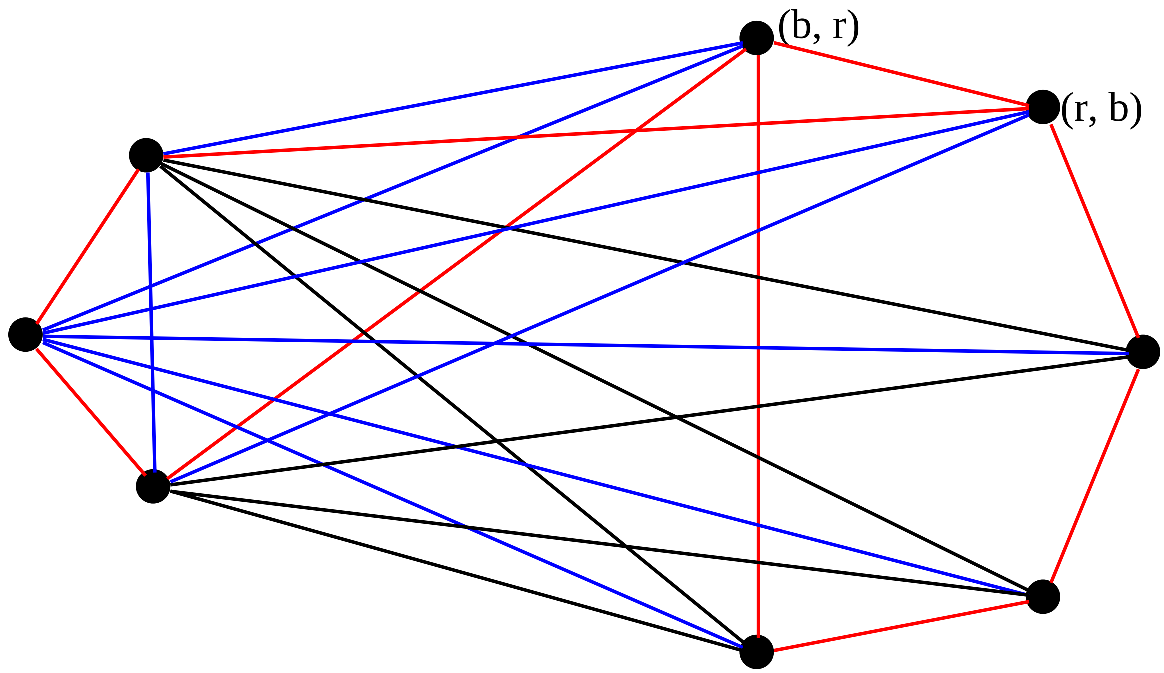

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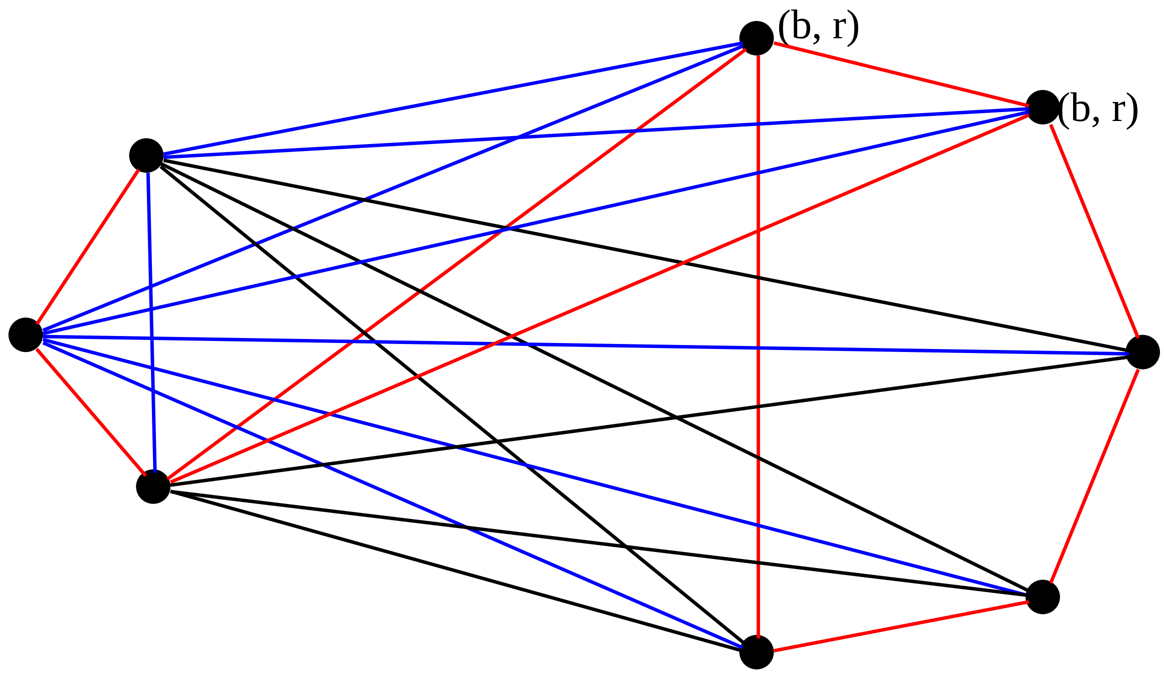

Label the vertices of C_5 by either (r, b) or (b, r).
A red triangle is unavoidable since $\chi(C_5) = 3$.

K_5-free G with $G \to (K_3)$

Year	Authors	$\|G\|$
1969	Schäuble	42
1971	Graham, Spencer	23
1973	Irving	18
1979	Hadziivanov, Nenov	16
1981	Nenov	15

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In 1998, Piwakowski, Radziszowski and Urbański used a computer-aided exhaustive search to rule out all possible graphs on less than 15 vertices.

General results

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These graphs are called Folkman Graphs.

General results

Folkman's theorem (1970): *For any $k_2 > k_1 \geq 3$, there exists a K_{k_2}-free graph G with $G \to (K_{k_1})$.*

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Nešetřil-Rödl's theorem (1976): *For $p \geq 2$ and any $k_2 > k_1 \geq 3$, there exists a K_{k_2}-free graph G with $G \to (K_{k_1})_p$.*

Here $G \to (H)_p$: if the edges of G are p-colored then there exists a monochromatic subgraph of G isomorphic to H.

$f(p, k_1, k_2)$

Let $f(p, k_1, k_2)$ denote the smallest integer n such that there exists a K_{k_2}-free graph G on n vertices with $G \to (K_{k_1})_p$.

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- What about $f(2, 3, 4)$?

Upper bound of $f(2,3,4)$

- **Folkman, Nešetřil-Rödl** 's upper bound is huge.
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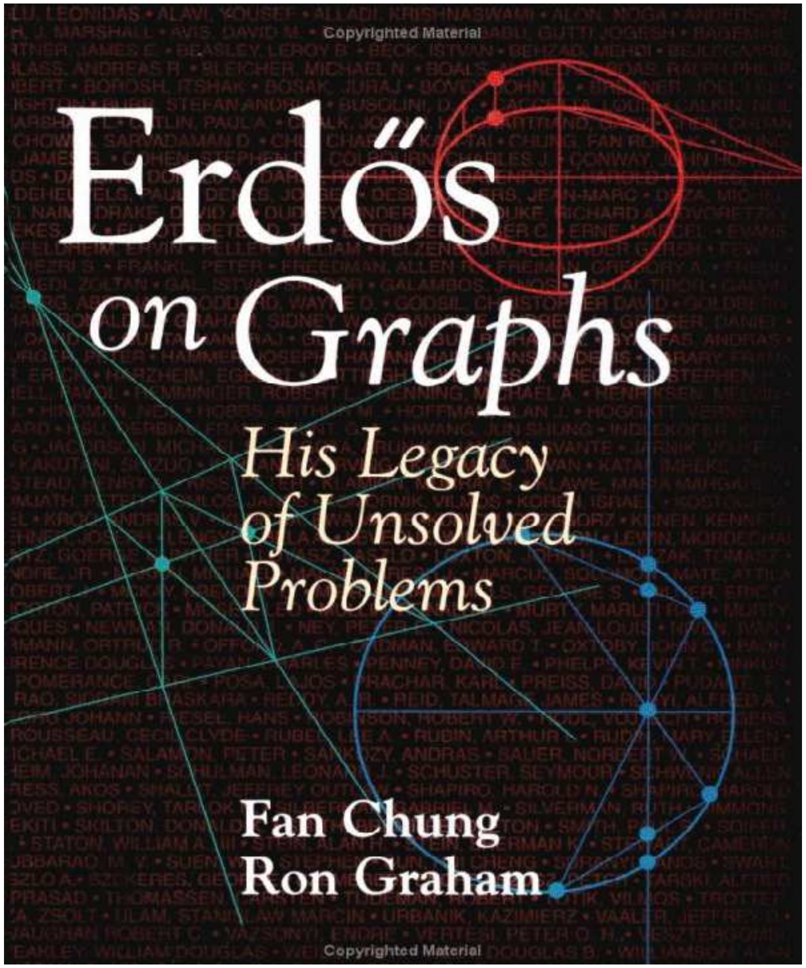
Most wanted Folkman Graph



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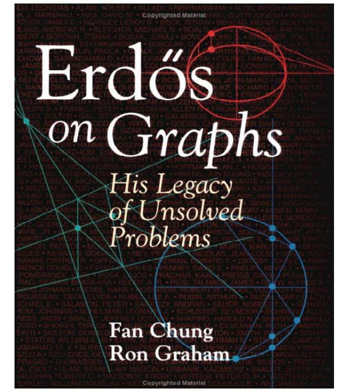

Problem on triangle-free subgraphs in graphs containing no K_4 $100
(proposed by Erdős)[48]
Let $f(p, k_1, k_2)$ denote the smallest integer n such that there is a graph G with n vertices satisfying the properties:
(1) any edge coloring in p colors contains a monochromatic K_{k_1};
(2) G contains no K_{k_2}.
Prove or disprove:

$$f(2,3,4) < 10^6.$$

Difficulty

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- For moderate n, Folkman graphs are very rare among all K_4-free graphs on n vertices.
- Probabilistic methods are generally good choices for asymptotic results. However, it is not good for moderate size n.

Our approach

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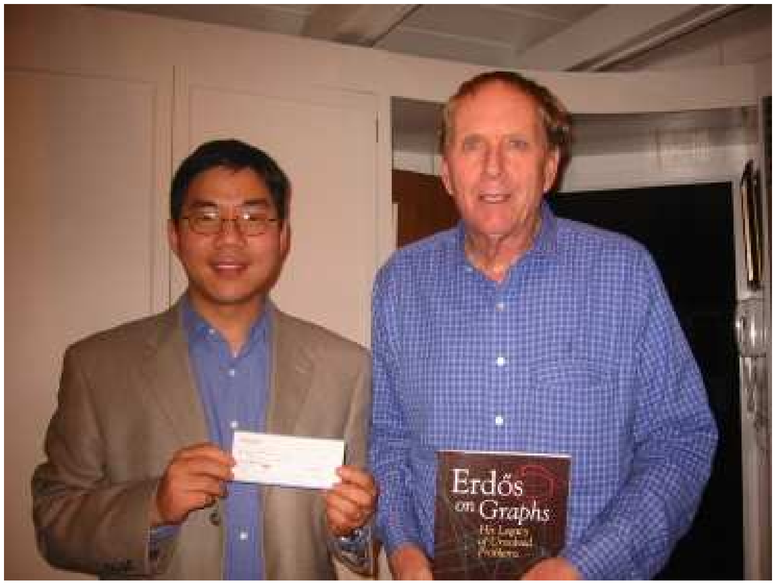
- Localization and δ-fairness.
- Circulant graphs and $L(m, s)$.

My result

I received \$100-award by proving

Theorem [Lu, 2007]: $f(2,3,4) \leq 9697.$



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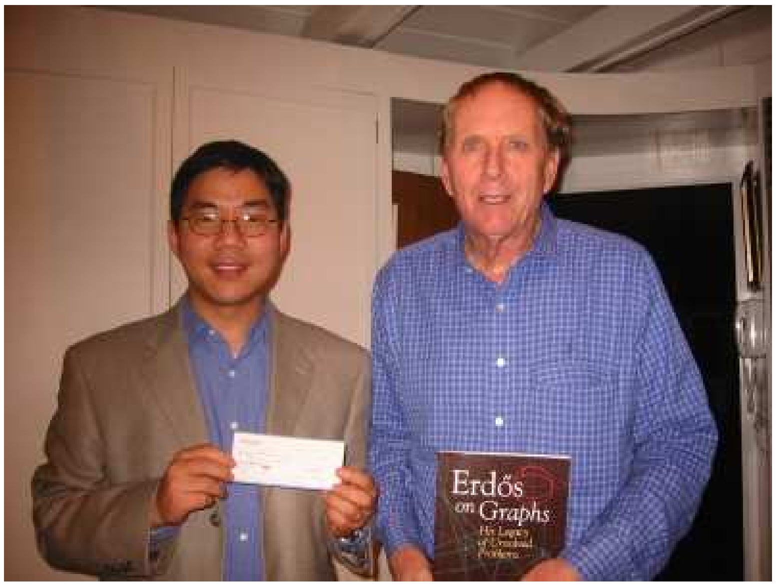

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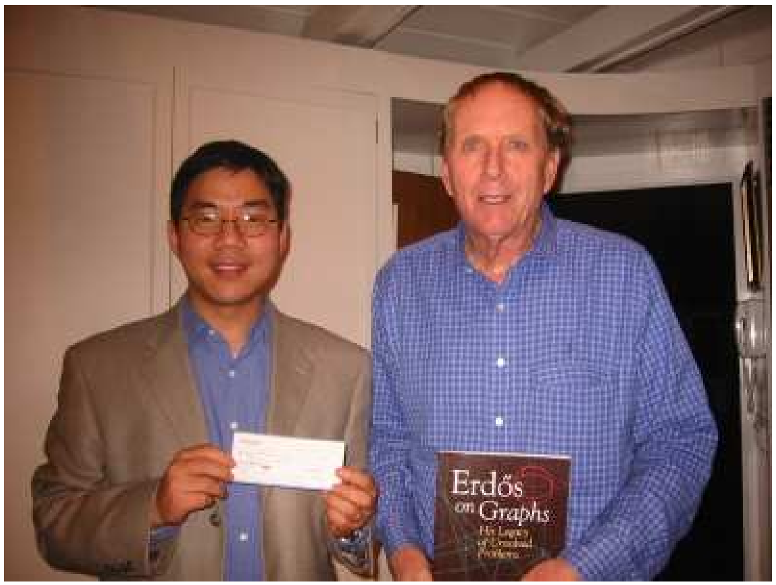

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Lange-Radziszowski-Xu [2012+]: $f(2,3,4) \leq 786$.

Spencer's Lemma

Notations:

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For vertex transitive graph G, all G_v's are isomorphic.

Spectral lemma

- H: a graph on n vertices
- A: the adjacency matrix of H
- $\mathbf{d} = (d_1, d_2, \ldots, d_n)$: degrees of H
- $\mathrm{Vol}(S) = \sum_{v \in S} d_v$: the volume of S
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Similar results hold for A and L. However, they are weaker than using M in experiments.

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Proof: We can replace M by A in the previous lemma.

- $\mathbf{1}$ is an eigenvector of A with respect to d.
- M is the projection of A to the hyperspace $\mathbf{1}^{\perp}$.
- M and A have the same smallest eigenvalues.

The proof of the Lemma

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- We observe $M\mathbf{1} = 0$.
- For each $t \in (0,1)$, let $\alpha(t) = (1-t)\mathbf{1}_X - t\mathbf{1}_Y$. We have

$$\alpha(t)' \cdot M \cdot \alpha(t) = -e(X,Y) + \frac{1}{\mathrm{Vol}(H)}\mathrm{Vol}(X)\mathrm{Vol}(Y).$$

The proof of the Lemma

Let ρ be the smallest eigenvalue of M. We have

$$e(X,Y) - \frac{\mathrm{Vol}(X)\mathrm{Vol}(Y)}{\mathrm{Vol}(H)} \leq -\alpha(t)' \cdot M \cdot \alpha(t) \leq -\rho \|\alpha_t\|^2.$$

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Choose $t = \frac{|X|}{n}$ so that $\|\alpha(t)\|^2$ reaches its minimum $\frac{|X||Y|}{n}$.

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$$\begin{aligned} e(X,Y) &\leq \frac{\text{Vol}(X)\text{Vol}(Y)}{\text{Vol}(H)} + \rho\frac{|X||Y|}{n}. \\ &\leq \frac{\text{Vol}(H)}{4} - \rho\frac{n}{4} \\ &< (\frac{1}{2} + \delta)|E(H)|, \text{ since } \rho > -2\delta\bar{d}. \qquad \square \end{aligned}$$

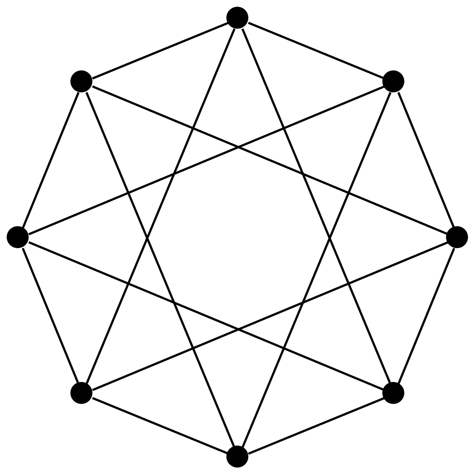
Circulant graphs

- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$
- S: a subset of $\mathbb{Z}_n$ satisfying $-S = S$ and $0 \notin S$.

We define a circulant graph H by

- $V(H) = \mathbb{Z}_n$
- $E(H) = \{xy \mid x - y \in S\}$.

Example: A circulant graph with $n = 8$ and $S = \{\pm 1, \pm 3\}$.



Spectrum of circulant graphs

Lemma: *The eigenvalues of the adjacency matrix for the circulant graph generated by* $S \subset \mathbb{Z}_n$ *are*

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Proof: Note $A = g(J)$, where

$$g(x) = \sum_{s \in S} x^s.$$

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Proof continues...

Let $\phi = e^{\frac{2\pi\sqrt{-1}}{n}}$ denote the primitive n-th unit root.
J has eigenvalues

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Thus, the eigenvalues of $A = g(J)$ are

$$g(1), g(\phi), \ldots, g(\phi^{n-1}).$$

For $i = 0, 1, 2, \ldots, n-1$, we have

$$g(\phi^i) = \Re(g(\phi^i)) = \sum_{s \in S} \cos \frac{2\pi i s}{n}. \qquad \square$$

Graph $L(m, s)$

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Proposition: The local graph G_v of $L(m, s)$ is also a circulant graph.

Algorithm

- For each $L(m, s)$, compute the local graph G_v.
- If G_v is not triangle-free, reject it and try a new graph $L(m, s)$.
- If the ratio the smallest eigenvalue verse the largest eigenvalue of G_v is less than $-\frac{1}{3}$, reject it and try a new graph $L(m, s)$.
- Output a Folkman graph $L(m, s)$.

Computational results

$L(m,s)$	σ
$L(127,5)$	$-0.6363\cdots$
$L(761,3)$	$-0.5613\cdots$
$L(785,53)$	$-0.5404\cdots$
$L(941,12)$	$-0.5376\cdots$
$L(1777,53)$	$-0.5216\cdots$
$L(1801,125)$	$-0.4912\cdots$
$L(2641,2)$	$-0.4275\cdots$
$L(9697,4)$	$-0.3307\cdots$
$L(30193,53)$	$-0.3094\cdots$
$L(33121,2)$	$-0.2665\cdots$
$L(57401,7)$	$-0.3289\cdots$

- σ is the ratio of the smallest eigenvalue to the largest eigenvalue in the local graph.

- All graphs on the left are K_4-free.
- Graphs in red are Folkman graphs.
- Graphs in black are good candidates.

Improvements

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- **Lange-Radziszowski-Xu [2012+]:** $f(2,3,4) \leq 786$.

Dudek and Rodl

Given a graph G, a triangle graph H_G is defined as

- $V(H_G) = E(G)$
- $e_1 \sim e_2$ in H_G if e_1 and e_2 belong to the same triangle of G.

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Dudek, Rodl, 2008

- If $b(H_G) < \frac{2}{3}|E(H_G)|$, then $G \to (K_3)$.
- If H_G is $\frac{1}{6}$-fair, then $G \to (K_3)$.

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Proof: Let $G = L(941, 12)$. Then G is 188 regular. The triangle graph H has $941 * 188/2 = 88454$ vertices and 2122896 edges.

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Using Matlab, they calculate the least eigenvalue

$$\mu_n \geq -15.196 > -(\frac{1}{2} + \frac{1}{6})24.$$

So H is $\frac{1}{6}$-fair. Done.

Lange-Radziszowski-Xu

Instead of spectral methods, they use semi-definite program (SDP) to approximate the MAX-CUT problem.

- First they try the graph G_1 obtained from $L(941, 12)$ by deleting 81 vertices. They showed

$$3b(H_{G_1}) < 1084985 < 1085028 = 2|E(H_{G_1})|.$$

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- Second they try the graph G_2 obtained from $L(785, 53)$ by one vertex and some 60 edges. They showed

$$3b(H_{G_2}) < 857750 < 857762 = 2|E(H_{G_2})|.$$

  This implies $f(2, 3, 4) \leq 786.$

Open questions

- Exoo conjectured $L(127, 5)$ is a Folkman graph.
- In 2012 SIAMDM, Ronald Graham announced a \$100 award for determining if $f(2, 3, 4) < 100$.
- A open problem on 3-colors: prove or disprove

$$f(3, 3, 4) \leq 3^{3^4}.$$

References

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Homepage: http://www.math.sc.edu/$\sim$ lu/

Thank You